Monetary Policy and Bubbles in a New Keynesian Model with Overlapping Generations

Jordi Gali *

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Abstract

I develop an extension of the basic New Keynesian model with overlapping generations, finite lives and retirement. In contrast with the standard model, the proposed framework allows for the existence of rational expectations equilibria featuring asset price bubbles. I examine the conditions under which bubbly equilibria may emerge and the implications for the design of monetary policy.

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*Centre de Recerca en Economia Internacional (CREI), Universitat Pompeu Fabra, and Barcelona GSE. E-mail: jgali@crei.cat. I am thankful for comments to Davide Debortoli, Alberto Martín, Jaume Ventura, Michael Reiter, Orazio Attanasio and seminar participants at CREI, NBER Summer Institute, U. of Mannheim, Rome MFB Conference, Seoul National University and U. of Vienna. I am grateful to Ángelo Gutiérrez, Christian Hoynick, Cristina Manea, and Matthieu Soupré for excellent research assistance. I acknowledge the European Research Council for financial support under the European Union’s Seventh Framework Programme (FP7/2007-2013, ERC Grant agreement no 339656) and the CERCA Programme/Generalitat de Catalunya for funding.
Speculative bubbles in asset prices are viewed by many economists and policymakers as an important source of macro-economic instability, with the bursting of some large bubble often pointed to as a key factor behind many financial crises. A monetary policy that focuses narrowly on inflation and output stability but which neglects the emergence and rapid growth of asset bubbles is often perceived as a potential risk to medium-term macroeconomic and financial stability.1

Interestingly, the recurrent reference to bubbles in the policy debate contrasts with their conspicuous absence in modern monetary models. A likely explanation for this seeming anomaly lies in the fact that standard versions of the New Keynesian model, the workhorse framework used in monetary policy analysis, leave no room for the existence of bubbles in equilibrium, and hence for any meaningful model-based discussion of their possible interaction with monetary policy.2

In the present paper I develop a modified version of the New Keynesian model featuring overlapping generations of finite-lived consumers with retirement.3 The assumption of an infinite sequence of generations makes it possible for the transversality condition of any individual consumer to be satisfied in equilibrium, even in the presence of a bubble that grows at the rate of interest.4 On the other hand, the assumption of retirement (or, more generally, of an eventual transition to inactivity) can generate an equilibrium rate of interest below the economy’s trend growth rate, which is a condition necessary for the size of the bubble to remain bounded relative to the size of the economy. Finally, and in contrast with most models with bubbles found in the literature, the assumption of sticky prices—a key feature of the New Keynesian model—makes monetary policy non-neutral, allowing it to influence the size of the bubble; on the other hand, price stickiness makes it possible for aggregate bubble fluctuations to influence aggregate demand and, hence, output and employment. An appealing feature of the framework developed here is that it nests the standard New Keynesian model as a limiting case, when the probability of death and that of retirement approach zero.

After deriving the equations describing the model’s equilibrium, I characterize the balanced growth paths consistent with that equilibrium and discuss the conditions under which a non-vanishing bubble may exist along those paths. If the probability of retirement is sufficiently low (relative to the consumer’s discount rate), there exists a unique balanced growth path, and it is a bubbleless one (as in the standard model). On the other hand, if the probability of retirement is sufficiently high (but plausibly so), a multiplicity of bubbly balanced growth paths is shown to exist, in addition to a bubbleless one (which always exists).

Once I characterize the existence and potential multiplicity of balanced growth paths—bubbly and bubbleless—I turn to the analysis of the stability properties of those paths and the role of monetary policy in determining those properties.

Several findings of interest emerge from that analysis. First, even in the absence of bubbles, the possibility of an interest rate below the growth rate, when combined with a sufficiently low responsiveness of inflation to the output gap, implies that

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1 See, e.g., Borio and Low (2002) for an early statement of that view. Taylor (2014) points to excessively low interest rates in the 2000s as a factor behind the housing boom that preceded the financial crisis of 2007-2008.
2 The reason is well known: the equilibrium requirement that the bubble grows at the rate of interest violates the transversality condition of the infinite-lived representative consumer assumed in the New Keynesian model (as well as most macro models). See, e.g., Santos and Woodford (1997).
3 Other authors have extended the New Keynesian model to incorporate overlapping generations of finite-lived agents into the New Keynesian framework, though none of them has allowed for the existence of bubbles. Piergallini (2006) develops a related model with money in the utility function to analyze the implications of the real balance effect on the stability properties of interest rate rules. Nisticò (2012) discusses the desirability of a systematic monetary policy response to stock price developments in a similar model, but in the absence of bubbles. Del Negro, Giannoni and Patterson (2015) propose a related framework as a possible solution to the "forward guidance puzzle." None of the previous authors allow for retirement in their frameworks. That feature plays a central role in the emergence of asset price bubbles in the model proposed here.
4 And even though that transversality condition does not hold for the economy as a whole.
a kind of "reinforced Taylor principle" is needed in order to guarantee a locally unique equilibrium.

A second finding of interest relates to the possibility of expectations-driven fluctuations in a neighborhood of a balanced growth path. I show that, in contrast with the standard New Keynesian model, the Taylor principle generally fails to guarantee the local uniqueness of the equilibrium. As a result, fluctuations in economic activity and inflation may often arise in association with fluctuations in the size of the bubble, even in the absence of any shocks to fundamentals. In some cases, such fluctuations are more likely to arise under a rule that has the central bank raise interest rates systematically in response to an increase in the size of the bubble (i.e. a "leaning against the bubble" policy). In other cases, bubble-driven sunspot fluctuations arise independently of the central bank’s response to bubbles.

The analysis of simulations of stochastic bubbly equilibria suggest that, contrary to conventional wisdom, a monetary policy that "leans against the bubble" is likely to generate more volatility in output and inflation than a policy that focuses exclusively on stabilizing inflation.

The rest of the paper is organized as follows. The next section summarizes the related literature. Section 2 describes the basic framework underlying the analysis in the rest of the paper. Section 3 characterizes the economy’s balanced growth paths, bubbleless and bubbly. Section 4 analyzes the equilibrium dynamics in a neighborhood of a balanced growth path, and the role of monetary policy in preventing indeterminacy of equilibria. Section 5 provides an example of aggregate fluctuations driven by a (stochastic) bubble, and of the possible consequences of "leaning against the bubble" policies. Section 6 summarizes and concludes.

1 Related Literature

Much of the analysis of rational bubbles in general equilibrium found in the literature has been based on real models. An early reference in that category is Tirole (1985), using a conventional overlapping generations (OLG) framework with capital accumulation. A more recent one is Martín and Ventura (2012), who modify the Tirole model by introducing financial frictions that are alleviated by the existence of a bubble.

There is also an extensive literature on bubbles using monetary models with fully flexible prices. In most of those models, including the seminal paper by Samuelson (1958), money itself is the bubbly asset. Asriyan et al. (2016) provide a more recent example, introducing the notion of a nominal bubble. While monetary policy is not always neutral in those models, the mechanism through which its effects are transmitted is very different from that emphasized in standard models with nominal rigidities.

A number of papers have modified the standard New Keynesian model by introducing overlapping generations à la Blanchard-Yaari, though none of them has considered the possible existence of bubbles. Piergallini (2006) develops a related model with money in the utility function to analyze the implications of the real balance effect on the stability properties of interest rate rules. Nisticò (2012) discusses the desirability of a systematic monetary policy response to stock price developments in a similar model, but in the absence of bubbles. Del Negro, Giannoni and Patterson (2015) propose a related framework as a possible solution to the "forward guidance puzzle." None of the previous authors allow for retirement in their frameworks. That feature plays a central role in the emergence of asset price bubbles in the model proposed here.
Bernanke and Gertler (1999, 2001) analyze the possible gains from "leaning against the wind" monetary policies in a New Keynesian model in which stock prices contain an ad-hoc deviation from their fundamental value. The properties of that deviation differ from those of a rational bubble, which cannot exist in their model, which assumes an infinite-lived representative consumer.

In Galí (2014) I carried out a similar analysis of monetary policy rules in a sticky price model in which rational bubbles may exist in equilibrium due to the assumption of an infinite sequence of overlapping generations. While closest in spirit to the present paper, the framework used in that paper differed significantly from the New Keynesian framework in many dimensions. In particular, employment and output were constant in equilibrium, independently of fluctuations in the aggregate bubble, which had only a redistributive effect. On the other hand, the assumption of two-period lived individuals, while convenient, cannot be easily reconciled with the frequency of observed asset boom-bust episodes (not to say with the observed duration of individual prices). By way of contrast, the model developed here displays endogenous fluctuations in output and employment in response to fluctuations in asset price bubbles, and it is consistent with a calibration of the model to a quarterly frequency (as is convention in the business cycle literature). Finally, an additional advantage of the framework developed below is that it nests the standard New Keynesian model as a limiting case.

2 A New Keynesian Model with Overlapping Generations

Next I describe the basic framework underlying the analysis in the rest of the paper.

2.1 Consumers

I assume an economy with overlapping generations of the "perpetual youth" type, as in Yaari (1965) and Blanchard (1984). The size of the population is constant and normalized to one. Each individual has a constant probability $\gamma$ of surviving into the following period, independently of his age and economic status ("active" or "retired"). A cohort of size $1 - \gamma$ is born (in an economic sense) and becomes active each period. Thus, the size in period $t \geq s$ of a cohort born in period $s$ is given by $(1 - \gamma)^{t - s}$.

At any point in time, active and retired individuals coexist in the economy. Active individuals supply labor and manage their own firms, which they set up when they join the economy. I assume that each active individual faces a constant probability $1 - v$ of permanently losing his job and quitting his entrepreneurial activities. That probability is independent of his age. For convenience, below I refer to that transition as "retirement," though it should be clear that it can be given a broader interpretation.\footnote{Gertler (1999) introduces retirement in a similar fashion in a model of social security. More recently, Carvalho et al (2016) have used a version of the Gertler model to analyze the sources of low frequency changes in the equilibrium real rate. Both papers develop real models, in contrast to the present one, and do not consider the possibility of bubbles.}

The previous assumptions imply that the size of the active population (and, hence, the measure of firms) at any point in time is constant and given by $\alpha \equiv (1 - \gamma)/(1 - v\gamma) \in (0, 1]$.

A representative consumer from cohort $s$, standing in period 0, chooses a consumption plan to maximize expected lifetime

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utility

\[ E_0 \sum_{t=0}^{\infty} (\beta \gamma)^t \log C_{t|s} \]

subject to the sequence of period budget constraints

\[ \frac{1}{P_t} \int_0^\infty P_t(i)C_{t|s}(i)di + E_t\{\Lambda_{t,t+1}Z_{t+1|s}\} = A_{t|s} + W_t N_{t|s} \quad (1) \]

for \( t = 0, 1, 2, \ldots \), \( \beta \equiv 1/(1+\rho) \in (0, 1) \) is the discount factor. \( C_{t|s} \equiv \left( \alpha^{-\frac{1}{\alpha}} \int_0^\infty C_{t|s}(i)^{1-\frac{1}{\alpha}}di \right)^{\frac{\alpha}{\alpha-1}} \) is a consumption index, with \( C_{t|s}(i) \) being the quantity purchased of good \( i \in [0, \alpha] \), at a price \( P_t(i) \).

Complete markets for state-contingent securities are assumed, with \( Z_{t+1|s} \) denoting the stochastic payoff (expressed in units of the consumption index) generated by a portfolio of securities purchased in period \( t \), with value given by \( E_t\{\Lambda_{t,t+1}Z_{t+1|s}\} \), where \( \Lambda_{t,t+1} \) is the stochastic discount factor for one-period-ahead (real) payoffs. Only individuals who are alive can trade in securities markets.

Variable \( A_{t|s} \) denotes financial wealth at the start of period \( t \), for a member of cohort \( s \leq t \). For individuals other than those joining the economy in the current period, \( A_{t|s} = Z_{t|s}/\gamma \), where the term \( 1/\gamma \) captures the additional return on wealth resulting from an annuity contract. As in Blanchard (1984), that contract has the holder receive each period from a (perfectly competitive) insurance firm an annuity payment proportional to his financial wealth, in exchange for transferring the latter to the insurance firm upon death.\(^6\)

Variable \( W_t \) denotes the (real) wage per hour, and \( N_{t|s} \) is the number of individual work hours. Both the wage and work hours are taken as given by each worker and assumed to be common to all active individuals, i.e. \( N_{t|s} = N_t/\alpha \), where \( N_t \) is aggregate employment.\(^7\) Note that \( N_{t|s} = 0 \) for retired individuals.

Finally, I assume a solvency constraint of the form \( \lim_{T \to \infty} \gamma^T E_t\{\Lambda_{t,t+T}A_{t+T|s}\} \geq 0 \) for all \( t \), where \( \Lambda_{t,t+T} \) is determined recursively by \( \Lambda_{t,t+T} = \Lambda_{t,t+T-1}\Lambda_{t+1-T,t} \).\(^8\)

The problem above yields a set of optimal demand functions

\[ C_{t|s}(i) = \frac{1}{\alpha} \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} C_{t|s} \quad (2) \]

for all \( i \in [0, \alpha] \), which in turn imply \( \int_0^\alpha P_t(i)C_{t|s}(i)di = P_tC_{t|s} \). Thus we can rewrite the period budget constraint as:

\[ C_{t|s} + \gamma E_t\{\Lambda_{t,t+1}A_{t+1|s}\} = A_{t|s} + W_t N_{t|s} \quad (3) \]

The consumer’s optimal plan must satisfy the optimality condition\(^9\)

\[ \Lambda_{t,t+1} = \beta \frac{C_{t|s}}{C_{t+1|s}} \quad (4) \]

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\(^6\)Thus, individuals who hold negative assets will pay an annuity fee to the insurance company. The latter absorbs the debt in case of death. The insurance arrangement can also be replicated through securities markets. In that case the individual will purchase a portfolio that generates a random payoff \( A_{t+1|s} \) if he remains alive, 0 otherwise. The value of that payoff will be given by \( E_t\{\Lambda_{t,t+1}\Lambda_{t+1|s}\} \) which is equivalent to the formulation in the main text, given that \( A_{t|s} = Z_{t|s}/\gamma \).

\(^7\)By not including hours of work in the utility function I effectively eliminate any wealth effects that would generate systematic counterfactual differences in the quantity of labor supplied by active individuals across age groups. Alternatively one may assume preferences that rule out those wealth effects, but at the cost of rendering the analysis below less tractable.

\(^8\)Note that \((\Lambda_t)^{-1}\) is the "effective" interest rate paid by a borrower in the steady state. The solvency constraint thus has the usual interpretation of a no-Ponzi game condition.

\(^9\)Note that in the optimality condition the probability of remaining alive \( \gamma \) and the extra return \( 1/\gamma \) resulting from the annuity contract cancel each other. Complete markets guarantee the same consumption growth rate between two different periods for all consumers alive in the two periods.
and the transversality condition

$$\lim_{T \to \infty} \gamma^T E_t \{ \Lambda_{t+T} A_{t+T|s} \} = 0$$

(5)

with (4) holding for all possible states of nature (conditional on the individual remaining alive in $t+1$).

The details of wage setting are not central to the main point of the paper. For convenience, I assume an ad-hoc wage schedule linking the productivity-adjusted real wage to average work hours:

$$W_t = \left( \frac{N_t}{\alpha} \right)$$

(6)

where $N_t \equiv \int_0^\alpha N_t(i) di$ denotes aggregate work hours.

### 2.2 Firms

Each individual is endowed with the know-how to produce a differentiated good, and sets up a firm with that purpose when he joins the economy. That firm remains operative until its founder retires or dies, whatever comes first.\(^{10}\) All firms have an identical technology, represented by the linear production function

$$Y_t(i) = \Gamma N_t(i)$$

(7)

where $Y_t(i)$ and $N_t(i)$ denote output and employment for firm $i \in [0, \alpha]$, respectively, and $\Gamma \equiv 1 + g \geq 1$ denotes the (gross) rate of productivity growth. Individuals cannot work at their own firms, and must hire instead labor services provided by others.\(^{11}\)

Aggregation of (2) across consumers yields the demand schedule facing each firm

$$C_t(i) = \frac{1}{\alpha} \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} C_t$$

where $C_t \equiv (1 - \gamma) \sum_{s=-\infty}^t \gamma^{t-s} C_{t|s}$ is aggregate consumption. Each firm takes as given the aggregate price level $P_t$ and aggregate consumption $C_t$.

As in Calvo (1983), each incumbent firm resets the price of its good with probability $1 - \theta$ in any given period, and keeps its unchanged with probability $\theta$. Those probabilities are independent of the time elapsed since the last price adjustment. I consider two alternative environments regarding price setting by newly created firms. Both guarantee identical initial wealth among members of a newly born cohort, but have different implications regarding the dynamics of inflation. Under the environment, which I refer to as baseline, newly created firms set a price equal to the economy’s average price in the previous period. In that case the aggregate price dynamics are described by the equation

$$P_t^{1-\epsilon} = (1 - v\gamma(1-\theta))P_{t-1}^{1-\epsilon} + v\gamma(1-\theta)(P_t^*)^{1-\epsilon}$$

where $P_t^*$ is the price set in period $t$ by firms reoptimizing their price.\(^{12}\) A log-linear approximation of the previous difference equation around the zero inflation equilibrium yields (letting lower case letters denote the logs of the original variables):

$$p_t = (1 - v\gamma(1-\theta))p_{t-1} + v\gamma(1-\theta)p_t^*$$

(8)

\(^{10}\)The assumption of finite-lived firms (or more generally, for firms whose dividends shrink relative to the size of the economy) is needed in order for bubbles to exist in equilibrium. By equating the probability of a firm’s survival to that of its owner remaining alive and active I effectively equate the rate at which dividends and labor income are discounted, which simplifies considerably the analysis below.

\(^{11}\)I assume that each firm newly set up in any given period inherits (through random assignment) the index of an exiting firm.

\(^{12}\)Note that the price is common to all those firms, since they face an identical problem.
Under the "alternative" price setting environment, newly created firms are assumed to set prices optimally when they start operating. In that case aggregate price dynamics are described by the difference equation

$$P_t^{1-\varepsilon} = \nu \varepsilon P_t^{1-\varepsilon} + (1 - \nu \varepsilon) (P_t^*)^{1-\varepsilon}$$

whose log-linear approximation about zero inflation equilibrium is

$$p_t = \nu \varepsilon p_{t-1} + (1 - \nu \varepsilon) p_t^*$$  \hspace{1cm} (9)

Note that in both cases the current price level is a weighted average of last period’s price level and the newly set price, all in logs, with the weights given by the fraction of firms that do not and do adjust prices, respectively.

In both environments, a firm adjusting its price in period $t$ will choose the price $P_t$ that maximizes

$$\max_{P_t} \sum_{k=0}^{\infty} (\nu \varepsilon)^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left( \frac{P_t^*}{P_{t+k}} - \mathcal{W}_{t+k} \right) \right\}$$

subject to the sequence of demand constraints

$$Y_{t+k|t} = \frac{1}{\alpha} \left( \frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} C_{t+k}$$ \hspace{1cm} (10)

for $k = 0, 1, 2, \ldots$ where $Y_{t+k|t}$ denotes output in period $t + k$ for a firm that last reset its price in period $t$ and $\mathcal{W}_t \equiv \mathcal{W}_t / \Gamma^t$ is the productivity-adjusted real wage (i.e. the real marginal cost). \hspace{1cm} (13)

The optimality condition associated with the problem above takes the form

$$\sum_{k=0}^{\infty} (\nu \varepsilon)^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left( \frac{P_t^*}{P_{t+k}} - \mathcal{M} \mathcal{W}_{t+k} \right) \right\} = 0$$ \hspace{1cm} (11)

where $\mathcal{M} \equiv \frac{\mathcal{W}_t}{\Gamma^t}$ is the optimal markup under flexible prices. Throughout I maintain the assumption that $\Lambda \nu \varepsilon \in [0, 1)$, so that the infinite sum in (11) converges in a neighborhood of the zero inflation balanced growth path.

A first-order Taylor expansion of (11) around the zero inflation steady state yields, after some manipulation:

$$p_t^* = \mu + (1 - \Lambda \nu \varepsilon \theta) \sum_{k=0}^{\infty} (\Lambda \nu \varepsilon \theta)^k E_t \{ \psi_{t+k} \}$$ \hspace{1cm} (12)

where $\psi_t \equiv \log P_t \mathcal{W}_t$ is the (log) marginal cost, $\mu \equiv \log \mathcal{M}$ and $\Lambda \equiv 1/(1 + r)$ is the steady state stochastic discount factor. Note that along a zero inflation balanced growth path, $w_t = w = -\mu$.

Letting $\mu_t \equiv p_t - \psi_t$ denote the average (log) price markup, and combining (8) (or (9)) and (12) yields the inflation equation:

$$\pi_t = \Phi E_t \{ \pi_{t+1} \} - \lambda (\mu_t - \mu)$$ \hspace{1cm} (13)

where $\pi_t \equiv p_t - p_{t-1}$ denotes inflation. Under the baseline price setting environment, $\Phi \equiv \frac{\Lambda \nu \varepsilon \theta}{\theta + (1 - \theta)(1 - \nu \varepsilon \theta)} \in (0, 1)$ and $\lambda \equiv \frac{\nu \varepsilon (1 - \theta)(1 - \Lambda \nu \varepsilon \theta)}{1 - \nu \varepsilon (1 - \theta)} > 0$. By contrast, under the alternative price setting environment $\Phi \equiv \Lambda \varepsilon$ and $\lambda \equiv \frac{(1 - \nu \varepsilon \theta)(1 - \Lambda \nu \varepsilon \theta)}{\varepsilon \theta} > 0$. \hspace{1cm} (14)

\hspace{1cm} 13\text{The firm’s demand schedule can be derived by aggregating (2) across cohorts.}

\hspace{1cm} 14\text{Note that in the standard model with a representative consumer, } \Phi \equiv \beta \text{ and } \lambda \equiv \frac{(1 - \theta)(1 - \beta \theta)}{\theta} \text{ which corresponds to the limit of the expressions for those coefficients as } \nu \varepsilon \rightarrow 1 \text{ under the two environments, and given that } \Lambda \varepsilon = \beta \text{ along a balanced growth path of the representative consumer economy.}
Wage schedule (6), together with the assumption of a constant gross markup $M$ under flexible prices, implies a natural level of output given by $Y_t^n = \Gamma' \gamma$, where $\gamma \equiv \alpha M^{-\frac{1}{2}}$. Log-linearizing (6), and combining the resulting expressions with (13) we obtain a version of the New Keynesian Phillips curve

$$\pi_t = \Phi E_t \{\pi_{t+1}\} + \kappa \tilde{y}_t \quad (14)$$

where $\kappa \equiv \lambda \varphi$, and $\tilde{y}_t \equiv \log(Y_t/Y_{t}^n)$ is the output gap. Note that, in contrast with the standard New Keynesian model, the coefficient on expected inflation is not pinned down by the consumer’s discount factor. In particular, if the real interest rate along a balanced growth path is lower than the growth rate, as is the case along many of the balanced growth paths analyzed below, coefficient $\Gamma$ will be larger than one, often implying some awkward (and likely counterfactual) joint dynamics of output and inflation.\(^{15}\) In what follows, much of the discussion on the baseline case, which guarantees $\Phi \in (0, 1)$, and is associated with well behaved dynamics.

### 2.3 Asset Markets

In addition to a complete set of state-contingent securities, I assume the existence of markets for a number of specific assets, whose prices and returns must satisfy some equilibrium conditions. In particular, the yield $i_t$ on a one-period nominally riskless bond purchased in period $t$ must satisfy\(^{16}\)

$$1 = (1 + i_t) E_t \left\{ \frac{A_{t+1}}{P_{t+1}} \right\} \quad (15)$$

thus implying that the relation $\lambda \equiv 1/(1 + r)$ between the discount factor and the real return on the riskless nominal bond $(r)$ will hold along a perfect foresight balanced growth path.

Stocks in individual firms trade at a price $Q^F_t(i)$, for $i \in [0, \alpha]$, which must satisfy the asset pricing equation:

$$Q^F_t(i) = D_t(i) + \gamma E_t \left\{ \Lambda_{t+1} Q^F_{t+1}(i) \right\} \quad (16)$$

where $D_t(i) \equiv Y_t(i) \left( \frac{P_t(i)}{P_t} - W_t \right)$ denotes firm $i$’s dividends, and $\gamma$ is the probability that firm $i$ survives into next period. Solving (16) forward under the assumption that $\lim_{k \to \infty} (\gamma)^k E_t \left\{ \Lambda_{t+k} Q^F_{t+k}(i) \right\} = 0$, and aggregating across firms:

$$Q^F_t = \int_0^\alpha Q^F_t(i) di = \sum_{k=0}^{\infty} (\gamma)^k E_t \left\{ \Lambda_{t+k} D_{t+k} \right\} \quad (17)$$

where $D_t \equiv \int_0^\alpha D_t(i) di$ denotes aggregate dividends. Note that firms’ death makes it possible for the aggregate value of firms to be finite even if the interest rate is below the economy’s growth rate, as long as $\Gamma \gamma < 1$.

Much of the analysis below focuses on intrinsically worthless assets i.e. assets generating no dividend, pecuniary or not.\(^{17}\) Let $Q^B_t(j)$ denote the price of one such asset. In equilibrium that price must satisfy the condition

$$Q^B_t(j) = E_t \left\{ \Lambda_{t+1} Q^B_{t+1}(j) \right\} \quad (18)$$

\(^{15}\)If one allows for a CRRA utility and secular growth the coefficient on expected inflation in the standard New Keynesian Phillips curve is given by $\Gamma_{\gamma} = \beta \gamma^{1-\sigma}$ which can also be larger than one if $\Gamma_{\gamma} > 1$ and $\sigma < 1$.

\(^{16}\)Note also that in the asset pricing equations the probability of remaining alive $\gamma$ and the extra return $1/\gamma$ resulting from the annuity contract cancel each other.

\(^{17}\)In Jean Tirole’s words, pure bubbly assets are "best thought of as pieces of paper."
as well as the non-negativity constraint $Q^B_t(j) \geq 0$ (given free disposal), for all $t$.

Let $Q^B_t$ denote the aggregate value of bubbly assets in period $t$. In equilibrium, that variable evolves over time according to the following two equations:

$$Q^B_t = U_t + B_t$$  \hspace{1cm} (19)

$$Q^B_t = E_t\{A_{t,t+1}B_{t+1}\}$$  \hspace{1cm} (20)

where $U_t \equiv Q^B_t \geq 0$ is the value of a new bubble introduced by cohort $t$ at birth, and $B_t \equiv \sum_{s=-\infty}^{t-1} Q^B_t \geq 0$ is the aggregate value in period $t$ of bubbly assets available for trade in period $t - 1$, with $Q^B_t$ denoting the period $t$ value of bubbly assets introduced in period $s \leq t$. Note that the introduction of new bubbly assets by incoming cohorts makes it possible for an aggregate bubble to re-emerge after a hypothetical collapse, thus overcoming a common criticism of early rational bubble models. A similar environment with bubble creation was first introduced and analyzed in Martín and Ventura (2012) in the context of an overlapping generations model with financial frictions.

Note that in the previous environment, the initial financial wealth of a member of a cohort born in period $t$ is given by:

$$A_{t|t} = Q^F_t + U_t/(1 - \gamma)$$

where $Q^F_t$ is the value in period $t$ of a newly created firm.

### 2.4 Monetary Policy

Unless otherwise noted, the central bank is assumed to follow a simple interest rate rule of the form

$$\hat{i}_t = \phi_p \pi_t + \phi_q \hat{q}^B_t$$  \hspace{1cm} (21)

where $\hat{i}_t = \log[(1 + i_t)/(1 + r)]$ and $q^B_t \equiv Q^B_t / (\Gamma^t Y)$ is the size of the aggregate bubble normalized by trend output, with $\hat{q}^B_t \equiv q^B_t - q^B$ denoting the deviation from its value along a balanced growth path. Note that the previous rule is consistent with a zero inflation target. In what follows, I assume the central bank takes $r$ and $q^B$ as given, as determined by the analysis below.

### 2.5 Market Clearing

Goods market clearing requires $Y_t(i) = (1 - \gamma) \sum_{s=-\infty}^{t} \gamma^{t-s} C_{t|s}(i)$ for all $i \in [0, \alpha]$. Letting $Y_t \equiv \left(\alpha^{-1/\nu} \int_0^\alpha Y_t(i)^{1-1/\nu} \, di\right)^{-\nu}$ denote aggregate output, we have:

$$Y_t = (1 - \gamma) \sum_{s=-\infty}^{t} \gamma^{t-s} C_{t|s}$$

$$= C_t$$  

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18 Think of pieces of paper of a cohort-specific color or stamped with the birth year of their creators.
19 The bubble introduced by each individual can be interpreted as being attached to the stock of his firm and hence to burst whenever the firm stops operating (i.e. with probability $1 - \nu \gamma$).
Note also that in equilibrium

\[ N_t = \int_0^\alpha N_t(i)di \]
\[ = \Delta_t^p \gamma \]
\[ \simeq \gamma \]

where \( \gamma \equiv Y_t/\Gamma^t \) is aggregate output normalized by productivity and \( \Delta_t^p \equiv \frac{1}{\alpha} \int_0^\alpha (P_t(i)/P_t)^\epsilon di \simeq 1 \) is an index of relative price distortions, which equals one up to a first-order approximation.

Assuming that all securities other than stocks and bubbly assets are in zero net supply, asset market clearing requires

\[ (1 - \gamma) \sum_{s = -\infty}^t \gamma^{t-s} A_{t|s} = Q_t^F + Q_t^B \]

Next I characterize the economy’s perfect foresight, zero inflation balanced growth paths.

3 Bubbles and Balanced Growth Paths

In a perfect foresight balanced growth (henceforth, BGP) the discount factor is constant and satisfies

\[ \Lambda = \frac{1}{1 + r} \]

as implied by (15), and where \( r \) is the real interest rate along a BGP. Note also that \( i = r \) in the BGP, given the zero inflation assumption.

Note also that \( W = 1/M \) in the zero inflation BGP. Combined with (6) the former condition implies:

\[ \frac{1}{M} = \left( \frac{\gamma}{\alpha} \right)^\varphi \]

Accordingly, output along the BGP, is given by \( Y_t^{BGP} = \Gamma^t \gamma \), which coincides with the natural level of output, as derived above.

Next I describe how aggregate consumption is determined. Details of the derivation can be found in the Appendix.

Let \( C_j \) and \( A_j^t \) denote, respectively, consumption and financial wealth along a BGP for an individual aged \( j \), normalized by productivity, i.e. Superindex \( i \in \{a, r\} \) denotes his status as active or retired. The intertemporal budget constraint for a consumer born in period \( s \) and who remains active at time \( t \), derived by solving (1) forward, can be evaluated at a BGP as:

\[ \sum_{k=0}^{\infty} (\Lambda\Gamma)^k C_{t+k-s} = A_{t-s}^a + \frac{1}{1 - \Lambda\Gamma\nu^\gamma} \left( \frac{WN}{\alpha} \right) \]

where \( N \) and \( W \) denote aggregate hours and the wage (the latter normalized by productivity) along the BGP.

Using the fact that \( C_{t+k-s} = [\beta(1 + r)/\Gamma^k]C_{t-s} \) - as implied by (4) evaluated at the BGP - the following consumption function can be obtained for an active individual aged \( j \):

\[ C_j = (1 - \beta\gamma) \left[ A_j^a + \frac{1}{1 - \Lambda\Gamma\nu^\gamma} \left( \frac{WN}{\alpha} \right) \right] \]
Thus, consumption for an active individual is proportional to the sum of his financial wealth, \( A_j^t \), and his current and future labor income (properly discounted), \( WN/[\alpha(1 - \Lambda \Gamma \nu \gamma)] \).

The corresponding consumption function for a retired individual is given by:

\[
C_j = (1 - \beta \gamma) A_j^r
\]  

(23)

Aggregating over all individuals, imposing the asset market clearing condition \( A = Q^F + Q^B \) (with these three variables now normalized by productivity) and using the fact that \( Q^F = D/(1 - \Lambda \Gamma \nu \gamma) \) and \( Y = WN + D \) along a BGP (with \( D \equiv (1 - 1/M)Y \) denoting aggregate profits normalized by productivity), we obtain:

\[
C = (1 - \beta \gamma) \left[ Q^B + \frac{1}{1 - \Lambda \Gamma \nu \gamma} Y \right]
\]  

(24)

which can be interpreted as an aggregate consumption function along the BGP. Finally, goods market clearing requires that \( C = Y \) thus implying the following equation relating the bubble-output ratio \( q^B \equiv Q^B/Y \) and the discount factor *

\[
1 = (1 - \beta \gamma) \left[ q^B + \frac{1}{1 - \Lambda \Gamma \nu \gamma} \right]
\]  

(25)

Next I turn to the analysis of "bubbleless" and "bubbly" BGPs.

3.1 Bubbleless Balanced Growth Paths

Consider first a "bubbleless" BGP, with \( q^B = 0 \). Imposing that condition in (25) implies

\[
\Lambda \Gamma \nu = \beta
\]

or, equivalently,

\[
r = (1 + \rho)(1 + g)\nu - 1
\]

Note that the real interest rate along a BGP is increasing in both \( \nu \) and \( g \). The reason is that an increase in either of those variables raises desired consumption by increasing the expected present discounted value of future income for currently active individuals. In order for the goods market to clear, an increase in the interest rate is called for.

When \( \nu = 1 \), the real interest rate is given (approximately) by the discount rate plus the growth rate, i.e. \( r \simeq \rho + g \), as in the standard model (with log utility, as assumed here).

Note also that an increase in the expected lifetime, as indexed by \( \gamma \), does not have an independent effect on the real interest rate along the bubbleless BGP. The reason is that, when \( \Lambda \Gamma \nu = \beta \), such a change increases in the same proportion the present value of consumption and that of income for any given real rate, making an adjustment in the latter unnecessary.\(^{20}\)

Finally, note for future reference that in the bubbleless BGP considered here the real interest rate \( r \) is lower than the growth rate \( g \) (i.e. \( \Lambda \Gamma > 1 \)) if and only if \( \nu < \beta \).

\(^{20}\)The independence of the steady state real interest rate from \( \gamma \) is a consequence of the log utility specification assumed here. That property is not critical from the viewpoint of the present paper, since there are other factors (the probability of retirement, in particular), that can drive real interest rate to values consistent with the presence of bubbles.
3.2 Bubbly Balanced Growth Paths

Next I consider the possibility of a BGP with an aggregate bubble growing at the same rate as output, thus implying a constant bubble-output ratio $q^B > 0$.

Letting $q^B_t = Q^B_t / (\Gamma' \mathcal{Y})$, $b_t = B_t / (\Gamma' \mathcal{Y})$ and $u_t = U_t / (\Gamma' \mathcal{Y})$, note first that (20) can be rewritten as:

$$q^B_t = E_t \{ \Lambda_{t,t+1} \Gamma b_{t+1} \}$$

$$= E_t \{ \Lambda_{t,t+1} \Gamma (q^B_{t+1} - u_{t+1}) \}$$

(26)

Along a bubbly BGP we must have $q^B_t = q^B > 0$, and $u_t = u \geq 0$ for all $t$. It follows from (25) and (26) that:

$$q^B = \frac{\gamma (\beta - \Lambda \Gamma v)}{(1 - \beta \gamma)(1 - \Lambda \Gamma v \gamma)}$$

(27)

$$u = \left(1 - \frac{1}{\Lambda \Gamma}\right) q^B$$

Thus, the existence of a BGP with a positive bubble, $q^B > 0$, requires that

$$\Lambda \Gamma v < \beta$$

On the other hand the non-negativity constraint on newly created bubbles $u \geq 0$ requires:

$$\Lambda \Gamma \geq 1$$

Accordingly, a necessary and sufficient condition for the existence of a bubbly BGP is given by

$$v < \beta$$

(28)

If the previous condition is satisfied, there exists continuum of bubbly BGPs $\{q^B, u\}$ indexed by $r \in (\Gamma v / \beta - 1, \Gamma - 1]$. Note that the condition for the existence of bubbly BGPs is equivalent to the real interest rate being less than the growth rate in the bubbleless BGP.

It can be easily checked that $q^B$ is increasing in $r$, with $\lim_{r \to g} q^B = \frac{\gamma (\beta - v)}{(1 - \beta \gamma)(1 - v \gamma)} = q^B$. Note also that $\partial q^B / \partial v < 0$, i.e. the upper bound on the size of the bubble is decreasing in $v$ over the range $v \in [0, \beta]$, and converges to zero as $v \to \beta$.

One particular such bubbly BGP has a constant supply of bubbly assets, i.e. $u = 0$. Note that in that case

$$\Lambda \Gamma = 1$$

or, equivalently,

$$r = g$$

with the implied bubble size given by $q^B = q^B$. Along that BGP any existing bubble will be growing at the same rate as the economy. By contrast, along a bubbly BGP with bubble creation, $r < g$ implies that the size of the aggregate pre-existing bubble will be shrinking over time relative to the size of the economy, with newly created bubbles filling up the gap so that the size of the aggregate bubble relative to the size of the economy remains unchanged.
Summing up, one can distinguish two regions of the parameter space relevant for the possible existence of bubbly BGPs:

(i) $\beta \leq v \leq 1$. In this case, the BGP is unique and bubbleless and associated with a real interest rate given by $r = \Gamma v / \beta - 1 > g$.

(ii) $0 < v < \beta$. In this case multiple BGPs coexist. One of them is bubbleless, with $r = \Gamma v / \beta - 1 < g$. In addition, there exists a continuum of bubbly BGPs, indexed by the real interest rate $r \in (\Gamma v / \beta - 1, \Gamma - 1]$, and associated with a bubble size (relative to output) $q^B \in (0, \bar{q}^B]$, given by (27).

Figure 1 summarizes graphically the two regions with their associated BGPs.

### 3.3 A Brief Detour: Bubbly Equilibria and Transversality Conditions

Equilibria with bubbles on assets in positive net supply can be ruled out in an economy with an infinite lived representative consumer.\(^{21}\) In that economy, any positive net supply of that asset must be necessarily held by the representative consumer, implying $\lim_{T \to \infty} E_t \{ A_{t+T} \} \geq \lim_{T \to \infty} E_t \{ A_{t+T} Q^B_{t+T} \}$. Given that the bubble component of any asset must satisfy $\lim_{T \to \infty} E_t \{ A_{t+T} Q^B_{t+T} \} = Q^B_t$, it follows that $\lim_{T \to \infty} E_t \{ A_{t+T} \} \geq Q^B_t$. But the consumer’s transversality condition requires that $\lim_{T \to \infty} E_t \{ A_{t+T} \} = 0$. Given that free disposal requires that $Q^B_t \geq 0$, it follows that $Q^B_t = 0$ for all $t$.

Note that the previous reasoning cannot be applied to an overlapping generations economy like the one developed above. The reason is that in that case it is no longer true that the positive net supply of any bubbly asset must be held (asymptotically) by any individual agent, since it can always be passed on to future cohorts (and it will in equilibrium).

In fact, it is easy to check that in the overlapping generations model above the individual transversality condition is satisfied along any BGP, bubbly or bubbleless. As shown in the appendix, for an individual born in period $s \leq t$ it must be the case that along any BGP

\[
\lim_{T \to \infty} \gamma^T E_t \{ A_{t+T} A_{t+T+s} \} = \lim_{T \to \infty} \gamma^T E_t \{ A_{t+T} A_{t+T+s} \} + (1 - v^T) E_t \{ A_{t+T} A_{t+T+s} \} = \lim_{T \to \infty} (\beta \gamma)^T \left[ A_{t+T+s} A_{t+T+s} + \frac{W_s N}{1 - \Gamma v \gamma} \left( 1 - \left( \frac{\Lambda v}{\beta} \right)^T \right) \right] = 0
\]

implying that the transversality condition is satisfied along any admissible BGP. It is straightforward to show that this will be the case along any equilibrium that remains in a neighborhood of a BGP, of the kind analyzed below.

### 3.4 Plausibility of Bubbly BGPs: Some Rough Numbers

A natural question one may raise at this point is whether the key necessary condition for the existence of bubbly equilibria, namely $v < \beta$, is likely to be satisfied in practice. Taking the model at face value, $v$ should be interpreted as the probability of (permanently) dropping out of the labor force and becoming inactive in any given quarter. If one takes a narrow perspective, that transition can be seen as corresponding to retirement. In that case information on the average age at retirement can be

\(^{21}\)See, e.g. Santos and Woodford (1997) for a discussion of the conditions under which rational bubbles can be ruled out in equilibrium.
used to calibrate $v$. In what follows I take the age at which individuals enter the economy ("are born" in the model) to be 20. In the U.S., the average age of retirement is 63. That corresponds to an expected active lifetime as of age 20 of $43 \times 4 = 172$ quarters (conditional on effectively retiring, i.e. not dying earlier). In the model the expected active lifetime (conditional on actual retirement) is given by $\frac{1}{1-\gamma}$. On the other hand, life expectancy at age 20 is $60 \times 4 = 240$ quarters, which is consistent with a value $\gamma = 1 - \frac{1}{240} = 0.9958$. It follows that $v = (1 - \frac{1}{172}) \frac{1}{0.9958} = 0.9983$, which should be viewed as an upper bound, since many individuals stop working permanently before retirement. Thus, the existence of bubbly balanced growth paths requires that $\beta > 0.9983$.

Unfortunately, the latter condition cannot be verified easily since that parameter is not identified by the above model once the existence of bubbles is allowed for. This is in contrast with the standard representative agent model, for which there is a tight connection between the discount rate and the real interest rate along a BGP.\(^2\) On the other hand, casual introspection suggests that a discount factor of $0.9342$ applied to utility 10 years from today (as implied by the lower bound $\beta = 0.9983$) would seem entirely plausible.

Alternatively, one may examine directly the relation between the average real interest rate, $r$, and the average growth rate of output, $g$, two observable variables. As discussed above, the existence of bubbly BGP\(s\) requires that $r \leq g$. Using data on 3-month Treasury bills, GDP deflator and (per capita) GDP, the average values for those variables in the U.S. over the period 1960Q1-2015Q4 are $r = 1.4\% \div 4 = 0.35\%$ and $g = 1.6\% \div 4 = 0.4\%$ (or, equivalently, $\Lambda = 0.9965$ and $\Gamma = 1.004$), values which satisfy the inequality condition necessary for the existence of bubbles. Note also that the above calibration implies $\Lambda \Gamma \gamma \approx 0.995$, thus satisfying the condition for a well defined intertemporal budget constraint.

Some of the quantitative analyses below require a calibrated value for $\beta$, the discount factor. I set $\beta = 0.998478$, which is consistent with $\gamma = 4$, i.e. with an upper bound of one for the size of the bubble relative to annual output. This is, admittedly, an arbitrary choice, but it allows me to rule out steady state bubbles with an implausible large size.

## 4 Bubbles, Monetary Policy and Equilibrium Fluctuations

Having characterized the BGP\(s\) of the model economy, in the present section I shift the focus to the analysis of the equilibrium dynamics in a neighborhood of a given BGP. In particular, I am interested in determining the conditions under which equilibrium fluctuations unrelated to fundamental shocks may emerge, as well as the role that variations in the size of the aggregate bubble and the nature of monetary policy may play in such fluctuations.

As in the standard analysis of the New Keynesian model with a representative agent, I restrict myself to equilibria that remain in a neighborhood of a BGP, with the local equilibrium dynamics being approximated using the log-linearized equilibrium conditions.\(^2\) I leave the analysis of the global equilibrium dynamics –including the possibility of switches between BGP\(\text{s}\), the existence of a zero lower bound on interest rates, and other nonlinearities– to future research. Secondly, in analyzing the model’s equilibrium I ignore the existence of fundamental shocks, and focus instead on the possibility of fluctuations driven by self-fulfilling expectations and on the role of bubbles as a source of those fluctuations.\(^2\)

\(^2\)Note that $\beta = 0.9983$ implies a discount factor of $0.9342$ applied to utility 10 years from today, which seems entirely plausible.


\(^2\)As the analysis of the equilibrium dynamics below will make clear, in the absence of bubbles and/or multiplicity of equilibria the economy’s behavior in response to fundamental shocks should involve no significant differences relative to that of the standard New Keynesian model with...
I start by deriving the log-linearized equilibrium conditions around a BGP. In contrast with the New Keynesian model with a representative agent, the individual consumer’s Euler equation and the goods market clearing condition are no longer sufficient to derive an equilibrium relation determining aggregate output as a function of interest rates (i.e. the so-called dynamic IS equation). Instead, the derivation of such a relation requires solving for an aggregate consumption function, by aggregating the individual consumption functions obtained by combining the consumer’s Euler equation and intertemporal budget constraint. Since no exact representation exists for the individual consumption function, I proceed by deriving a log-linearized approximation of that function around a perfect foresight, zero inflation balanced growth path. Then I aggregate the approximate individual consumption functions to obtain an (approximate) aggregate consumption function. See the Appendix for detailed derivations.

Letting $\tilde{c}_t \equiv \log(C_t/T^4C)$ denote log deviation of aggregate consumption from its value along a BGP, the goods market clearing condition can be written as:

$$\tilde{y}_t = \tilde{c}_t$$

(29)

As shown in the appendix, the aggregate consumption function can be written, up to a first order approximation, as follows:

$$\tilde{c}_t = (1 - \beta \gamma)(\tilde{q}_t^B + \tilde{x}_t)$$

(30)

where $\tilde{q}_t^B \equiv q_t^B - q^B$ and

$$\tilde{x}_t \equiv \sum_{k=0}^{\infty} (\Lambda \Gamma \nu \gamma)^k E_t \{ \hat{y}_{t+k} \} - \frac{\Lambda \Gamma \nu \gamma}{1 - \Lambda \Gamma \nu \gamma} \sum_{k=0}^{\infty} (\Lambda \Gamma \nu \gamma)^k E_t \{ \hat{r}_{t+k} \}$$

is the non-bubbly component of aggregate wealth (i.e. current and future discounted income), with $\tilde{y}_t \equiv \log(Y_t/T^4Y)$ denoting the output gap and $\hat{r}_t = \hat{\gamma}_t - E_t \{ \pi_{t+1} \}$ the real interest rate, all expressed in log deviations from their values along a BGP. Note that $\tilde{x}_t$ can also be rewritten in recursive form as:

$$\tilde{x}_t = \Lambda \Gamma \nu \gamma E_t \{ \tilde{x}_{t+1} \} + \hat{y}_t - \frac{\Lambda \Gamma \nu \gamma}{1 - \Lambda \Gamma \nu \gamma} (\hat{r}_t - E_t \{ \pi_{t+1} \})$$

(31)

Log-linearization of (26) around a BGP yields the equations describing fluctuations in the aggregate bubble.

$$\tilde{q}_t^B = \Lambda \Gamma E_t \{ \hat{b}_{t+1} \} - q^B (\hat{\gamma}_t - E_t \{ \pi_{t+1} \})$$

(32)

$$\tilde{q}_t^B = \hat{b}_t + \tilde{u}_t$$

(33)

where $q^B = \frac{\gamma(\beta - \Lambda \Gamma \nu)}{(1 - \beta \gamma)(1 - \Lambda \Gamma \nu \gamma)} \geq 0$ and $\hat{u}_t \equiv u_t - u$.

The New Keynesian Phillips curve derived above, given by

$$\pi_t = \Phi E_t \{ \pi_{t+1} \} + \kappa \hat{y}_t$$

(34)

completes the description of the non-policy block of the system of difference equations describing the model’s equilibrium in a neighborhood of a BGP, where the latter is defined by a pair $(q^B, \Lambda)$ satisfying the BGP conditions derived in the previous section.

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25 Formally, one can use the definition of aggregate consumption and the individual Euler equations to derive the aggregate relation:

$$E_t \{ \tilde{c}_{t+1} \} = \gamma \left( \tilde{c}_t + \hat{\gamma}_t - E_t \{ \pi_{t+1} \} \right) + (1 - \gamma) E_t \{ \tilde{c}_{t+1|t+1} \}$$

where $E_t \{ \tilde{c}_{t+1|t+1} \}$ cannot be immediately as a function of $\tilde{c}_t$.  

---

a representative agent.
4.0.1 The Size of Bubble Effects on Aggregate Demand

As made clear by (30), consumption moves in proportion to fluctuations in total wealth, bubbly and non-bubbly. Through this channel, a rise in the size of the bubble leads, ceteris paribus, to an increase in aggregate demand and output. Note, however, that the marginal propensity to consume out of wealth, $\frac{1}{1 + \beta \gamma}$, is likely to be very small for any plausible values of the discount factor and the survival rate. In particular, under the baseline calibration introduced above, $1 - \beta \gamma = 0.00571$. Thus, the direct effect on aggregate demand of a 10 percent temporary (one-quarter) blip in the bubble would be $0.0571\%$, a negligible figure. Adding the indirect effect resulting from the induced increase in current output, the total effect is given by $\left(1 - \beta \gamma\right) / \beta \gamma = 0.00574$, a value only marginally larger. On the other hand, starting from the bubbleless BGP ($\Lambda \Gamma u = \beta$), the implied total effect of a hypothetical aggregate bubble that survives into the following period with probability $\delta$ is given by the multiplier $\left(1 - \beta \gamma\right) / \beta \gamma(1 - \delta)$ which can be made arbitrarily large as $\delta \to 1$. For $\delta = 0.99$, however, that multiplier takes the value 0.009, still very small and only slightly larger than in the case of a purely transitory bubble.

Next I turn to the analysis of the properties of equilibria in a neighborhood of a BGP. I analyze separately the case of fluctuations around bubbleless and bubbly BGPs.

4.1 Equilibrium Dynamics around a Bubbleless BGP

As shown above, in a bubbleless BGP $q^B = 0$ and $\Lambda \Gamma u = \beta$. Imposing those conditions on the equations above, the non-policy block of the system describing the economy’s equilibrium is given by:

$$\hat{y}_t = (1 - \beta \gamma)(q^B_t + \hat{x}_t) \quad (35)$$

$$\hat{x}_t = \beta \gamma E_t\{\hat{x}_{t+1}\} + \hat{y}_t - \frac{\beta \gamma}{1 - \beta \gamma}(\hat{i}_t - E_t\{\pi_{t+1}\}) \quad (36)$$

$$\pi_t = \Phi E_t\{\pi_{t+1}\} + \kappa \hat{y}_t \quad (37)$$

$$q^B_t = (\beta / \nu)E_t\{b_{t+1}\} \quad (38)$$

$$q^B_t = b_t + u_t \quad (39)$$

where $(q^B_t, u_t) \geq 0$ for all $t$.

Next I analyze separately bubbly and bubbleless equilibria in a neighborhood of the bubbleless BGP, and discuss the role of monetary policy.

4.1.1 Bubbleless Equilibria

I start by analyzing the case of bubbleless equilibria, i.e. $q^B_t = 0$ for all $t$. The non-policy block of the model now simplifies to:

$$\hat{y}_t = E_t\{\hat{y}_{t+1}\} - (\hat{i}_t - E_t\{\pi_{t+1}\}) \quad (40)$$

$$\pi_t = \Phi E_t\{\pi_{t+1}\} + \kappa \hat{y}_t \quad (41)$$
which has a structure "similar" to that of the standard New Keynesian model. I assume a simple interest rate rule of the form:

$$\hat{i}_t = \phi_x \pi_t$$

(42)

where $\phi_x \geq 0$. Using (42) to eliminate the interest rate in (40) one can write the resulting system (after some manipulation) as:

$$\begin{bmatrix} \hat{y}_t \\ \pi_t \end{bmatrix} = A \begin{bmatrix} E_t \{ \hat{y}_{t+1} \} \\ E_t \{ \pi_{t+1} \} \end{bmatrix}$$

(43)

where

$$A \equiv \Omega \begin{bmatrix} 1 & 1 - \Phi \phi_x \\ \kappa & \kappa + \Phi \end{bmatrix}$$

with $\Omega \equiv 1/(1 + \kappa \phi_x)$.

The solution to (43) is (locally) unique (and equal to $\hat{y}_t = \pi_t = 0$ for all $t$) if and only if the two eigenvalues of matrix $A$ are inside the unit circle. As shown in the appendix, this will be the case if the following condition is satisfied:

$$\phi_x > \max\{1, (\Phi - 1) / \kappa\}$$

(44)

Thus, under that condition, and in the absence of exogenous shocks, the only (bubbleless) equilibrium that remains in a neighborhood of the steady state is given by $\hat{y}_t = \pi_t = 0$ for all $t$.

Under the baseline price setting environment, $\Phi \equiv \beta / \nu$ which is potentially larger than one. As long as $\nu \geq \beta / (1 + \kappa)$, (44) also simplifies to $\phi_x > 1$. On the other hand, when $\nu < \beta / (1 + \kappa)$, the uniqueness condition takes the form $\phi_x > (\beta / \nu - 1) / \kappa > 1$, which can be thought of as a "reinforced Taylor principle," requiring that the central bank adjust the policy rate more aggressively than implied by the usual "more than one-for-one" condition in order to guarantee the uniqueness of the equilibrium. Note that a low value for $\nu$ reduces the average interest rate, making current inflation more sensitive to expectations, and hence a stable backward solution to (43) becomes more likely, especially when $\kappa$ is small, unless the central bank responds more aggressively to inflation thus inducing a sufficiently large decrease in the output gap.

Figure 2 represents the regions of the parameter space for $\phi_x$ and $\nu$, consistent with a determinate and an indeterminate equilibrium, using the baseline setting for $\beta$, and assuming $\kappa = 0.0434$, the calibration for the slope of the Phillips curve introduced and motivated below. Given that calibration the "reinforced Taylor principle" becomes relevant for values of $\nu$ below 0.957, a value substantially smaller than the baseline setting based on retirement data, but not one that is fully implausible under a broader interpretation of the transition to inactivity.
If condition (44) is not satisfied, there exist equilibria involving stationary sunspot fluctuations around the bubbleless BGP, driven by self-fulfilling revisions of expectations. If \( \phi_{\pi} < 1 \) then only one eigenvalue of \( A \) is larger than one, implying indeterminacy of dimension one.\(^{26}\) On the other hand, if \( v < \beta / (1 + \kappa) \) and \( 1 \leq \phi_{\pi} < (\beta / v - 1) / \kappa \) both eigenvalues of \( A \) have moduli larger than one, with the resulting indeterminacy being two-dimensional, a case that can be ruled out in the New Keynesian model with a representative agent. See the appendix for a discussion of the representation of such equilibria.

The previous results point to a first dimension in which the introduction of finite lives and an OLG structure in an otherwise standard New Keynesian model affects the properties of the equilibrium dynamics and their connection to monetary policy, even in the absence of bubbles.

4.1.2 Bubbly Equilibria

Next I consider the possibility of bubbly equilibria in a neighborhood of the bubbleless BGP. To illustrate the possibility of bubble-driven fluctuations and the potential role of monetary policy in shaping those fluctuations, I specify a process for the bubble which satisfies the conditions above while displaying a boom-bust pattern characteristic of conventional accounts of historical bubble episodes. Consider a bubbly asset with positive value that is introduced in a fully unanticipated way by a newly born cohort (say, in period 0). As long as it doesn’t burst it evolves over time according to the equation

\[
q_t^B = \frac{v}{\beta \delta} q_{t-1}^B
\]

for \( t = 1, 2, 3, \ldots \) where \( \delta < v / \beta < 1 \) is the bubble’s survival probability, assumed to be constant. Thus, with probability \( \delta \) the bubble grows at a rate \( v / \beta \delta > 1 \). With probability \( 1 - \delta \) the bubble bursts, and does not re-emerge again. Note that the previous specification satisfies (38) and (39) as well as the non-negativity condition. Furthermore, the assumption \( v / \beta < 1 \) guarantees that \( \lim_{T \to \infty} E_t \{ q_{t+T}^B \} = 0 \). Henceforth, I assume \( v < \beta \), a condition required to rule out an explosive bubble that would drift away from the bubbleless steady state, as made clear by (38).

Combining (35), (40) and the assumed process for the bubble, one can derive the dynamic IS equation:

\[
\tilde{\gamma}_t = E_t \{ \tilde{\gamma}_{t+1} \} - (\tilde{i}_t - E_t \{ \pi_{t+1} \}) + \Theta q_t^B
\]

where \( \Theta \equiv (1 - \beta \gamma)(1 - \nu \gamma) / \beta \gamma > 0 \). Note that one can rewrite (46) as follows:

\[
\tilde{\gamma}_t = E_t \{ \tilde{\gamma}_{t+1} \} - (\tilde{i}_t - E_t \{ \pi_{t+1} \} - \tilde{\gamma}_t^n)
\]

where \( \tilde{\gamma}_t^n = \Theta q_t^B \) has the interpretation of a natural rate of interest, i.e. the equilibrium real interest rate that would prevail under flexible prices.

The previous equation, together with (14) and a process for the bubble satisfying (38) and (39) describe the non-policy block of the economy. In addition, I assume an interest rate rule of the form

\[
\tilde{i}_t = \phi_{\pi} \pi_t + \phi_q q_t^B
\]

where \( \phi_{\pi} > 1 \), which guarantees the (local) uniqueness of the equilibrium in the absence of a bubble.

\(^{26}\)Such sunspot equilibria are similar in nature to those analyzed in, e.g. Clarida, Gali and Gertler (2000).
By eliminating the interest rate, we can rewrite the corresponding system of difference equations more compactly as

\[
\begin{bmatrix}
\hat{y}_t \\
\pi_t
\end{bmatrix}
= \begin{bmatrix}
A \\
B
\end{bmatrix}
\begin{bmatrix}
E_t\{\hat{y}_{t+1}\} \\
E_t\{\pi_{t+1}\}
\end{bmatrix}
+ B\hat{q}_t^B
\]

where \(\{\hat{q}_t^B\}\) evolves according to (45), \(A\) and \(\Omega\) are defined as above, and \(B \equiv \Omega(\Theta - \phi_q)[1 - \kappa]'.

Based on the analysis above, the condition for (local) uniqueness of the equilibrium, conditional on a given bubble process, is given by (44) and is thus independent of the presence of the bubble or the policy response to it, as measured by \(\phi_q\). The latter parameter, however, is key in determining the effects of the bubble on the real economy.

Note that in this case, and up to a first order approximation, there is no feedback from monetary policy to the bubble. Hence, fluctuations in the latter can be viewed as playing a role similar to an exogenous demand shock, albeit one that must satisfy conditions (38) and (39), as well as the non-negativity condition.

Assuming that the uniqueness condition (44) is satisfied, I use the method of undetermined coefficients to derive the following expressions for inflation and the output gap as a function of the bubble:

\[
\hat{y}_t = (1 - \Phi\psi/\beta)\Psi(\Theta - \phi_q)\hat{q}_t^B
\]

\[
\pi_t = \kappa\Psi(\Theta - \phi_q)\hat{q}_t^B
\]

where \(\Psi \equiv 1/[(1 - \Phi\psi/\beta)(1 - \psi/\beta) + \kappa(\phi_x - \psi/\beta)] > 0\). Thus, the emergence of the bubble will trigger movements in inflation and the output gap, as long as \(\phi_q \neq \Theta\). The sign of the joint comovement of those variables is not invariant to the price setting environment assumed. Under the baseline price setting environment, \(\Phi \in (0, 1)\), implying that the sign of the response of both the output gap and inflation to fluctuations in the bubble is the same, and corresponds to the sign of \(\Theta - \phi_q\). Thus, if \(\phi_q < \Theta\), both variables will increase with the bubble. On the other hand, if the central bank overreacts to the growth of the bubble (\(\phi_q > \Theta\)), both inflation and the output gap will decline as the bubble grows. By setting \(\phi_q = \Theta\) the central bank will succeed in insulating the output gap and inflation from fluctuations in the bubble. Note that such a policy is equivalent to following the rule

\[
\hat{i}_t = \hat{r}_t^m + \phi_x \pi_t
\]

where \(\hat{r}_t^m = \Theta\hat{q}_t^B\).

Such a policy can be thought of as a "leaning against the bubble" rule, since it requires raising interest rates when the bubble is growing, and lowering them when it bursts. That policy, however, does not have any effect on the bubble itself, it just offsets the effects of the bubble on aggregate demand. In fact, it should be clear that the same outcome can be achieved by any other policy rule that succeeds in stabilizing inflation, for in that case, and independently of how the policy is implemented, \(\hat{i}_t = \Theta\hat{q}_t^B\) will have to hold in equilibrium. An example of such a policy is given by the limit of the simple rule \(\hat{i}_t = \phi_x \pi_t\) as \(\phi_x \to +\infty\). It follows that observability of the bubble and a systematic response to the latter is not a requirement for insulating the economy from bubble fluctuations, at least in the environment analyzed here. That result is reminiscent of the main findings in Bernanke and Gertler (1999, 2001). On the other hand, an inappropriate choice of coefficient \(\phi_q\) and/or a "more flexible" form of inflation targeting will fail to insulate the "real economy" from the booms and busts experienced by the bubble.
In the previous boom and bust episode is triggered by a one-off emergence and rise of a bubble. For an example with recurrent bubble-driven booms consider an economy where the bubble evolves exogenously according to the following process.

\[
q_t^B = \begin{cases} 
\frac{\delta}{\beta} q_{t-1}^B + u_t & \text{with probability } \delta \\
u_t & \text{with probability } 1 - \delta
\end{cases}
\]  

(48)

where \(\{u_t\}\) follows a white noise process with positive support and constant mean \(\mu > 0\). Again, the previous process satisfies (38) and (39), as well as the non-negativity condition. Furthermore, \(E_t\{q_{t+T}^B\} = \pi/(1 - \nu/\beta)\), the size of the bubble converges in expectation to a value arbitrarily close to zero. Equilibrium output gap and inflation can be shown to be given by

\[
\hat{y}_t = \chi_y + (1 - \Phi \nu/\beta) \Psi (\Theta - \phi_q) q_t^B \\
\pi_t = \chi_\pi + \kappa \Psi (\Theta - \phi_q) q_t^B
\]

where \(\lim_{\pi \to 0} \chi_y = \lim_{\pi \to 0} \chi_\pi = 0\). Note that all the qualitative findings regarding the role of monetary policy in shaping the effects of bubble fluctuations found in the discussion above apply also to the present case.

Figure 3 displays a simulated equilibrium path. In addition to the baseline parameter settings introduced above, I set \(\delta = 0.95\) as values describing the process for the bubble and draw \(\{u_t\}\) from an exponential distribution with mean 0.01(1 - \(\nu/\beta\)). Furthermore, I set \(\kappa = 0.0434\) as a baseline value for the slope of the New Keynesian Phillips curve. That value is consistent with \(\theta = 0.75\) and \(\varphi = 0.5\). The former corresponds to an average duration of individual prices of 4 quarters, in accordance with much of the micro evidence. The latter is consistent with (6) and the observation that the standard deviation of the (log) real wage is roughly half the size that of (log) hours worked (both HP-detrended), based on postwar U.S. data. Finally, I assume a "flexible inflation targeting" policy, with \(\phi_\pi = 1.5\) and \(\phi_q = 0\).

Figure 4 shows the standard deviation of output as a function of \(\phi_\pi\) and \(\phi_q\), with the standard deviation of the bubble normalized to unity. The figure illustrates the dangers of choosing too large a value for \(\phi_q\), which contrasts with the unambiguous stability gains from a strong anti-inflationary stance.

### 4.2 Equilibrium Dynamics around a Bubbly BGP

Next I analyze the properties of the model’s equilibrium dynamics in a neighborhood of a bubbly BGP. For simplicity I assume that the newly created bubbles follow a martingale difference process about a constant mean \(\pi\), so that \(E_t\{\tilde{u}_{t+1}\} = 0\) for all \(t\). By combining (30), (31), (32), (33) and (29) the following version of a dynamic IS equation can be derived:

\[
\hat{y}_t = \frac{\Lambda \Gamma \nu}{\beta} E_t\{\hat{y}_{t+1}\} - \frac{\Upsilon \nu}{\beta} (\hat{u}_t - E_t\{\pi_{t+1}\}) + \Theta \tilde{q}_t^B
\]  

(49)

where \(\Theta \equiv \frac{(1 - \beta \gamma)(1 - \nu/\gamma)}{\beta \gamma} > 0\) and \(\Upsilon \equiv 1 + \frac{(1 - \beta \gamma)(\Lambda \Gamma - 1)}{1 - \Lambda \Gamma \nu \gamma} \geq 1\). The previous equation has a form similar to the dynamic IS equation in the standard New Keynesian model with preference shocks, with fluctuations in the bubble playing the role of the latter. In contrast with the "standard" model though, note that the coefficients on expected inflation and interest rate will be different from unity. In particular, the fact that \(\Lambda \Gamma \nu/\beta < 1\) in a bubbly BGP makes the influence of an expected change in the interest rate on current output decline with the horizon of the policy intervention, thus overcoming the so called "forward guidance puzzle." Note also that the largest possible theoretical value for coefficient \(\Lambda \Gamma \nu/\beta\) determining the
rate of decline is \( v \), which under a narrow interpretation will take a value too close to one to make a significant difference quantitatively for plausible horizons.

Thus, the non-policy block of the system describing the equilibrium dynamics around a bubbly BGP are described by (49), the bubble equation

\[
\hat{q}_t^B = \Lambda \Gamma E_t\{\hat{q}_{t+1}^B\} - q^B (\hat{y}_t - E_t\{\pi_{t+1}\})
\]

(50)
together with the New Keynesian Phillips curve (14). Note that, in contrast with the system describing the equilibrium around the bubbleless steady state, the size of the bubble in the dynamical system above has an endogenous component, which depends on the interest rate.

Again, I assume the central bank follows the interest rate rule (47), repeated here for convenience:

\[
\hat{i}_t = \phi_x \pi_t + \phi_q q^B
\]

By setting \( \phi_q = \Theta \beta / (\gamma \nu) \), the central bank can fully offset the effects of the bubble on aggregate demand, output and inflation. In that case, the equilibrium dynamics are described by the system

\[
\begin{bmatrix}
\hat{y}_t \\
\pi_t
\end{bmatrix} = A_*
\begin{bmatrix}
E_t\{\hat{y}_{t+1}\} \\
E_t\{\pi_{t+1}\}
\end{bmatrix}
\]

where

\[
A_* \equiv \Omega_* \left[ \frac{\Lambda \Gamma v}{\Lambda \Gamma v \kappa} \frac{\gamma \nu \beta}{\kappa \beta} (1 - \Phi_\pi) \right]
\]

with \( \Omega_* \equiv \frac{1}{1+ (\gamma \nu / \beta) \kappa \phi_x} \). The necessary and sufficient condition for (local) uniqueness in that case can be shown to be given by:

\[
\phi_\pi > 1 - \frac{(1 - \Phi)(1 - \Lambda \Gamma v / \beta)}{\kappa \gamma \nu / \beta}
\]

The lower threshold can be shown to be decreasing the BGP interest rate and bubble size. It is exactly equal to unity at the bubbleless BGP (where \( \Lambda \Gamma v / \beta = 1 \)), in a way consistent with the findings above. At the other extreme, corresponding to a BGP with the largest possible bubble and interest rate (corresponding to \( \Lambda \Gamma = 1 \)) that lower bound is given by

\[
1 - \frac{(1 - \Phi)(1 - \Lambda \Gamma v / \beta)}{\kappa \gamma \nu / \beta}
\]

which under the baseline calibration equals 0.99997.

Thus, an interest rate rule \( \hat{i}_t = \phi_x \pi_t + \frac{\phi_\beta}{\gamma \nu} q^B \) with \( \phi_\pi > 1 \) guarantees full stabilization of the output gap and inflation in the face of fluctuations in the aggregate bubble. The latter will evolve according to the difference equation:

\[
\hat{q}_t^B = \Xi \hat{q}_{t-1}^B + \xi_t
\]

where \( \Xi \equiv \left( \frac{1}{\Lambda \Gamma} \right) \left( 1 + \frac{(\beta - \Lambda \Gamma v)(1 - \nu \gamma)}{\nu [1 - \Lambda \Gamma v \beta + (1 - \beta) (1 - \Lambda \Gamma)]} \right) \) and \( \xi_t \equiv \hat{b}_t - E_{t-1}\{\hat{b}_t\} + \hat{u}_t \) is the aggregate bubble innovation.

It is easy to show that \( \lim_{q^B \to 0} \Xi = v / \beta < 1 \), \( \lim_{q^B \to 0} \Xi = \beta / v > 1 \). Thus as long as the bubble along the BGP is not too large, the aggregate bubble may display non-explosive fluctuations around that constant value. Furthermore, the bubble's volatility will be enhanced by the "leaning against the bubble" policy.
As in the case of fluctuations around the bubbleless BGP, an identical outcome will obtain if the central bank follows any other rule that succeeds in stabilizing inflation (and hence the output gap), for in that case (49) requires that in equilibrium
\[ \hat{t}_t - E_t \{ \pi_{t+1} \} = \hat{\tau}_t = \frac{\Theta}{1-q^B} q^B. \]

What are the implications of more general calibrations of the interest rate rule (47)? Using (??) to eliminate the interest rate in (49) and (32), we can rewrite the system as follows:
\[ A_0 x_t = A_1 E_t \{ x_{t+1} \} \]
where \( x_t = [\hat{y}_t, \pi_t, \hat{q}_t^B]^\prime \) and
\[
A_0 = \begin{bmatrix}
1 -\frac{\tau_u}{\beta} \phi_x & -\Theta + \frac{\tau_u}{\beta} \phi_q \\
-\kappa & 1 \\
0 & q^B \phi_x + 1 + q^B \phi_q
\end{bmatrix}; \quad A_1 = \begin{bmatrix}
\frac{\Lambda \tau_u}{\beta} & \frac{\tau_u}{\beta} & 0 \\
0 & \Phi & 0 \\
0 & q^B & \Lambda \Gamma
\end{bmatrix}
\]

The solution to the previous dynamical system is locally unique if and only if the three eigenvalues of the companion matrix \( A = A_0^{-1} A_1 \) lie inside the unit circle. Figures 5a-5c display the uniqueness and indeterminacy regions on the \((\phi_q, \phi_y)\) plane, under alternative assumptions regarding the real interest rate in the underlying BGP, and given the baseline calibration introduced above as well as the baseline price setting.

Figure 5a displays the determinacy and indeterminacy regions around four different BGPs (indexed by the real interest rate) under the baseline price setting environment. Figure 5b shows the corresponding findings under the alternative price setting environment [Discussion to be added].

Figure 6 displays simulated stationary fluctuations corresponding to a model calibration consistent with one-dimensional indeterminacy, under the baseline price setting environment. As the figure illustrates, the implied fluctuations are highly persistent and involve "proportional" variations in output, inflation and the bubble.

### 5 Concluding Comments

The New Keynesian model has become the workhorse framework for monetary policy analysis, even though it is unsuitable—in its standard formulation—to accommodate the existence of asset price bubbles and hence to address one of the key questions facing policy makers, namely, how monetary policy should respond to those bubbles. That shortcoming, however, is not tied to any key ingredient of the model (e.g. staggered price setting), but to the convenient (albeit unrealistic) assumption of an infinite-lived representative consumer. In the present paper I have developed an extension of the basic New Keynesian model featuring overlapping generations, finite lives and retirement. The analysis of the properties of that model has yielded several insights, which I briefly summarize next.

Firstly, the introduction of an overlapping generation structure, even when one abstracts from the possibility of bubbly equilibria, may change the conditions under which simple interest rate rules will guarantee equilibrium uniqueness. In particular, under some parameter configurations, the so-called Taylor principle has to be "reinforced," calling for a stronger anti-inflationary stance in order to avoid indeterminacy.

Secondly, and most importantly, the proposed framework allows for the existence of rational expectations equilibria with asset price bubbles, in contrast with the standard model. In particular, plausible calibrations of the model’s parameters are
consistent with the existence of a continuum of bubbly balanced growth paths, as well as a bubbleless one (which always exists).

Thirdly, a "leaning against the bubble" interest rate policy, if precisely calibrated, can insulate output and inflation from bubble fluctuations, without necessarily bursting the bubble or dampening its fluctuations. However, mismeasurement of the bubble or an inaccurate response may end up amplifying the economy's fluctuations. But even if the central bank observes the bubble correctly and calibrates its optimal response to it accurately, there exist conditions under which such a "leaning against the bubble" policy may end up increasing the volatility and persistence of bubble fluctuations. On the other hand, the same stabilization objectives can be attained by targeting inflation directly, without the risks associated with direct responses to the bubble.

Three additional remarks, pointing to possible future research avenues are in order. Firstly, the analysis of the equilibrium dynamics above has assumed "rationality" of asset price bubbles. That assumption underlies the equilibrium conditions that individual and aggregate bubbles must satisfy, i.e. (18) and (20), respectively, and the implied log-linear approximation (32). However, the remaining equilibrium conditions, including the modified dynamic IS equations, are still valid even if the process followed by the aggregate bubble \( \{\tilde{q}_t^B\} \) is inconsistent with a rational bubble. That observation opens the door to analyses of the macroeconomic effects and policy implications of alternative specifications of the aggregate bubble's behavior.

Secondly, the analysis of the model suggests that balanced growth paths characterized by a larger bubble size are associated with a higher real interest rate. Thus, the presence of a bubble along a BGP should make it less likely for the zero lower bound on the nominal interest rate to become binding, ceteris paribus and conditional on the bubble not bursting. Similarly, the bursting of the bubble would bring along a reduction in the natural rate of interest that could pull the interest rate toward the zero lower bound. The analysis of the interaction of bubble dynamics with the zero lower bound seems an avenue worth exploring in future research.

Finally, the analyses of the equilibrium dynamics above has assumed that the central bank takes as given the BGP on which the economy settles and, hence, its associated real interest rate. That assumption is reflected in the implicit exogeneity of the real interest rate term embedded in the policy rule through the term \( \hat{i}_t \simeq \bar{i}_t - r \). But while that assumption is a natural one in the context of economies whose real interest rate along a BGP is uniquely pinned down (as it is the case in the standard New Keynesian model with a representative consumer), it is no longer so in an economy like the one analyzed above, in which a multiplicity of real interest rates may be consistent with a perfect foresight BGP. In future research I plan to explore the implications of relaxing the assumption of an exogenously given steady state real interest rate.
References


1. Transversality condition in a Bubbly BGP

The consumption function for an active individual born in period $s$ is:

$$C^a_{t+T|s} = (1 - \beta \gamma) \left[ A^a_{t+T|s} + \frac{W_{t+T} N/\alpha}{1 - \Gamma \nu \gamma} \right]$$

In particular:

$$C^a_{t|s} = (1 - \beta \gamma) \left[ A^a_{t|s} + \frac{W_t N/\alpha}{1 - \Gamma \nu \gamma} \right]$$

In addition, $C_{t+T|s} = (\beta/\Lambda)^T C_{t|s}$ thus implying

$$A^a_{t+T|s} = (\beta/\Lambda)^T \left[ A^a_{t|s} + (1 - (\Lambda/\beta)^T) \frac{W_t N/\alpha}{1 - \Gamma \nu \gamma} \right]$$

On the other hand, for a retired individual

$$C^r_{t|s} = (1 - \beta \gamma) A^r_{t|s}$$

Using the fact that $C^r_{t|s} = C^a_{t|s}$ we have:

$$A^r_{t|s} = A^a_{t|s} + \frac{W_t N/\alpha}{1 - \Gamma \nu \gamma}$$

The transversality condition for an active individual takes the form:

$$\lim_{T \to \infty} (\gamma \Lambda)^T E_t \{ A_{t+T|s} \} = \lim_{T \to \infty} (\gamma \Lambda)^T [v^T A^a_{t+T|s} + (1 - v^T) A^r_{t+T|s}]$$

$$= \lim_{T \to \infty} (\gamma \Lambda)^T [A^a_{t+T|s} + (1 - v^T) \Gamma^T \frac{W_t N/\alpha}{1 - \Lambda \nu \gamma}]$$

$$= \lim_{T \to \infty} (\beta \gamma)^T \left[ A^a_{t|s} + \frac{W_t N/\alpha}{1 - \Lambda \nu \gamma} (1 - (\Lambda/\beta)^T) \right] + \lim_{T \to \infty} (\Lambda \gamma)^T (1 - v^T) \frac{W_t N/\alpha}{1 - \Lambda \nu \gamma}$$

$$= \frac{W_t N/\alpha}{1 - \Lambda \nu \gamma} \lim_{T \to \infty} [((\beta \gamma)^T - (\Lambda \gamma)^T + (\Lambda \gamma)^T - (\Lambda \nu \gamma)^T]$$

$$= 0$$

where the maintained assumption $\Lambda \Gamma \nu \gamma < 1$ has been used.

2. Log-linearized individual intertemporal budget constraint

The intertemporal budget constraint for an individual born in period $s$ and still active in period $t \geq s$ can be derived by iterating (3) forward to yield:

$$\sum_{k=0}^{\infty} \gamma^k E_t \{ A_{t+k|s} C_{t+k|s} \} = A^a_{t|s} + \frac{1}{\alpha} \sum_{k=0}^{\infty} (\gamma v)^k E_t \{ A_{t+k|s} W_{t+k} N_{t+k} \}$$

For retired individuals, the corresponding constraint is:

$$\sum_{k=0}^{\infty} \gamma^k E_t \{ A_{t+k|s} C_{t+k|s} \} = A^r_{t|s}$$
Letting lowercase letters with a "^" symbol denote the log deviation of a variable from its value along a perfect foresight balanced growth path (BGP), the left hand side term of (51) and (52) can be approximated in a neighborhood of the BGP as:

\[
\sum_{k=0}^{\infty} \gamma^k E_t \{ \Lambda_{t,t+k} C_{t+k|s} \} \approx \frac{\Gamma^t C_{t-s}}{1 - \beta \gamma} + \frac{\Gamma^t \sum_{k=0}^{\infty} (\Lambda \gamma)^k C_{t+k-s}}{1 - \beta \gamma} E_t \{ \hat{C}_{t+k|s} + \hat{\lambda}_{t,t+k} \}
\]

where \( C_j \) denotes consumption of an individual aged \( j \) (normalized by current productivity) along a BGP, \( \hat{\lambda}_{t,t+k} = \log(\Lambda_{t,t+k}/\Lambda^k) \), and where I have made use of the fact that \( C_{t+k-s} = [\beta(1 + r)/\Gamma]^k C_{t-s} \) and \( E_t \{ \hat{\lambda}_{t,t+k} + \hat{\lambda}_{t,t+k} \} = \hat{\lambda}_{t,s} \) (resulting from (4)).

The second term on the right hand side of (51), which is relevant only for active individuals, can be approximated around the BGP as:

\[
\frac{1}{\alpha} \sum_{k=0}^{\infty} (\nu \gamma)^k E_t \{ \Lambda_{t,t+k} W_{t+k} N_{t+k} \} \approx \frac{\Gamma^t W N}{\alpha (1 - \Lambda \nu \gamma)} + \left( \frac{\Gamma^t W N}{\alpha} \right) \sum_{k=0}^{\infty} (\Lambda \nu \gamma)^k E_t \{ \hat{w}_{t+k} + \hat{n}_{t+k} + \hat{\lambda}_{t,t+k} \}
\]

where \( W \) is the wage along the BGP, normalized by productivity.

### 3. Aggregation and derivation of log-linearized aggregate consumption Euler equation

Let \( \mathcal{C} \) denote aggregate consumption (normalized by current productivity), along the BGP. One can derive the approximate relations \( \mathcal{C}_t = (1 - \gamma) \sum_{s=-\infty}^{t} \gamma^{t-s} C_{t-s} \hat{C}_{t|s} \) and the BGP relation \( \mathcal{C} = (1 - \gamma) \sum_{s=-\infty}^{t} \gamma^{t-s} C_{t-s} \). Those relations can be used to aggregate the log-linearized individual consumption functions across all individuals to yield:

\[
\frac{\Gamma^t \mathcal{C}}{1 - \beta \gamma} + \frac{\Gamma^t \mathcal{C}}{1 - \beta \gamma} \hat{C}_t = (Q^F_t + Q^B_t) + \frac{\Gamma^t W N}{1 - \Lambda \nu \gamma} + \Gamma^t W N \sum_{k=0}^{\infty} (\Lambda \nu \gamma)^k E_t \{ \hat{w}_{t+k} + \hat{n}_{t+k} + \hat{\lambda}_{t,t+k} \}
\]

Note also that in a neighborhood of the BGP,

\[
Q^F_t = \sum_{k=0}^{\infty} (\nu \gamma)^k E_t \{ \Lambda_{t,t+k} D_{t+k} \}
\]

\[
\approx \frac{\Gamma^t D}{1 - \Lambda \nu \gamma} + \frac{\Gamma^t D}{1 - \Lambda \nu \gamma} \sum_{k=0}^{\infty} (\Lambda \nu \gamma)^k E_t \{ \hat{d}_{t+k} + \hat{\lambda}_{t,t+k} \}
\]

where \( D \) denotes aggregate dividends (normalized by productivity), along a BGP.

Thus, using the approximation \( \hat{y}_t = (W N/\gamma)(\hat{w}_t + \hat{n}_t) + (D/\gamma) \hat{d}_t \), the BGP relation \( \frac{\Gamma^t \mathcal{C}}{1 - \beta \gamma} = Q^B + \frac{\Gamma^t (W N + D)}{1 - \Lambda \nu \gamma} \), and the goods market clearing condition \( Y = C \), we obtain the log-linearized aggregate consumption function

\[
\hat{C}_t = (1 - \beta \gamma) \left[ \hat{q}^B_t + \sum_{k=0}^{\infty} (\Lambda \nu \gamma)^k E_t \{ \hat{y}_{t+k} + \hat{\lambda}_{t,t+k} \} \right]
\]

\[
= (1 - \beta \gamma) \left[ \hat{q}^B_t + \sum_{k=0}^{\infty} (\Lambda \nu \gamma)^k E_t \{ \hat{y}_{t+k} \} - \sum_{k=1}^{\infty} (\Lambda \nu \gamma)^k \sum_{j=0}^{k-1} E_t \{ \hat{r}_{t+j} \} \right]
\]

\[
= (1 - \beta \gamma) \left[ \hat{q}^B_t + \sum_{k=0}^{\infty} (\Lambda \nu \gamma)^k E_t \{ \hat{y}_{t+k} \} - \frac{\Lambda \nu \gamma}{1 - \Lambda \nu \gamma} \sum_{k=0}^{\infty} (\Lambda \nu \gamma)^k E_t \{ \hat{r}_{t+k} \} \right]
\]
where \( \hat{q}_t^B \equiv q_t^B - q^B \), with \( q_t^B \equiv Q_t^B / (\Gamma^t Y) \) and \( q^B \) its corresponding value along a BGP, and \( \hat{\tau}_t \equiv \hat{\tau}_t - E_t \{ \pi_{t+1} \} \).

Note that we can write the consumption function more compactly as:

\[
\hat{c}_t = (1 - \beta \gamma) (\hat{q}_t^B + \hat{x}_t) \tag{53}
\]

where \( \hat{x}_t \equiv \sum_{k=0}^{\infty} (\Lambda \Gamma \nu \gamma)^k E_t \{ \hat{y}_{t+k} \} - \frac{\Lambda \Gamma \nu \gamma}{1 - \Lambda \Gamma \nu \gamma} \sum_{k=0}^{\infty} (\Lambda \Gamma \nu \gamma)^k E_t \{ \hat{\tau}_{t+k} \} \) the non-bubbly component of wealth, expressed in log deviations from its value along a BGP. It satisfies the recursive equation:

\[
\hat{x}_t = \Lambda \Gamma \nu \gamma E_t \{ \hat{x}_{t+1} \} + \hat{y}_t - \frac{\Lambda \Gamma \nu \gamma}{1 - \Lambda \Gamma \nu \gamma} \hat{\tau}_t
\]

Furthermore, log-linearization of (53) and (54) about a BGP yields

\[
\hat{q}_t^B = \Lambda \Gamma E_t \{ \hat{q}_{t+1}^B \} - q^B \hat{\tau}_t \tag{54}
\]

In a neighborhood of the bubbleless BGP \( q^B = 0 \) and \( \Lambda \Gamma \nu = \beta \), thus implying the approximate aggregate Euler equation:

\[
\hat{c}_t = \beta \gamma E_t \{ \hat{c}_{t+1} \} + (1 - \beta \gamma)(1 - \nu \gamma)q_t^B + (1 - \beta \gamma)\hat{y}_t - \beta \gamma \hat{\tau}_t
\]

Imposing goods market clearing (\( \hat{c}_t = \hat{y}_t \)) and rearranging terms, we obtain a dynamic IS equation:

\[
\hat{y}_t = E_t \{ \hat{y}_{t+1} \} + \Phi \hat{q}_t^B - \hat{\tau}_t
\]

where \( \Phi \equiv (1 - \beta \gamma)(1 - \nu \gamma) / \beta \gamma > 0 \) and \( \{ q_t^B \} \) must satisfy \( q_t^B = (\beta / \nu) E_t \{ q_{t+1}^B \} \) and \( q_t^B \geq 0 \) for all \( t \).

Note that in a bubbleless equilibrium around the bubbleless BGP:

\[
\hat{y}_t = E_t \{ \hat{y}_{t+1} \} - \hat{\tau}_t
\]

which is identical to the standard dynamic IS equation of the representative consumer model.

In any bubbly steady state, \( q^B = \frac{1}{1 - \beta \gamma} - \frac{1}{1 - \Lambda \Gamma \nu \gamma} \). Combining (53) and (54) yields the Euler equation:

\[
\hat{c}_t = \Lambda \Gamma \nu \gamma E_t \{ \hat{c}_{t+1} \} + (1 - \beta \gamma)(1 - \nu \gamma)\hat{q}_t^B + (1 - \beta \gamma)\hat{y}_t - \gamma \nu \gamma \hat{\tau}_t
\]

where \( \gamma \equiv \left( 1 + \frac{(1 - \beta \gamma)(\Lambda \Gamma - 1)}{1 - \Lambda \Gamma \nu \gamma} \right) > 1 \) and \( \Lambda \in \left( 1, \frac{\beta}{1 - \beta \gamma} \right) \). Imposing goods market clearing:

\[
\hat{y}_t = \frac{\Lambda \Gamma \nu}{\beta} E_t \{ \hat{y}_{t+1} \} + \Phi \hat{q}_t^B - \frac{\gamma \nu}{\beta} \hat{\tau}_t \tag{55}
\]

In the particular case of \( \delta = 0 \), we have \( \Lambda \Gamma = \gamma = 1 \) thus implying a dynamic IS equation of the form:

\[
\hat{y}_t = \frac{\nu}{\beta} E_t \{ \hat{y}_{t+1} \} + \Phi \hat{q}_t^B - \frac{\nu}{\beta} \hat{\tau}_t
\]

4. Conditions for equilibrium uniqueness around the bubbleless BGP.
The necessary and sufficient conditions for $\mathbf{A}_T$ to have two eigenvalues within the unit circle are: (i) $|\det(\mathbf{A}_T)| < 1$ and (ii) $|\text{tr}(\mathbf{A}_T)| < 1 + \det(\mathbf{A}_T)$ (LaSalle (1986)). Note that in the case of a bubbleless BGP, $\det(\mathbf{A}_T) = \frac{\phi}{1+\kappa\phi}$, while $\text{tr}(\mathbf{A}_0) = \frac{1+\kappa+\Phi}{1+\kappa\phi} > 0$. Condition (i) requires $\phi_\pi > (\Phi - 1) / \kappa$. Condition (ii) corresponds to $\phi_\pi > 1$.

5. Sunspot Fluctuations

Let the equilibrium be described by the system of difference equations

$$\mathbf{x}_t = \mathbf{A}E_t\{\mathbf{x}_{t+1}\}$$

where $\mathbf{x}_t \equiv [\tilde{y}_t, \pi_t, \tilde{q}^P_t]'$ are all non-predetermined variables. Let $\mathbf{A}$ have $q \leq 3$ eigenvalues with modulus less than one.

Consider the transformation $\mathbf{x}_t = \mathbf{Q}\mathbf{v}_t$ where $\mathbf{QJQ}^{-1} = \mathbf{A}$ where $\mathbf{J}$ is the canonical Jordan matrix and $\mathbf{Q} \equiv [\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \mathbf{q}^{(3)}]$ is the matrix of generalized eigenvectors, corresponding to the three eigenvalues. Thus,

$$\mathbf{v}_t = \mathbf{J}E_t\{\mathbf{v}_{t+1}\}$$

where $\mathbf{J} = \begin{bmatrix} \mathbf{J}_u & 0 \\ 0 & \mathbf{J}_s \end{bmatrix}$ and $\mathbf{v}_t = [\mathbf{v}_t^u, \mathbf{v}_t^s]'$.

Consider the case where $\mathbf{A}$ has two eigenvalues with modulus less than one and one with modulus greater than one. For concreteness, assume $|\lambda_1| < 1$, $|\lambda_2| < 1$, and $|\lambda_3| > 1$. If all eigenvalues are real $\mathbf{J}_u = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\mathbf{J}_s = \lambda_3$, and $\mathbf{q}^{(k)}$ corresponds to the eigenvector associated with eigenvalue $k$, for $k = 1, 2, 3$. Otherwise, if $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ are complex conjugates, $\mathbf{J}_u = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $\mathbf{J}_s = \lambda_3$ and $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are, respectively, the imaginary and real components of the eigenvector associated with the complex eigenvalues.

Thus, $\mathbf{v}_t^u = 0$ all $t$. In addition $\mathbf{v}_t^s = \lambda_3E_t\{\mathbf{v}_{t+1}^s\}$, which has a stable solution:

$$\mathbf{v}_t^s = \lambda_3^{-1}\mathbf{v}_{t-1}^s + \xi_t$$

where $\xi_t$ is a martingale-difference (univariate) process.

Thus, we have $\mathbf{x}_t = \mathbf{q}^{(3)}\mathbf{v}_t^s$ is the sunspot solution. The three variables in $\mathbf{x}_t$ will be perfectly correlated (positively or negatively), and will display a first-order autocorrelation $\lambda_3^{-1}$. We can normalize the the third element of $\mathbf{q}^{(3)}$ to unity.

Consider next the case where $\mathbf{A}$ has one eigenvalue with modulus less than one and two with modulus greater than one. For concreteness, assume $|\lambda_1| < 1$, $|\lambda_2| > 1$, and $|\lambda_3| > 1$. If all eigenvalues are real $\mathbf{J}_u = \lambda_1$, $\mathbf{J}_s = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{bmatrix}$ and $\mathbf{q}^{(k)}$ corresponds to the eigenvector associated with eigenvalue $k$, for $k = 1, 2, 3$. Otherwise, if $\lambda_2 = a + bi$ and $\lambda_3 = a - bi$ are complex conjugates, $\mathbf{J}_u = \lambda_1$, $\mathbf{J}_s = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $\mathbf{q}^{(2)}$ and $\mathbf{q}^{(3)}$ are, respectively, the imaginary and real components of the eigenvector associated with the complex eigenvalues, while $\mathbf{q}^{(1)}$ is the eigenvector associated with $\lambda_1$.

Thus, $\mathbf{v}_t^u = 0$ all $t$. In addition $\mathbf{v}_t^s = \lambda_3E_t\{\mathbf{v}_{t+1}^s\}$, which has a stable solution:

$$\mathbf{v}_t^s = \lambda_3^{-1}\mathbf{v}_{t-1}^s + \xi_t$$

where $\xi_t$ is a (bivariate) martingale-difference process.

Thus, we have $\mathbf{x}_t = [\mathbf{q}^{(2)}, \mathbf{q}^{(3)}]\mathbf{v}_t^s$ is the sunspot solution. We can normalize the third element of $\mathbf{q}^{(2)}$ and $\mathbf{q}^{(3)}$ to unity.
Figure 1. Balanced Growth Paths

- Bubbly without bubble creation
- Bubbly with bubble creation
- Bubbleless
Figure 2. Determinacy and Indeterminacy Regions for Bubbleless Equilibria (Alternative Price Setting)
Figure 3. Simulated Bubble-Driven Fluctuations around the Bubbleless BGP
Figure 4. Bubble-driven Fluctuations: Monetary Policy and Macro Volatility
Figure 5a. Determinacy and Indeterminacy Regions around a Bubbly BGP (Baseline Price Setting)

$r = 0.003866$

$r = 0.0039$

$r = 0.00395$

$r = 0.004$
Figure 5b. Determinacy and Indeterminacy Regions around a Bubbly BGP (Baseline Price Setting)

- $r = 0.003866$
- $r = 0.00395$
- $r = 0.004$

1. Indeterminacy (1-dim)
2. Indeterminacy (2-dim)
3. Indeterminacy (3-dim)
Figure 6. Simulated Bubble-driven Fluctuations
Around a Bubbly BGP