Over-the-Counter Markets with Bargaining Delays: 
The Role of Public Information in Market Liquidity

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Abstract

In many over-the-counter asset markets, prices are negotiated bilaterally and 
bargaining over prices takes time. We show that bargaining delays arise when 
investors have precise private information about the asset quality, but the public 
information (e.g. credit ratings, benchmarks, past quotes) is coarse. We incorporate 
this type of bargaining delays into the standard dynamic equilibrium model of over-
the-counter markets with search delays à la Duffie, Gärleanu and Pedersen (2005) 
and derive implications of both delays for prices and liquidity. Search and bargaining 
delays have opposite effects on the range of traded assets showing that the current 
approach that views search delays as a proxy for all types of delays is with a loss. 
Conditional on the public information, the liquidity is U-shaped in the quality and 
assets in the middle of the quality range may not be traded, which contrasts with 
the decreasing liquidity in asymmetric-information models.

Keywords: search friction, bargaining delay, liquidity, over-the-counter mar-
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1 Introduction

Many important asset markets are decentralized. Examples include over-the-counter (OTC) markets for real estate, asset-backed securities, derivatives, bonds, credit-default swaps, private equity, sovereign debt, bank loans, etc. In such markets, prices are negotiated bilaterally and it takes parties time to agree on the price. These bargaining delays can range from months as in the real estate or private equity to hours, minutes, or even seconds as in the most liquid parts of the bond market. The existing literature pioneered by Duffie, Gärleanu and Pedersen (2005) adopts the generalized Nash bargaining solution, and thus, abstracts from bargaining delays focusing instead on search delays. In fact, search delays are thought of as a reduced form for all types of trade delays. The question remains: how justified is this reduced form approach? For example, one may conjecture that increased market uncertainty or better quality of public information would affect bargaining delays. Is it justified to use search delays as a proxy and derive liquidity predictions from the existing models? This paper disentangles the effect of both search and bargaining delays on asset liquidity and pricing, and shows that the two operate quite differently.

The novelty of our approach is in the departure from the generalized Nash bargaining solution. The key theoretical observation is that in the standard alternating-offer bargaining game (see Binmore et al. 1986), bargaining delays arise even when parties can make offers almost continuously and have very precise private signals about the asset quality, as long as the public information is coarse (formally, the quality is not common knowledge). We introduce and micro-found the screening bargaining solution, which captures in a reduced form bargaining delays arising in the limit as the precision of private signals increases to infinity, while holding fixed the amount of public information. Despite precise private signals, parties are not able to credibly reveal their private information to the opponent other than through the bargaining delay.

This approach has two advantages. On the one hand, it captures the fact that in many OTC markets both sides have some private information about the asset, and this private information is more refined as compared to the public information. Existing empirical studies often reject predictions of adverse selection models and point to the two-sided private information (see Garmaise and Moskowitz 2004 for the commercial real estate, Merlo and Ortalo-Magne 2004 for the residential real estate, and Pérignon et al. 2015

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Duffie (2012) summarizes the current approach as follows: “[s]earch delays ... proxy for delays associated with reaching an awareness of trading opportunities, arranging financing and meeting suitable legal restrictions, negotiating trades, executing trades, and so on.”
for the wholesale funding market). Further, the gap between coarse public and precise private information of parties is relevant in many OTC markets. For example, in OTC markets for financial assets, credit ratings, benchmarks, past quotes, etc. provide rough bounds on the asset value, which investors further refine based on their own information sources and their expertise in assessing risks (see e.g. Adelino 2009, Ashcraft et al. 2010 for the mortgage-backed securities market). This aspect is well captured in the following account of the OTC trade by the Committee on the Global Financial System (2005):

“Interviews with large institutional investors in structured finance instruments suggest that they do not rely on ratings as the sole source of information for their investment decisions ... Indeed, the relatively coarse filter a summary rating provides is seen, by some, as an opportunity to trade finer distinctions of risk within a given rating band. Nevertheless, rating agency ‘approval’ still appears to determine the marketability of a given structure to a wider market.”

Similarly, in the commercial real estate, investors widely utilize publicly available property tax assessments, which provide only a noisy value estimates (see Garmaise and Moskowitz 2004), while both sides get much better estimates from hired private appraisers, their previous property ownership (for sellers), or their professional expertise in evaluating the particular property (for buyers).2

On the other hand, focusing on the case where both sides get private signals and these signals become infinitely precise allows us to abstract from learning about the quality of assets,3 and thus, incorporate in a tractable way bargaining delays into otherwise standard dynamic equilibrium model of OTC markets à la Duffie et al. (2007). Specifically, we consider an economy, in which investors are occasionally hit by or recover from liquidity shocks. Thus, there are natural buyers and sellers, and to share risks, they can trade a continuum of assets of various qualities in the market with search and bargaining delays. The quality reflects the heterogeneity in asset payoffs not captured by the public information, e.g. differences in the default probability among mortgage-backed securities with AAA credit rating and same maturity. Bargaining delays associated with each asset quality are captured by the screening bargaining solution and determined in equilibrium

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2Garmaise and Moskowitz (2004) shows that when the public information is noisy, buyers tend to trade properties that are closer to them or have longer price histories, and informed brokers are more likely to trade with each other. This suggests that both sides have relatively precise private information about the property value.

3In this respect, we complement the ongoing research on learning in OTC markets, which abstracts from trading for liquidity purposes to focus on the information percolation in such markets (see Duffie and Manso 2007, Duffie et al. 2009, 2010 among others).
by investors’ valuations, which in turn depend on their ability to sell quickly assets in the future as well as the steady-state distribution of assets among investors in the economy. We solve for the unique steady-state equilibrium of this economy and derive implications of both search and bargaining delays for asset prices and standard liquidity measures, such as bid-ask spread, trade volume, turnover, trade delays, and the range of tradeable assets.

The screening bargaining solution can be described in terms of the continuous-time bargaining game, in which the seller and the buyer gradually make, resp., decreasing and increasing price offers that are not conditioned on the asset quality, and both sides choose only the acceptance time. In this game, the first offers of the buyer and the seller are naturally interpreted as the bid and ask prices, resp., as they guarantee immediate trade when accepted at the start of bargaining. We show that the heterogeneity of asset payoffs translates in the endogenous variation of bargaining delays across assets, and the degree of this payoff heterogeneity determines the size of the bid-ask spread. Unlike in Duffie et al. (2005), where bid-ask spreads arise from the market makers’ ability to provide immediacy to investors, in our model, the bid-ask spread reflects purely the quality of public information about assets, and it expands when the public information deteriorates (e.g. during periods of market uncertainty), and shrinks when the public information improves (e.g. when benchmarks are introduced).

Our liquidity implications are drastically different from that in the adverse selection models (e.g. Guerrieri and Shimer 2014) or search-and-bargaining models of OTC liquidity (e.g. Duffie et al. 2005). Conditional on the public information about the asset quality, the trade volume and turnover are U-shaped in the quality, while the bargaining delay is hump-shaped. As was mentioned, the screening bargaining solution is a reduced form for bargaining when sides have precise private information about the quality, but lack common knowledge. Precise private information leads to the non-monotone liquidity. In the dynamics of negotiation behind the screening bargaining solution, the buyer of a high quality asset and the seller of a low quality asset are willing to accept early on an offer close to maximal and minimal prices, resp., while buyers and owners of assets in the middle of the quality range prefer to delay trade to hold out for a more favorable price offer. Our analysis sheds light on the liquidity when both sides have private information and suggests that the liquidity is non-monotone, which is in contrast with the decreasing pattern common in adverse selection models of liquidity.4

4In adverse selection models, the liquidity is decreasing if the seller has private information, and increasing if the buyer has private information, and normally, these models view the former as the most
Bargaining delays operate quite differently from search delays. Bargaining delays endogenously vary across asset qualities, which creates an extensive trade margin reflected in the range of asset qualities that can be traded in equilibrium. When the buyer finds the seller, she compares the bargaining delay associated with the seller’s asset with the outside option of searching further. We show that in equilibrium buyer’s strategy takes a simply threshold form: For assets that involve lengthy negotiation, the buyer prefers to continue the search, while she accepts assets with sufficiently short negotiation times. Importantly, focusing exclusively on the intensive margin (the length of trade delay) can be misleading. When search delays for buyers are very short, buyers accept for trade only the most quickly negotiated assets. As a result, the market is illiquid, as many sellers cannot liquidate their positions, while by looking only at the intensive margin, the market may seem liquid, as observed search and bargaining delays are short.

We further show that search and bargaining delays have opposite effects on the extensive margin. Higher bargaining delays, resulting from the deterioration in the public information about assets, reduce the set of traded assets. This is consistent with the dried-up liquidity during periods of heightened market uncertainty, when infrequently updated credit ratings become less reliable in assessing the risks associated with the asset.\(^5\) Perhaps surprisingly, an increase in search delays expands the set of traded assets. When it is harder to find a trade partner, investors’ outside option of searching for an alternative partner deteriorates, which makes buyers willing to accept a wider range of assets for trade.

The connection between the quality of public information and liquidity has important implications for the design of OTC markets. On the one hand, better quality of credit ratings, standardization of products, benchmarks, etc. improve the quality of public information of assets, and thus, reduce bargaining delays. On the other hand, electronic trade platforms or active brokerage improve the search. Our analysis shows that former policies expand the set of actively traded assets, while the latter reduce it. Thus, making the search more efficient can improve the risk-sharing through shorter search times, but the improvement need not be Pareto, as less sellers are able to liquidate their positions. This finding can explain why complex and non-standardized assets are traded over-the-counter as opposed to centralized exchanges: inefficient search for counter-parties allows for a greater variety of such assets to be tradeable. Our finding is also consistent with relevant case.

\(^5\)In the recent financial crisis, the significant increase in downgrades of financial products (see Benmelech and Dlugosz 2010, Ashcraft et al. 2010) indicates the drop in the accuracy of credit ratings before the ratings’ adjustments.
the existing mixed evidence on the effect on liquidity of the post-trade transparency in the corporate bonds market (see Bessembinder et al. 2006, Edwards et al. 2007, Goldstein et al. 2007, Asquith et al. 2013), as the post-trade transparency improves both the quality of public information and the search efficiency.

In the analysis of liquidity, different assets act as substitutes for risk-sharing. In the recent financial crisis of 2007-2008, traders reacted to the increase in market uncertainty by shifting towards more liquid assets, a phenomenon known as flight-to-liquidity (Dick-Nielsen et al. 2012, Friewald et al. 2012). Similarly, opponents of greater transparency in OTC markets point out that it causes the migration of trade to certain asset classes hurting the liquidity of the market as a whole. We extend the baseline model to take into account the substitutability between asset classes. We show that an increase in bargaining delays in one asset class, caused by less precise public information about assets in this class, results in the flight-to-liquidity wherein investors migrate to trading assets with lower bargaining delays. This exacerbates the negative effect of the increased bargaining delays on the liquidity of the former asset class. Similar flight-to-liquidity effect arises when the reduction in the bargaining friction is uneven across asset classes, which reveals that gradual transparency policies can have negative liquidity effects. This is consistent with the effect on liquidity of the recent introduction of mandatory trade reporting in the corporate bonds market. The reform was done in several phases spread over time, and Asquith et al. (2013) shows that it hurt most trade volumes of high-yield bonds, for which the post-trade transparency was introduced later than for investment grade bonds.

Related literature This paper contributes to the growing literature on search and bargaining models of OTC markets pioneered by Duffie et al. (2005) and further developed to account for risk-aversion (Duffie et al. 2007), unrestricted asset holdings (Lagos and Rocheteau 2007, 2009), asset heterogeneity (Vayanos and Weill 2008, Weill 2008), agent heterogeneity (Vayanos and Wang 2007, Shen et al. 2015, Hugonnier et al. 2014, Üslü 2015). This literature adopts the generalized Nash bargaining solution, and thus, abstracts from bargaining delays. To the best of our knowledge, our paper is the first to study implications of micro-founded bargaining delays for OTC liquidity and show that they operate quite differently from search delays.

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6Similarly, the generalized Nash bargaining solution has been used in the monetary search literature (e.g. Trejos and Wright 1995) and labor search literature (e.g. Hosios 1990).

7If anythings, existing results suggest that search and bargaining frictions are similar. E.g. Lagos and Rocheteau (2009) show that the bargaining power of market makers operates similarly to the search friction.
Our paper offers a dynamic information-based theory of liquidity arising from the lack of public information about assets, rather than the adverse selection. The screening bargaining solution is a reduced form for bargaining between privately informed parties, who nonetheless, engage in lengthy negotiation, because of the lack of common knowledge about the asset quality. Guerrieri and Shimer (2014), Kurlat (2013), Daley and Green (2012), Chang (2014), Daley and Green (2016) study adverse selection in dynamic asset trading models, Guerrieri et al. (2010), Kurlat (2016) introduce adverse selection into the Walrasian competitive equilibrium.\footnote{Our U-shaped liquidity pattern contrasts with the decreasing liquidity in these models. With adverse selection, in order to provide incentives for sellers of low quality assets to reveal their quality, such assets should be more liquid. In Lester et al. (2012), investors acquire information to overcome the adverse selection, which leads to the endogenous liquidity of various asset classes. In our model, bargaining delays result in the variation of liquidity both within and across asset classes.}

Our paper contributes to the theoretical literature that studies the effect of transparency, or more generally better public information, on the efficiency and liquidity of OTC markets. Duffie et al. (2015), Asriyan et al. (2015) show that greater transparency reduces the information asymmetry between agents, and hence, may lead to more efficient risk sharing and higher liquidity. Our paper shows that the effect of transparency on liquidity is ambiguous depending on whether it leads to the reduction in the bargaining or search friction. In the setting with static trading mechanisms, Morris and Shin (2012) stress the role of public information in establishing the common knowledge of quality among investors, and this way, ensuring trade efficiency. Unlike them, we show that the lack of common knowledge leads to inefficient delays in bargaining and study implications of such delays in a fully-fledged dynamic equilibrium model.

Our micro-foundation for the screening bargaining solution contributes to the literature on bargaining with two-sided independent private information about values (Cramton 1984, Cho 1990, Ausubel and Deneckere 1992, 1993, Larsen 2013),\footnote{Independence assumption is also common in bargaining with two-sided private information about discount factors (Watson 1998) and players’ rationality (see Kambe 1999, Abreu and Gul 2000 among others).} and bargaining with interdependent values and one-sided private information (Deneckere and Liang 2006, Fuchs and Skrzypacz 2013, Gerardi et al. 2014). In contrast, we study bargaining with two-sided private information about correlated (through the unobserved asset quality) values and

\footnote{He and Milbradt (2014), Chen et al. (2014) analyze the feedback loop between the liquidity and default, and show that assets closer to default are associated with higher bid-ask spreads. Both their channel and prediction are different from ours.}
show that bargaining delays arise even when values are almost perfectly correlated (and in fact, are necessary in separating outcomes).

Finally, our paper is related to the theoretical literature on search-and-bargaining pioneered by Rubinstein and Wolinsky (1985). Most of this literature focuses on the case of complete information and hence immediate agreement (see Osborne and Rubinstein 1990, Gale 2000 for an excellent survey) with the exception of Satterthwaite and Shneyerov (2007) and Lauermann and Wolinsky (2014) who study conditions for convergence to the Walrasian outcome in search models with incomplete information. This literature studies static allocation mechanisms, while our focus is on delays in dynamic bargaining and our model does not converge to the competitive outcome even as the search friction vanishes. Atakan and Ekmekci (2014) also explicitly incorporate bargaining delays into the search market. In their model, investors can imitate exogenously given commitment types requesting a fixed share of the surplus, while in our model all investors are rational.

The paper is organized as follows. Section 2 presents the OTC model. Section 3 provides game-theoretic foundations for the screening bargaining solution. Section 4 characterizes the equilibrium of the OTC model. Section 5 provides pricing and liquidity implications of bargaining delays. Section 6 shows how substitutability of asset classes leads to flights-to-liquidity and adverse effects of gradual transparency policies. Section 7 concludes. All proofs are relegated to the Appendix.

2 Model

Time $t \geq 0$ is continuous. There is a continuum of asset qualities $\theta \in [0,1]$ each in the fixed unit supply. An asset quality $\theta$ brings a flow payoff $d + k\theta$ where $k > 0$ and so, assets of higher quality are associated with higher flow payoffs. The interpretation is that assets are traded within asset classes defined by the public information. Examples of such asset classes are AAA mortgage-backed securities maturing in 10 years; investment grade zero-coupon bonds with long maturities issued by biotech firms; renovated studios in downtown Boston. The quality $\theta$ is the index that aggregates various factors that affect asset payoffs, but are not captured by the public information, and $k$ reflects the asset heterogeneity conditional on the public information. For example, the MBSs with

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10For real assets, such as real estate, planes, etc., the flow payoff is the instantaneous utility to the asset owner. For financial fixed-income assets, such as corporate bonds, sovereign debt, etc., the flow payoff reflects both the fixed coupon and the risk and costs of default (see Online Appendix for an example).
the same characteristics and credit rating still vary in the default risk determined by the quality of underlying mortgages; the identity of the bond issuer affects the riskiness of the bond beyond the maturity, credit rating, and industry; the safety and expected developments in the area affect the value of the residential property along with publicly available assessments of the property.

There is a continuum of infinitely-lived investors of mass $a > 1$. Investors are risk-neutral and discount at rate $r$. There are two intrinsic types of investors, which we call in anticipation of their equilibrium behavior buyers and sellers. Sellers experience a transitory liquidity shock interpreted as a hedging or liquidity need. For them, holding the asset is associated with additional (flow) holding costs, which we normalize to 1. Sellers recover from the liquidity shock (and become buyers) with the Poisson intensity $y_u$, and buyers are hit by the liquidity shock (and become sellers) with the Poisson intensity $y_d$. Liquidity shocks and recoveries are independent across investors. The initial distribution of intrinsic types is stationary with a mass $\frac{y_u}{y_u + y_d}a$ of buyers and a mass $\frac{y_d}{y_u + y_d}a$ of sellers. To focus on risk-sharing motives, we abstract from investors’ portfolio choices and restrict that each investor can hold at most one asset. Assets are initially randomly distributed among investors. Investors can borrow and lend freely at interest rate $r$ so that the value of their savings stays bounded below, ruling out Ponzi schemes.

Investors can trade assets in the market with the search friction. Search is costless, and all unmatched investors participate in search. Searching investors are randomly matched. The matching process is independent of the evolution of intrinsic types and is given by the matching technology commonly used in the search-and-bargaining literature: Buyers of mass $m_b$ contact sellers of mass $m_s$ with intensity $\lambda m_b m_s$, and so the total meeting rate of these two groups of agents is $\lambda m_b m_s$. The contact intensity $\lambda$ reflects the severity of the search friction.

After the match is found, the investors involved choose whether to participate in the bargaining stage or continue the search by simultaneously saying “yes” or “no” to their current match. If both choose “yes”, the bargaining starts, otherwise, both return to the search market. Note that at this stage no offers are allowed. In particular, we rule out conditional offers, e.g. when the buyer threatens to leave, if her offer is not accepted, or the seller promises to offer a low price, if the buyer agrees to start the negotiation.

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11 Flow payoffs and hence prices can be negative, which can be interpreted as the buyer is compensated for holding the assets on the balance sheet.
12 See e.g. Lagos and Rocheteau (2009) for the OTC model with endogenous choice of asset holdings.
13 Since $a > 1$, not all investors own assets.
14 Duffie and Sun (2007) provide probabilistic foundations for this matching technology.
This assumption can be motivated by the limited commitment of investors before the bargaining starts. We assume that sellers always choose to participate in the bargaining stage, which is essentially without loss of generality.\footnote{This rules out uninteresting equilibria where the buyer rejects the trade simply because she anticipates that the seller will also reject it. We will show that in equilibrium the seller always gets a higher utility from bargaining than from continuing the search.}

The screening bargaining solution that we apply (described below) is the reduced form for the limiting bargaining outcome between investors with noisy private signals about the quality, as the noise goes to zero. This approach is advantageous in two respects. On the one hand, we show in the next section that the presence of an arbitrarily small noise is enough to generate bargaining delays arising from the lack of common knowledge about $\theta$ (rather than asymmetric information). On the other hand, by considering the limit as the noise goes to zero, we abstract from complex learning about the quality. In particular, in the OTC model, investors act as if they observe $\theta$ and can condition their strategies on $\theta$. However, one needs to keep in mind the interpretation that investors observe (and condition on) the almost-perfect private signals about the asset quality, rather than the quality itself.

The mixed strategy of the buyer $\sigma(\theta) \in [0, 1]$ specifies the probability, with which the buyer matched with the seller of asset quality $\theta$ participates in the bargaining stage.\footnote{As is standard, we interpret mixed strategies as investors conditioning the actions on their idiosyncratic preferences over asset characteristics that do not affect payoffs.} After investors proceed to the bargaining stage, they trade an asset $\theta$ with delay $t(\theta)$ at price $p(\theta)$. Investors do not search during the bargaining stage,\footnote{One can think of this restriction as investors cannot make side calls while on the phone negotiating the price with the current match.} and once agents complete the trade or the intrinsic type of one of the matched agents switches (and there are no gains from trade any more), the match is destroyed. Since holding costs do not depend on $\theta$, if we set $t(\cdot) \equiv 0$ and let $p(\cdot)$ proportionally split the surplus, our model reduces to Duffie et al. (2007). The novelty of our approach is that instead of using the generalized Nash bargaining solution, we use the novel screening bargaining solution to pin down functions $p(\cdot)$ and $t(\cdot)$, which generally leads to positive, quality-dependent bargaining delays.

**Screening Bargaining Solution**

We first introduce the SBS for a general class of bargaining problems, and then show how we apply it in our model. Consider the following general bargaining problem described
by the tuple \((\rho, v, c)\). For each asset quality \(\theta \in [0, 1]\), the buyer’s valuation is \(v(\theta)\) and the seller’s cost is \(c(\theta)\). In the OTC model, \(v\) and \(c\) will correspond to endogenous buyer’s gains from buying the asset and seller’s losses from selling the asset, resp. Assume that \(v\) and \(c\) are weakly increasing, almost everywhere continuously differentiable, and \(v(\theta) - c(\theta) = \xi > 0\) for all \(\theta\).\(^{18}\) Time is continuous, and parties discount at rate \(\rho\). If parties trade at time \(t\) at price \(p\), then the payoff to the buyer is \(e^{-\rho t} (v(\theta) - p)\) and the payoff to the seller is \(e^{-\rho t} (p - c(\theta))\). The SBS is defined as follows.

**Definition 1.** The screening bargaining solution \((p(\cdot), t(\cdot))\) to the bargaining problem \((\rho, v, c)\) with the surplus split \(\alpha \in (0, 1)\) satisfies:

1. for all \(\theta \in [0, 1]\),
   \[ p(\theta) = (1 - \alpha)v(\theta) + \alpha c(\theta); \quad (2.1) \]

2. \(t(1) = t(0) = 0\) and for some \(\theta^*\):
   \[ \theta \in \arg\max_{\theta' \in [\theta^*, 1]} \left\{ e^{-\rho t(\theta')} (v(\theta) - p(\theta')) \right\}, \text{ for } \theta \geq \theta^*, \quad (2.2) \]
   \[ \theta \in \arg\max_{\theta' \in [0, \theta^*]} \left\{ e^{-\rho t(\theta')} (p(\theta') - c(\theta)) \right\}, \text{ for } \theta \leq \theta^*. \quad (2.3) \]

Let us intuitively describe how the SBS works out in terms of a related continuous-time bargaining game \(G(p_s, p_b)\). In \(G(p_s, p_b)\), the seller continuously decreases her price offer \(p_s^t\) starting from \(p_s^0\), and the buyer continuously increases his price offer \(p_b^t\) starting from \(p_b^0\) (see Figure 1). Both sides take paths of offers as exogenous (offers are not conditioned on \(\theta\)), but choose times when they accept the opponent’s offer strategically (acceptance times are conditioned on the asset quality \(\theta\)). The trade happens once one of the sides accepts the price offer of the opponent. Initial offers \(p_s^0\) and \(p_b^0\) have a natural interpretation of bid and ask prices, as by accepting such offers parties can guarantee an immediate trade. However, generally parties prefer to wait for a more favorable price offer from the opponent. The continuous-time bargaining game \(G(p_s, p_b)\) is a realistic description of the actual OTC negotiations, where parties often start from extreme price offers and gradually moderate their offers until one of the sides accepts.\(^{19}\) The next section shows

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\(^{18}\)The latter assumption is not necessary for the results in this and the next section, but it simplifies certain steps in proofs and it holds for the endogenous \(v\) and \(c\) in the OTC model (see equations (A.12) and (A.13) in Appendix).

\(^{19}\)E.g., in the residential real estate sellers adjust listing prices and buyers often make offers below the listing price (Merlo and Ortalo-Magne 2004). Lewis (2011) (pp. 212-213) describes the negotiation between Morgan Stanley and Deutsche Bank over the price of subprime CDOs:
Figure 1: Illustration of the SBS. For an asset quality $\theta > \theta^*$, the buyer accepts the seller’s offer $p^s_{t(\theta)} = p(\theta)$ at time $t(\theta)$; for an asset quality $\theta' < \theta^*$, the seller accepts the buyer’s offer $p^b_{t(\theta')} = p(\theta')$ at time $t(\theta')$.

that $\mathcal{G}(p^s, p^b)$ is closely related to the standard alternating-offer bargaining game, in which parties get very precise signals about the quality.

In the pure-strategy Nash equilibrium of $\mathcal{G}(p^s, p^b)$, for any asset quality $\theta$ corresponds the bargaining outcome consisting of the price $p(\theta)$ and the bargaining delay $t(\theta)$. Of course, the outcome would depend on the choice of paths $p^s$ and $p^b$. Fix price offers such that in the equilibrium outcome, the surplus is split proportionally as in the generalized Nash bargaining solution (equation (2.1)). This choice uniquely pins down the bargaining delay, which is characterized by (2.2) and (2.3). For asset qualities above $\theta^*$, the buyer gives in first and accepts the seller’s offer at time $t(\theta)$. Condition (2.2) ensures that for the buyer of quality $\theta > \theta^*$ accepting at time $t(\theta)$ is preferred to accepting any other offer corresponding to a different asset quality. Symmetrically, for asset qualities below $\theta^*$, the seller gives in first and accepts the buyer’s offer at time $t(\theta)$ and condition (2.3) ensures the optimality of the acceptance time $t(\theta)$ (see Figure 1). In other words, the buyers is screened for high qualities, and the seller is screened for low qualities.

We apply the SBS in the OTC model as follows. Call the status quo $(\hat{v}, \hat{c}(\theta))$, the outcome that gives parties the same utility as they receive during bargaining if they

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What do you mean seventy? Our model says they are worth ninety-five, said one of the Morgan Stanley people on the phone call.

Our model says they are worth seventy, replied one of the Deutsche Bank people.

Well, our model says they are worth ninety-five, repeated the Morgan Stanley person, and then went on about how the correlation among the thousands of triple-B-rated bonds in his CDOs was very low, ... he didn’t want to take a loss, and insisted that his triple-A CDOs were still worth 95 cents on the dollar.

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$^{20}$The single-crossing property of payoffs implies that the Nash equilibrium of $\mathcal{G}(p^s, p^b)$ is unique.
never agree on price. In Section 4, we introduce value functions $V_{bu}(\theta)$ and $V_{su}(\phi)$ of the unmatched buyer with asset of quality $\theta$ and the unmatched seller without asset, resp. Then the gain from trade for the buyer $v(\theta)$ and the loss from trade for the seller $c(\theta)$ are given by

$$c(\theta) = \hat{c}(\theta) - V_{su}(\phi), \quad (2.4)$$

$$v(\theta) = V_{bu}(\theta) - \hat{v}. \quad (2.5)$$

Define by $\rho = r + y_u + y_d$ the discount rate adjusted for the fact that the match can be destroyed if the intrinsic types switch, which happens with the Poisson intensity $y_u + y_d$. To determine the price of trade and bargaining delay, we apply the SBS to the bargaining problem $(\rho, v, c)$ to determine the price of trade $p(\theta)$ and bargaining delay $t(\theta)$ for every quality $\theta$. We restrict attention to equilibria of the OTC model with weakly increasing functions $v$ and $c$, which is a natural assumption given the monotonicity of flow payoffs. We also assume that equilibrium functions $v$ and $c$ are continuous on $\Theta_L \cup \Theta_M$, which rules out self-sustained illiquidity. If some asset is expected to trade with a significant bargaining delay, this would lead to a discontinuity in $v$ and $c$ at $\theta$, which could in turn justify the long bargaining delay. While interesting, this mechanism is not the focus of this paper and we prefer to abstract from it (see e.g. Guerrieri and Shimer 2014 for the relevant analysis).

3 Microfoundation

This section provides the game-theoretic foundation for the screening bargaining solution (SBS). The generalized Nash bargaining solution commonly used in the literature is derived from the static axiomatic approach (Nash 1950, Roth 1979, Binmore 1987). It predicts the proportional split of the surplus, which we refer to as the Nash split, but is silent about the delay required to reach this split. Rubinstein (1982) and Binmore et al. (1986) take the non-cooperative approach to show that the Nash split is attained without delay, when the information about values is public and offers are frequent. We relate the SBS outcome to the outcome of bargaining when the information about values is almost public and offers are frequent, and show that in such a model, the bargaining delay is

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21 The way we define $v, c, \rho$ is the same as in the application of the generalized Nash bargaining solution in the existing search-and-bargaining models of OTC markets (see e.g. Duffie et al. 2007).

22 Readers more interested in implications of bargaining delays for the OTC liquidity may skip it on first reading.
necessary to attain the Nash split.

Consider the following discrete-time bargaining game \( G(F, \Delta) \). The seller’s type \( \theta^s \) and the buyer’s type \( \theta^b \) are jointly distributed on \([0, 1]^2\) according to the CDF \( F \) with strictly positive, continuously differentiable density \( f \). Types are affiliated, i.e. \( f \) is log-supermodular. We can think of types as noisy private signals about the underlying asset quality \( \theta \). The affiliation of signals captures the correlation of signals with the underlying asset quality: a buyer is more likely to receive a high signal \( \theta^b \) when the asset quality \( \theta \) is high, and thus, the seller’s signal \( \theta^s \) is likely to be high as well. Such a signal structure is similar to that used in the global games literature (see e.g. Morris and Shin 1998).

An example of \( F \) is the joint distribution of \( \theta^s = \theta + \varepsilon^s \) and \( \theta^b = \theta + \varepsilon^b \) conditional on \((\theta^s, \theta^b) \in [0, 1]^2\) where \( \theta, \varepsilon^s, \varepsilon^b \) are independent normal random variables with zero means and variances \( \sigma^s, \sigma^b \) and \( \sigma^s \), resp.

This information structure allows us to capture the coarseness of the public information about the asset quality in OTC markets. Even if \( F \) puts most of probability mass on the diagonal and so, private signals are very precise, it is only common knowledge among players that \((\theta^s, \theta^b)\) are drawn from \( F \) (what they knew before receiving signals), which provides only coarse information about \( \theta^s \) and \( \theta^b \). Credit ratings put only crude bounds on the risks associated with financial assets, while experienced traders can significantly refine these bounds, or in the real estate, experienced realtors go beyond public assessments of the property to determine more precisely its value based on demographic trends and developments in the area.

Values are private and \( v(\theta^b) \) is the buyer’s value and \( c(\theta^s) \) is the seller’s cost. Since values depend only on signals, but not directly on \( \theta \), the quality \( \theta \) only serves for the purpose of interpretation, and we will further identify the asset quality with the seller’s signal \( \theta^s \). We assume that functions \( v \) and \( c \) are strictly increasing, continuously differentiable, and \( v(x) - c(x) = \xi \) for all \( x \in [0, 1] \).\(^{23}\) Both sides discount time at a constant rate \( \rho \). Thus, if trade happens at time \( t \) at price \( p \), the buyer’s utility is \( e^{-\rho t}(v(\theta^b) - p) \) and the seller’s utility is \( e^{-\rho t}(p - c(\theta^s)) \). Bargaining happens in discrete rounds of length \( \Delta \). In the beginning of the round, the seller makes a price offer or accepts the last price offer of the buyer. After delay \( \Delta_b \in (0, \Delta) \), the buyer either accepts the last price offer of the seller or makes a counter-offer. After that, time \( \Delta_s = \Delta - \Delta_b \) elapses and the new

\(^{23}\)In the equilibrium of our OTC model, functions \( v \) and \( c \) are only guaranteed to be weakly increasing and they may have discontinuous jumps. Such functions can be approximated in \( L^1 \) norm by strictly increasing, continuously differentiable functions. Thus, when \( v \) and \( c \) are constant on some intervals or have discontinuities, we provide the microfoundation for the SBS in the sense that the SBS describes the bargaining outcome for arbitrarily close specifications of \( v \) and \( c \).
round starts. The ratio \( \frac{\Delta b}{\Delta b + \Delta s} = \alpha \) captures the bargaining strength of the buyer. The game stops when one of the parties accepts the price offer of the opponent with the trade occurring at the accepted price. Note that as \( \Delta \to 0 \), parties are able to make offers and respond almost continuously.

The solution concept is Perfect Bayesian equilibrium (PBE). We focus on PBEs in strategies that have the following simple interval form: after any history, the set of types that pool with each other and make the same counter-offer (or accept) is an interval.\(^{24}\) We additionally introduce the following refinement. Call a party informed after a history if its posterior beliefs assign probability 1 to a single type of opponent. We require that the support of players’ posterior beliefs about the opponent’s type cannot expand over time, unless there is an informed party, in which case the beliefs of only the informed party are not allowed to expand. This refinement is a weaker version of the requirement that the support of beliefs does not expand (the support restriction), which is standard in the bargaining literature (see Cramton 1984, Rubinstein 1982, Bikhchandani 1992).\(^{25}\)

The SBS is intended to capture bargaining when parties make offers almost continuously and signals about the quality are very precise. We next formalize this idea. The bargaining outcome consists of a pair of functions \((\tilde{t}, \tilde{p})\) where \(\tilde{t}(\theta^s, \theta^b)\) and \(\tilde{p}(\theta^s, \theta^b)\) are the time and price, resp., at which types \(\theta^s\) and \(\theta^b\) trade. A PBE outcome is the outcome generated by PBE strategies. For a fixed distribution of types \(F\), consider a sequence of PBE outcomes \((t_{F,\Delta}, p_{F,\Delta})\) indexed by \(\Delta \to 0\), and say that \((t_{F,\Delta}, p_{F,\Delta}) \stackrel{p}{\to} (t_F, p_F)\).\(^{26}\)

Let \(F^*\) be the uniform distribution on the diagonal \(\theta^s = \theta^b\). Under \(F^*, \theta^s = \theta^b\) and the asset quality \(\theta^s\) is public information, or formally, common knowledge. Consider a sequence of distributions \(F \stackrel{p}{\to} F^*\) such that for any \(\varepsilon > 0\), \(\sup_{(\theta^s, \theta^b); |\theta^s - \theta^b| > \varepsilon} \max \{f(\theta^b | \theta^s), f(\theta^s | \theta^b)\} < \)

\(^{24}\)This requirement is stronger than pure strategies, as it rules out strategies in which two types pool with each other, but separate from some types in between. However, it still allows for rich signaling possibilities. In the literature on bargaining with one-sided private information and two-sided offers (e.g. Gul and Sonnenschein 1988, Ausubel and Deneckere 1989, Grossman and Perry 1986), the cheap-talk messages that are not accepted, but reveal information are normally ruled out by assumption. Our restriction to interval strategies allows for such cheap-talk messages.

\(^{25}\)Existing PBE constructions in bargaining games with one-sided uncertainty and two-sided offers, however, do not always satisfy this requirement (e.g. Grossman and Perry 1986). Thus, in order to guarantee the existence of PBEs satisfying our refinement, we slightly weaken the support restriction for the case when one party fully revealed its type. The first part of Theorem 1 below holds without this modification of the support restriction, and we only use it in the proof of the second part (see footnote 42 in the proof of Lemma 20).

\(^{26}\)Here and further, \(\overset{p}{\to}\) denotes convergence in probability, e.g. \((t_{F,\Delta}, p_{F,\Delta}) \overset{p}{\to} (t_F, p_F)\) as \(\Delta \to 0\) if for all \(\varepsilon > 0\), \(\lim_{\Delta \to 0} P_F (|t_{F,\Delta} - t_F| < \varepsilon\) and \(|p_{F,\Delta} - p_F| < \varepsilon) = 1\).
ε for all F sufficiently far in the sequence. As $F \rightarrow F^*$, the quality $\theta^*$ is almost public information, or formally, almost common knowledge.

The main result of this section is that bargaining outcomes are quite different when $\theta$ is public information ($F = F^*$), and when it is almost public information ($F \approx F^*$). Let us start with the former. Denote the price of the Nash split by $p(\theta^*, \theta^b) = (1 - \alpha)v(\theta^b) + \alpha c(\theta^s)$. Binmore et al. (1986) show that the frequent-offer limit $(t_{F^*}, p_{F^*})$ of the sequence of PBEs when values are common knowledge is given by $(t_{F^*}, p_{F^*}) = (0, p)$, i.e. trade is immediate and the split of surplus is proportional. Consider now the case of almost public information.

**Theorem 1.**

1. Consider a sequence of PBE continuous-time limits $(t_F, p_F)$ indexed by $F \rightarrow F^*$. If $p_F \rightarrow p$ as $F \rightarrow F^*$, then there exist $0 < x_l < x_h < 1$ and $0 < \bar{\theta}^s < \bar{\theta}^s < 1$ such that

$$x_l > \limsup_{F \rightarrow F^*} E_F[e^{-\rho t_F}]$$

$$x_h > \liminf_{F \rightarrow F^*} E_F[e^{-\rho t_F} | \theta^s < \bar{\theta}^s \text{ or } \theta^s > \bar{\theta}^s].$$

2. Suppose $c(1) - c(0) < \min\{\frac{1-\alpha}{\alpha}, \frac{\alpha}{1-\alpha}\} \xi$ and denote by $t(\theta^s)$ the delay associated with quality $\theta^s$ in the SBS as given by (2.2) – (2.3). Then there exists a sequence of PBE continuous-time limits $(t_F, p_F)$ such that for all $\varepsilon > 0$,

$$\lim_{F \rightarrow F^*} P_F(|t_F - t| < \varepsilon \text{ and } |p_F - p| < \varepsilon) = 1.$$  

Theorem 1 shows that the bargaining outcome when the asset quality is almost common knowledge drastically differs from when the quality is common knowledge. First, the bargaining delay is necessary to attain the Nash split of the surplus (inequality (3.1)). Second, the bargaining delay is generally non-monotone: it is lower for qualities closer to extremes of the distribution (0 and 1) and higher in the middle (inequality (3.2)). Finally, when the surplus from trade is sufficiently large, the SBS can be approximated by PBE outcomes as offers become frequent and the information about quality becomes almost common knowledge. As we will show in the next section, the SBS outcome captures positive and non-monotonic bargaining delays that is characteristic for all limits approximating the Nash split.

27 See Online Appendix A for an example of such a sequence.
Let us provide the intuition for Theorem 1. First, why the delay is necessary to attain the Nash split? Although the formal proof is quite involved, the underlying idea is simple. Suppose to contradiction, for any $\Delta$ and $F$ arbitrarily close to 0 and $F^*$, resp., there were a PBE in which trade happens with a high probability without a significant delay. Since sufficiently different asset qualities should be traded at sufficiently different prices (to match the Nash split), it is necessary that at least one side, say the buyer, reveals quite precisely its type. Then the buyer can relatively quickly convince the seller that its value is relatively low. But this implies that high types of the buyer can mimic lower types and get a more favorable price by only slightly delaying the trade, which is a contradiction to the sequential rationality.

The non-monotonicity of the bargaining delay is also quite intuitive. We show that in equilibrium, the buyer has the option to trade immediately at price close to $p(1)$, which is the complete-information price of trade when both players’ types equal 1. Since $p_F \rightarrow p$, the buyer’s types close to 1 expect to trade with a high probability at a price close to $p(1)$, thus, for them the expected bargaining delay cannot be too long. Symmetric argument shows that for the seller’s types close to 0, the expected bargaining delay is relatively low, as they have the option to trade immediately at a price close to $p(0)$ (which is the complete-information price of trade when both players’ types equal 0). Therefore, types close to the extremes of the range are guaranteed to trade relatively quickly, which gives inequality (3.2).

Now, let us turn to why the SBS can be approximated by the PBEs for $F \approx F^*$, but not when $F = F^*$? When the information about the quality is public, there is a unique split of the surplus sustainable in any continuation equilibrium and so, it is not possible to reward or punish players to sustain the delay. This is, however, possible when the information is noisy. We construct PBEs approximating the SBS in grim trigger strategies. We specify that if e.g. the seller deviates from the equilibrium path, then the buyer infers that the seller’s signal is very low and the seller is very desperate to trade (formally, the buyer believes that the seller’s type is the lowest remaining type). After such an optimistic updating, in the continuation equilibrium the buyer almost immediately

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28 In fact, one can generalize Theorem 1 to show that the delay is necessary for any sequence of frequent-offer PBE limits, in which in the limit as $F \rightarrow F^*$, $p_F \rightarrow p$ where $p(\theta^s, \theta^b)$ is some strictly increasing function of both arguments.

29 Here, the assumption that the support cannot expand is crucial: once the buyer signals that his value is relatively low, this gives him a guarantee of relatively low price in any continuation play. Lemma 12 in Online Appendix A shows how bounds on the price of trade depend on the support of types remaining in the game.

30 See Lemma 12 in Online Appendix A.
gets the maximal share of the surplus. By specifying such a punishment path, we can sustain the equilibrium path that involves delay. Despite the fact there is an efficiency loss due to the bargaining delay on the equilibrium path and both parties assign a high probability to it, nobody wants to seem desperate and deviate from the equilibrium path.

Finally, we motivate our focus on PBEs of the bargaining game approximating the SBS. In our bargaining model, generally there is a variety of other PBEs that do not approximate the Nash split. E.g. whenever \( c(1) < v(0) \) along with PBEs described in Theorem 1, there is a continuum of PBEs, in which trade happens immediately at some price in \( (c(1), v(0)) \). We focus on PBEs that approximate the Nash split, as it, first, allows us to contrast our results to that in the existing OTC literature. Second, Nash (1950) provides axiomatic foundations for the Nash split of the surplus, and it is reasonable to suppose that such a split would be a natural focal point for the surplus split, when players have very precise information about the quality.\(^{31}\)

4 Equilibrium

This section characterizes the equilibrium of the OTC model.

Steady-State Equilibrium We first describe the distribution of asset holdings among investors and define the steady-state equilibrium. At every time, each investor can be either matched \((m)\) or unmatched \((u)\). We refer to the intrinsic type of the investor and his match status as the type \( \tau \in \{bu, su, bm, sm\} \). The asset position of the investor \([0, 1]\cup\{\phi\}\) is the quality of the asset that the investor owns or bargains over \((\phi\) denotes investors who neither own nor bargain over an asset\). The distribution of assets among different types of investors at time \( t \) is given by \( M_t = \{M_{\tau,t} \in \Delta([0, 1]), \tau \in \{bm, bu, sm, su\}\} \) where for any measurable set \( \Theta \subset [0, 1] \), \( M_{\tau,t}(\Theta) \) gives the mass of type \( \tau \) investors with asset positions in \( \Theta \). We suppose that there is the mass density function \( \mu_{\tau,t} \) of \( M_{\tau,t} \) such that \( M_{\tau,t}(\Theta) = \int_\Theta \mu_{\tau,t}(\theta) d\theta \). \( M_t \) satisfies following balance conditions:

1. for any \( \theta \), the mass of investors with the asset position \( \theta \) is equal to the (unit) supply of the asset of quality \( \theta \),

\[
\mu_{su,t}(\theta) + \mu_{bu,t}(\theta) + \mu_{bm,t}(\theta) = 1; \tag{4.1}
\]

\(^{31}\) Tsay (2016) shows that the SBS outcome is also robust to the assumption about the distribution of types.
2. the mass of investors with the asset position $\phi$ is equal to $a - 1$,

$$M_{su,t}(\phi) + M_{bu,t}(\phi) + M_{bm,t}([0, 1]) = a - 1; \quad (4.2)$$

3. for any $\theta$, the masses of matched buyers and matched sellers coincide,

$$\mu_{sm,t}(\theta) = \mu_{bm,t}(\theta). \quad (4.3)$$

We focus on steady states of the economy in which $\frac{\partial}{\partial t} M_{\tau,t}(\Theta) = 0$ for all $\Theta$ and $\tau$, and in what follows, we omit from the notation the subscript $t$.

**Definition 2.** A tuple $(\sigma(\cdot), M)$ constitutes an equilibrium if the buyer’s strategy $\sigma(\cdot)$ is optimal given $M$, and $M$ is the steady-state distribution of assets generated by $\sigma(\cdot)$.

**Equilibrium Characterization** We now characterize the equilibrium of the OTC model. We first introduce several quantities central in the characterization. Denote $\Theta_L = \{\theta: \sigma(\theta) = 1\}$, $\Theta_I = \{\theta: \sigma(\theta) = 0\}$, and $\Theta_M = \{\theta: \sigma(\theta) \in (0, 1)\}$. Call assets in $\Theta_L$ unconditionally liquid or simply liquid, assets in $\Theta_M$ conditionally liquid, and assets in $\Theta_I$ illiquid. Let $\Lambda_s = \lambda M_{bu}(\phi)$ be the Poisson intensity of contact with unmatched buyers without assets, and $\Lambda_b = \lambda M_{su}(\Theta_L)$ be the Poisson intensity of contact with sellers of liquid assets. Both are measures of market thickness and capture how easily each side of the market can find a trade partner. We will show that two are closely related, and by convention, we will only refer to $\Lambda_s$ as the market thickness. Let $F_L \in \Delta(\Theta_L)$ be the steady-state probability distribution of qualities in the pool of unmatched sellers of liquid assets and $L = |\Theta_L|$ be the mass of liquid assets. Let $x(\theta) = e^{-\rho t(\theta)}$ be the factor by which the surplus from trade of the asset $\theta$ is dissipated due to the bargaining delay. We refer to $x(\theta)$ as the liquidity index of asset $\theta$.

Let us outline the steps of the analysis. First, in Lemma 1 we derive for fixed $x(\cdot)$ and $\sigma(\cdot)$ the steady-state distribution $M$. Then for fixed $x(\cdot)$ and $M$ we derive investors’ value functions and optimal strategy $\sigma(\cdot)$. Finally, in Lemma 3, given investors’ value functions we derive the liquidity profile $x(\cdot)$. Generally, finding the equilibrium would require solving a system of functional equations in $(M, \sigma(\cdot), x(\cdot))$. However, we show that the equilibrium analysis can be reduced to finding the market thickness $\Lambda_s$ and the mass of liquid assets $L$, which pin down $M$ and $\sigma(\cdot)$.

**Step 1:** We first characterize the unique steady-state distribution $M$. 

19
Lemma 1. For any $\sigma(\cdot)$ and $x(\cdot)$, there exists a unique steady-state distribution $M$ characterized by $(\sigma(\cdot), x(\cdot), \Lambda_s)$, and

- $F_L$ is uniform on $\Theta_L$;
- $\Lambda_s$ is the unique solution to
  \[
  \frac{\Lambda_s}{\lambda} = \frac{y_u(a - 1)}{y_u + y_d} - \frac{y_d}{y_u + y_d} \int_0^1 \frac{\Lambda_s \sigma(\theta)}{y_u + y_d + \Lambda_s \sigma(\theta)} d\theta; \tag{4.4}
  \]
- $\Lambda_b$ is given by
  \[
  \Lambda_b = \frac{\lambda y_d L}{y_u + y_d + \Lambda_s}. \tag{4.5}
  \]

Moreover, a pointwise weak increase in $\sigma(\cdot)$ leads to a decrease in $\Lambda_s$ and an increase in $\Lambda_b$.

The steady-state distribution of assets is pinned down by the market thickness $\Lambda_s$. Moreover, $\Lambda_s, \Lambda_b$, and $F_L$ depend only on $\sigma(\cdot)$, but not on $x(\cdot)$.$^{32}$ The fact that $F_L$ coincides with the distribution of assets’ supply conditional on $\theta \in \Theta_L$ greatly simplifies equilibrium analysis and may be surprising at first sight. Indeed, one may expect that assets with shorter bargaining delays are more abundant in the market and $F_L$ should reflect this. The key observation is that the length of the bargaining delay does not affect the likelihood of forming or dissolving the match involving qualities in $\Theta_L$, but only the distribution of investors between those who have already completed the trade and those still bargaining.

To interpret equation (4.4) suppose $|\Theta_M| \approx 0$. Then

\[
M_{bu}(\phi) \approx \frac{y_u(a - 1)}{y_u + y_d} - \lambda L \frac{y_d}{y_u + y_d} M_{bu}(\phi) \frac{1}{y_u + y_d + \Lambda_s}, \tag{4.6}
\]

that is, the mass of unmatched buyers without assets ($M_{bu}(\phi)$) equals the total mass of buyers without assets ($\frac{y_u(a - 1)}{y_u + y_d}$) minus the term proportional to the number of matches between sellers of liquid assets (of mass $L \frac{y_d}{y_u + y_d}$) and unmatched buyers without assets (of mass $M_{bu}(\phi)$) (the adjustment term accounts for the fact that some of the sellers are already matched). Notice that the fact that buyers reject some asset qualities in equilibrium results in the endogenous length of search delays. In equation (4.6), the

$^{32}$Unlike $\Lambda_s, \Lambda_b, F_L$, the steady-state distribution $M$ derived explicitly in the Appendix depends on the profile $x(\cdot)$.
contact intensity is effectively $\lambda L$ adjusted for the fact that only a fraction $L$ of assets is accepted for trade by buyers.

Note that Lemma 1 holds for any $\sigma(\cdot)$ and $x(\cdot)$, not necessarily those analysed in this paper, and can be used to study other forms of post-match trade delay, such as the time for parties to familiarize themselves with the asset, time to execute the trade, etc.

**Step 2:** Now, given the steady-state distribution $M$ and $x(\cdot)$, we determine investors’ value functions and equilibrium $\sigma(\cdot)$. For $\tau \in \{bu, su, bm, sm\}$, let $V_\tau(\theta)$ be the expected utility of an investor of type $\tau$ owning (or bargaining over) asset $\theta$, and for $\tau \in \{bu, su\}$, let $V_\tau(\phi)$ be the expected utility of an investor of type $\tau$ owning no asset. Value functions during the search stage are determined by the following Bellman equations,

$$
\begin{align*}
 rV_{su}(\phi) &= y_u(V_{bu}(\phi) - V_{su}(\phi)), \\
 rV_{bu}(\theta) &= d + k\theta + y_d(V_{su}(\theta) - V_{bu}(\theta)), \\
 rV_{bu}(\phi) &= y_d(V_{su}(\phi) - V_{bu}(\phi)) + \Lambda_b (\mathbb{E}[V_{bm}(\theta)|\theta \in \Theta_L] - V_{bu}(\phi)), \\
 rV_{su}(\theta) &= d + k\theta - 1 + y_u(V_{bu}(\theta) - V_{su}(\theta)) + \sigma(\theta)\Lambda_s (V_{sm}(\theta) - V_{su}(\theta)).
\end{align*}
$$

The depreciation of value functions in the left-hand side of equations (4.7) – (4.10) equals the sum of flow payoffs and changes in value functions due to either switches of intrinsic types or the formation of matches. For example, in equation (4.9), the flow payoff of the searching buyer without an asset is zero. If the buyer is hit by a liquidity shock, his value function drops to $V_{su}(\phi)$, while if he is matched to a seller, then his value function increases to $\mathbb{E}[V_{bm}(\theta)|\theta \in \Theta_L]$. Notice that if a buyer is matched to a seller of an asset in $\Theta_M$, then his continuation utility is $V_{bu}(\phi)$ irrespective of whether he starts to negotiate or continues to search. Therefore, in equation (4.9), it is sufficient to consider only assets in $\Theta_L$. This is, however, not the case in equation (4.10), which describes the value function of the unmatched seller of asset quality $\theta$. In equilibrium, such a seller strictly prefers to start the negotiation, and hence, the probability with which her asset is accepted is important for her.

To determine $V_{bm}(\theta)$ and $V_{sm}(\cdot)$, let $V_{bm}(t, \theta)$ and $V_{sm}(t, \theta)$ be the value functions of the matched seller and buyer, resp., when time $t$ has passed since the start of bargaining.
Functions $V_{bm}(t, \theta)$ and $V_{sm}(t, \theta)$ satisfy the Bellman equations

\[ rV_{bm}(t, \theta) = y_u(V_{bu}(\phi) - V_{bm}(t, \theta)) + y_d(V_{su}(\phi) - V_{bm}(t, \theta)) + \frac{\partial}{\partial t} V_{bm}(t, \theta), \tag{4.11} \]

\[ rV_{sm}(t, \theta) = d + k\theta - 1 + y_u(V_{bu}(\theta) - V_{sm}(t, \theta)) + y_d(V_{su}(\theta) - V_{sm}(t, \theta)) + \frac{\partial}{\partial t} V_{sm}(t, \theta). \tag{4.12} \]

Equation (4.11) (analogously equation (4.12)) reflects that with intensity $y_u$ and $y_d$ the intrinsic type of one side switches and the value function becomes $V_{bu}(\phi)$ and $V_{su}(\phi)$, resp., and the value function increases over time as the time of trade $t(\theta)$ approaches. After parties have spent time $t(\theta)$ in the bargaining stage, they trade and get utilities $V_{bm}(t(\theta), \theta) = V_{bu}(\theta) - p(\theta)$ and $V_{sm}(t(\theta), \theta) = p(\theta) + V_{su}(\phi)$, which are the terminal conditions for (4.11) and (4.12), resp. With $c(\theta)$ and $v(\theta)$ defined in (2.4) and (2.5), we plug $p(\theta) = (1 - \alpha)v(\theta) + \alpha c(\theta)$ into terminal conditions. Finally, we combine (4.7)–(4.10) and (4.11)–(4.12) to express value functions through $x(\theta)$ and $\sigma(\theta)$. This allows us to determine optimal strategies described.

**Lemma 2.** The asset of quality $\theta$ is always accepted by buyers ($\theta \in \Theta_L$), whenever

\[ x(\theta) > \bar{x} \equiv \frac{\Lambda_b}{\rho + \Lambda_b} \left( \frac{1}{L} \int_{\Theta_L} x(\theta) d\theta \right), \tag{4.13} \]

and it is always rejected ($\theta \in \Theta_I$) whenever the inequality in (4.13) is reversed. Moreover, $\xi \equiv \frac{1}{\rho} = v(\theta) - c(\theta)$ and

\[ V_{bu}(\phi) = \alpha r + y_u \alpha x \bar{x}. \tag{4.14} \]

From Lemma 2, buyers search for sufficiently liquid assets in the market. Thus, the search is non-trivial: the buyer may reject several assets before he finds a sufficiently liquid asset, for which he proceeds to the bargaining stage. The threshold of the buyer depends on the average liquidity and the ability to find liquid assets. If the search is fast ($\Lambda_b$ is large), then this threshold is close to the average liquidity, i.e. the outside option of the buyer is essentially to go back to the market and get a random draw from the pool $\Theta_L$. If the search is slow ($\Lambda_b$ is small), then the buyer accepts a wide range of assets, as finding another asset entails a significant delay.

Note that buyers trade-off search and bargaining delays and by varying their strategy $\sigma(\cdot)$ they effectively control the length of their search delays. When buyers accept a smaller range of assets for trade, their search delays increase as they reject more assets and it takes longer time to find “sufficiently” liquid asset.
Step 3: Finally, we use the definition of the SBS to determine the liquidity profile $x(\cdot)$ for given $M$ and $\sigma(\cdot)$.

Lemma 3. Either $\Theta_L = [0, 1]$ or there exist $0 < \tilde{\theta} < \underline{\theta} \leq \theta^* \leq \hat{\theta} < 1$ such that $\Theta_L = [0, \tilde{\theta}] \cup [\hat{\theta}, 1]$ and $\Theta_M = (\tilde{\theta}, \theta^*)$. Moreover,

$$
x(\theta) = \begin{cases} 
\exp \left( \frac{v(1) - v(\theta)}{\alpha \xi} \right), & \text{for } \theta > \hat{\theta}, \\
\exp \left( -\frac{c(\theta) - c(0)}{(1 - \alpha) \xi} \right), & \text{for } \theta \leq \tilde{\theta}.
\end{cases}
$$

The next theorem combines equilibrium conditions for $M, \sigma, x$ derived above to characterize the equilibrium and provide conditions for existence and uniqueness.

Theorem 2. The equilibrium of the OTC model is characterized by the unique solution $(\Lambda_s, L)$ to

$$
\begin{cases}
\Lambda_s \geq \frac{\lambda y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\rho} L - 1} \right) - L \right) - y_u - y_d, & \text{with equality iff } L < 1 \quad (4.16) \\
\frac{\Lambda_s}{\lambda} = \frac{y_u(a - 1)}{y_u + y_d} - \frac{y_d}{y_u + y_d} I(L, \Lambda_s),
\end{cases}
$$

where

$$
I(L, \Lambda_s) = \frac{\Lambda_s L}{y_u + y_d + \Lambda_s} + (1 - \alpha) \frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k}{\rho} L} \int_0^{\min \left\{ \frac{\rho \Lambda_s}{\lambda} \frac{(1 - \alpha) \rho}{y_u + y_d} e^{\frac{k}{\rho} L}, 1 \right\}} \frac{1 - s}{1 + \frac{y_u + y_d}{\Lambda_s} - \frac{r}{\rho} s} ds.
$$

The equilibrium exists whenever $\alpha \geq \frac{y_d}{r + y_d}$ and is unique if in addition $1 - \alpha < \frac{r^2}{y_u(r + y_u + y_d)}$.

Moreover, the equilibrium distribution of asset holdings $M$, liquidity profile $x(\cdot)$ and strategy $\sigma(\cdot)$ do not depend on the level of flow payoffs $d$.

The equilibrium is pinned down by the market liquidity $L$ and the market thickness $\Lambda_s$. Equation (4.16) describes the optimal buyers’ strategy (derived in Lemma 2) and it gives an increasing relationship between $\Lambda_s$ and $L$. When the market thickness $\Lambda_s$ is higher, it becomes harder for buyers to find trade partners (see equation (4.5)) and so, they optimally expand the range of acceptable assets. Equation (4.17) describes the steady-state market thickness (derived in Lemma 1) and it gives a decreasing relationship between $\Lambda_s$ and $L$. When buyers accept more assets for trade, fewer buyers are searching in the market, as more of them have already traded or are in the process of negotiation,
which leads to lower market thickness $\Lambda_s$. When the seller’s share is not too high $(1 - \alpha < \frac{x^2}{y_u(r + y_u + y_d)})$, both relations are strictly monotone proving the uniqueness of the equilibrium.

Observe that the level of flow payoffs $d$ does not affect the liquidity. Thus, in our liquidity analysis, we interpret an increase in $k$ as capturing the increase in asset heterogeneity and will abstract from the fact that such an increase also changes the level of asset payoffs.\(^{33}\)

5 Bargaining Delays and Liquidity

This section derives implications of bargaining delays for asset liquidity and prices.

Liquidity Measures  We focus on the following liquidity measures. First, in the bargaining game $G(p^s, p^b)$ describing the screening bargaining solution, each side can immediately accept the first offer of the opponent and so, offers $p^b_0$ and $p^s_0$ are essentially the bid and ask prices, resp., and their difference $ba = p^s_0 - p^b_0$ is the bid-ask spread. Second, we look at how liquidity varies across assets. The extensive margin is captured by $\sigma(\theta)$, the probability that the buyer accepts the asset. The intensive margin is captured by the liquidity index $x(\theta) = e^{-\rho t(\theta)}$ and the instantaneous trade volume which equals\(^{34}\)

$$w(\theta) = \frac{\Lambda_s \sigma(\theta) y_d}{y_u + y_d + \Lambda_s \sigma(\theta)} x(\theta) \frac{y_u + y_d}{\rho}.$$  \hspace{1cm} (5.1)

Since each asset is in the unit supply, $w$ is also the asset turnover.

Third, we study marketwide liquidity by looking at the mass of liquid assets $L = |\Theta_L|$, and the average and aggregate liquidity index of liquid assets $\bar{x} = \frac{1}{L} \int_{\theta \in \Theta_L} x(\theta) dF_L(\theta)$ and $X = \int_{\theta \in \Theta_L} x(\theta) d\theta$, resp. The former captures the marketwide extensive margin, while the latter two capture the marketwide intensive margin. We also look at $W = \frac{\Lambda_s y_d}{y_u + y_d + \Lambda_s} X \frac{y_u + y_d}{\rho}$ as a proxy for the market trade volume.\(^{35}\) In the analysis, we focus on $\Theta_L$ rather than $\Theta_L \cup \Theta_M$ as it allows for a cleaner analytic results, and as our simulations indicate,\(^{36}\) the difference between the two is often small.

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\(^{33}\)In particular, an increase in $k$ can be accompanied by a decrease in $d$ so that the average level of asset payoffs is the same (or even drops), and this would not affect our conclusions about the effect of $k$ on liquidity.

\(^{34}\)See equation (A.3) and the characterization of $M$ in the Appendix.

\(^{35}\) $W$ approximates the aggregate trade volume when $r$ is small relative to $y_u + y_d$ and the set of conditionally liquid assets $\Theta_M$ is small.

\(^{36}\)See Figures 2b and 2c below.
Asset Prices and Bid-Ask Spread  We next proposition describes the equilibrium bid-ask spread and the asset price decomposition.

**Proposition 1.** The bid-ask spread equals $\frac{k}{r}$ and prices of assets are given by

$$p(\theta) = \frac{1}{x}(d + k\theta - (r + y_d)\xi) + (1 - \alpha)\xi + (1 - \alpha)\frac{y_d}{r \rho + \sigma(\theta)\Lambda_s} \xi x(\theta) - \alpha \frac{y_u}{r \rho + \Lambda_b} \xi x(\theta).$$

Equation (5.2) provides an intuitive asset price decomposition. The first component is the price, if neither side had an option to search. The other two components reflect the outside options created by the search market. The liquidity premium depends on the costs of bargaining delay $x(\theta)$, and it reflects the seller’s outside option. For a more liquid asset, less surplus is dissipated due to bargaining delay, which increases the value of the seller’s outside option, and hence, the price. This outside option is more valuable when the seller can more easily locate buyers in the market (higher $\Lambda_s$). This dependence of the sensitivity of the price to liquidity on aggregate market conditions has been documented empirically (Bao et al. 2011, Friewald et al. 2012). The third component depends on the average liquidity $\bar{x}$ of liquid assets (in $\Theta_L$) and it reflects the buyer’s outside option. When the buyer expects that his next trading partner will have more liquid asset, the value of the buyer’s outside option increases, which in turn decreases the price. This effect is larger the easier it is for the buyer to find a trade partner (higher $\Lambda_b$). Longstaff et al. (2005) show that the non-default component of corporate spreads varies with liquidity measures in the cross-section of assets, but also depends on the marketwide liquidity in the time series analysis, which is in line with our decomposition.

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37 We abstract from market makers to focus on the novel feature of the model, bargaining delays. In the recent years, increased capital requirements diminished the role of market makers in markets where they traditionally were central, e.g. bonds market, and big banks nowadays prefer to act as brokers matching buyers and sellers rather than provide immediacy to their clients.
U-shaped Liquidity We next describe the dependence of the liquidity on the asset quality.

**Proposition 2.** The liquidity index $x(\cdot)$ and trade volume $w(\cdot)$ are U-shaped in quality $\theta$ (or the price $p(\theta)$). Moreover, $\Theta_I$ is an interval in the interior of $[0, 1]$.

The U-shaped asset liquidity immediately follows from the bargaining dynamics in $G(p^s, p^b)$. For lower qualities, the seller’s value of the asset is low and so, she prefers to accept a larger discount earlier, rather than wait longer for more favorable offers. Symmetrically, the buyer of higher qualities has a high value, and so he prefers to accept a high price early on, rather than wait for the lower offers. Qualities in the middle, however, trade with the longest delay, as for them, investors prefer to wait until either the opponent gives in or reduces his/her demand (see Figure 1). Such qualities are less liquid, and in fact, may not be traded at all. For qualities in the middle that are rejected, the buyer prefers to continue the search rather than start the lengthy negotiation over the price. We stress that the U-shaped liquidity holds conditional on the public information about assets. E.g., it does not state that AAA assets and unrated assets are the most liquid, but rather that within the AAA asset class, the liquidity of assets of the highest quality and assets that barely pass the AAA threshold will be the highest.

The U-shaped prediction is a novel empirical implication and it differs from the decreasing liquidity in adverse selection models of liquidity (e.g. Guerrieri and Shimer 2014). When the seller has superior information about the quality (e.g. the bank selling its loans to external investors), to incentivize the seller to reveal her private information, assets of higher quality are traded at higher prices, but with lower probability compared to the lower-quality assets. In contrast, as shown in Section 3, our model describes environments where both parties have precise private information about the quality, but lack of common knowledge about the asset quality which causes bargaining delays. Our analysis allows researchers to distinguish environments with adverse selection from that with precise two-sided private information by observing the correlation between the liquidity (trade volume or turnover) and quality (or prices). We expect that the adverse selection models are most relevant in primary markets (e.g. mortgage originators v.s. outside investors buying MBSs), while our model captures the gap between scarce public information and private information of sophisticated investors prevalent in secondary markets (e.g. hedge funds trading MBSs with each other or the commercial real estate market).
Figure 2: **Bargaining delay as a function of asset quality.** Parameters $y_u = 70, y_d = .2, r = 12\%, \alpha = .7, a = 1.5$. Bargaining delay has an inverse U-shaped form, as $x(\theta)$ is U-shaped. The gap in the graph depicts the set of illiquid assets which expands as the bargaining friction increases (from panel (a) to panel (b)) and shrinks as the search friction increases (from panel (b) to panel (c)). Panels (b) and (c) also illustrate that $\Theta_I$ (set of assets with $t(\theta) = -\frac{1}{r} \log x$) is often a small set.

**Market Liquidity** We next show that search and bargaining frictions have opposite effects on the market liquidity. Thus, we conclude that using one friction as a proxy can be misleading for the liquidity analysis.

**Proposition 3.** Suppose in equilibrium $\Theta_I \neq \emptyset$. Then

1. (bargaining friction) for any $\tilde{L} > 0$, there exists $\tilde{\alpha} \in (0, 1)$ such that if $\alpha > \tilde{\alpha}$ and in equilibrium $L > \tilde{L}$, then $\frac{dL}{dk} < 0, \frac{dx}{dk} < 0, \frac{dx}{dk} < 0, \frac{dX}{dk} < 0, \frac{d\Lambda_s}{dk} > 0$;

2. (search friction) $\frac{dL}{d\lambda} < 0, \frac{dx}{d\lambda} > 0, \frac{dx}{d\lambda} > 0, \frac{dX}{d\lambda} < 0, \frac{d\Lambda_s}{d\lambda} > 0$.

To see the effect of the bargaining friction, let us start with the case $k \approx 0$. Small differences in flow payoffs across assets translate into small difference in prices across asset qualities. As a result, investors can gain little by delaying trade and negotiations are short. An increase in the asset heterogeneity $k$ leads to larger differences in prices across qualities. Hence, the negotiation starts from offers that are farther apart (bid-ask spread increases by Proposition 1) and investors have more incentives to delay trade in bargaining and wait for a more favorable price offer. Since buyers trade-off search and bargaining delays, the increase in bargaining delays makes fewer assets attractive for trade to buyers and $L$ drops (see Figure 2b). However, buyers decrease their required liquidity threshold $\underline{x}$ and bear part of the increase in bargaining delays, as otherwise, buyers would accept too few assets and would end up searching longer for sufficiently liquid assets.
Note that as the bargaining friction becomes more severe, it becomes easier for sellers to find trade partners ($\Lambda_s$ increases). This, however, is accompanied by the reduction in the set of acceptable assets and so, leads to a higher match intensity only for sellers of liquid assets. The effect on the trade volume proxy $W$ is ambiguous. Higher contact intensity for sellers positively affects the trade volume proxy $W$. However, increased negotiation times and fewer actively traded assets reduce $X$, and thus, negatively affect $W$.

The search friction has the opposite effect on the range of accepted assets $L$. When it is easier to find a trade partner ($\lambda$ increases), the outside option of searching further in the market appreciates, and buyers increase their required liquidity threshold $\bar{x}$. Hence, fewer assets are accepted and $L$ drops (see Figure 2c). This increases the average liquidity index of acceptable assets $\bar{\pi}$, but negatively affects the aggregate liquidity index $X$. As a result, more efficient search positively affects aggregate trade volume $W$ through higher contact intensity for sellers $\Lambda_s$, but it also makes buyers more selective, which is a countervailing force negatively affecting $W$ through $X$.

One of the criticisms of using search delays to explain OTC liquidity is that in many OTC markets, e.g. the corporate bonds market, the practitioners’ view is that trade delays are not significant and do not have a first-order effect on liquidity. Proposition 3 shows that short trade delays reflect only the intensive margin and may also indicate market illiquidity. Even when trade delays for executed trades are small, this may imply that a vast amount of trades cannot be executed as they are rejected by buyers. In particular, when the search delays are reduced ($\lambda$ increases), few assets are actively traded ($L$ decreases), while the average bargaining delays of observed trades are short ($\bar{x}$ increases). Thus, our results stress that short observed search and bargaining delays do not mean that assets can be quickly sold, and it is important to look at both extensive and intensive margins.

Market Uncertainty and Transparency The opposite effect on the extensive margin of two frictions again suggests that search delays may not be the right proxy for bargaining delays. We next demonstrate that this difference is important for the analysis of the market uncertainty and the design of OTC markets.

Our model captures the effect of the quality of public information on market liquidity, the dimension that is not present in the current OTC models based on search delays. In the recent financial crisis, credit ratings became less reliable in assessing the quality of structured financial products, such as mortgage-backed securities, collateralized debt
obligations, etc., which coincided with the dried-up liquidity of structured finance products (see Brunnermeier 2008, Benmelech and Dlugosz 2010, Ashcraft et al. 2010). This is consistent with the effect of an increased bargaining friction on market liquidity: the drop in the quality of ratings exacerbates the bargaining friction, which in turn results in the reduction in the range of actively traded assets. Notice that more severe search friction leads to the opposite conclusion. While the aggregate trade volume may fall, the range of traded assets expands as the search becomes harder.

The analysis of two trade frictions adds to the debate about the design of OTC markets. On the one hand, such policies as more accurate and frequently updated credit ratings, introduction of benchmarks, standardization of products, dissemination of past quotes, etc., improve the quality of public information. As a result, conditional on better public information, the asset heterogeneity is reduced which decreases the bargaining friction, and thus, increases the market liquidity $L$. On the other hand, more efficient trading platforms and greater post-trade transparency reduce the search friction, and thus, lead to lower market liquidity. Thus, while market changes that improve the search can make risk-sharing more efficient, there are winners and losers of such measures, as some sellers are not able to liquidate their positions in the more transparent markets. This is consistent with the mixed evidence on effect of post-trade transparency on the liquidity of corporate bonds market (Bessembinder et al. 2006, Edwards et al. 2007, Asquith et al. 2013). It can also explain why many OTC markets are opaque. OTC markets by definition provide trading platforms for a variety of non-standardized assets. Our analysis shows that the opaque structure of such markets that does not allow for an efficient search is in line with their role in supporting liquidity of a greater variety of assets.

**Selling Pressure**  The extensive trade margin can be interpreted both literally as assets in $\Theta_I$ are not traded in the market and as assets in $\Theta_I$ are more sensitive to market conditions, e.g. the liquidity of assets in the middle can suffer from the selling pressure. This interpretation is formalized in the following proposition.

**Proposition 4.** Suppose $\Theta_I \neq \emptyset$. Consider an increase in $y_d$ and a decrease in $y_u$ so that $y_u + y_d$ stays constant. Then $L$ decreases.

In Proposition 4, the selling pressure is associated with the increase in the proportion of sellers in the market captured via a simultaneous offsetting increase in $y_d$ and decrease in $y_u$. The selling pressure results in a wider range of assets being illiquid. Thus, assets in $\Theta_I$ may be traded during normal times, but the ability to sell such assets is impaired.
when market conditions worsen.

**Only One Type of Delay**  We next study the difference between the search and bargaining delays by looking at the limits when only one type of delays is present. In the following proposition, we mark with one star equilibrium quantities in the limit $k \to 0$, and with two stars equilibrium quantities in the limit $\lambda \to \infty$.

**Proposition 5.** 1. The limit of the equilibrium as $k \to 0$ is characterized by $L^* = 1, x^* = \overline{x}^* = 1, \text{ and } \Lambda_s^* \in (0, \infty)$. Moreover, $\lim_{\lambda \to \infty} \Lambda_s^* = \infty$ only if $\frac{y_u}{y_u + y_d} a > 1$, and $\lim_{\lambda \to \infty} \Lambda_s^* < \infty$ only if $\frac{y_u}{y_u + y_d} a < 1$.

2. The limit of the equilibrium as $\lambda \to \infty$ is characterized by $(L^{**}, \Lambda^{**})$ satisfying:

   (a) If $L^{**} \in (0, 1)$, then $x^{**} < 1, \overline{x}^{**} < 1, \Lambda_b \to \Lambda_b^{**} < \infty$, and $\Lambda_s \to \infty$. Moreover, $L^{**} = 1$ if $\frac{y_u}{y_u + y_d} a \geq 1 + \frac{y_d}{\rho} \left( \frac{x}{k} \left( e^{\frac{k}{\overline{L}}} - 1 \right) - 1 \right)$.

   (b) If $L^{**} = 0$, then $x^{**} = \overline{x}^{**} = 1, \Lambda_s^{**} < \infty$, and $\Lambda_b \to \infty$. Moreover, $|\Theta_I| > 1 - \frac{u_k^b}{k}$ and $L^{**} = 0$ if $\frac{y_u}{y_u + y_d} a < 1$.

   (c) If $\Theta_I \neq \emptyset$ for some $\lambda$, then $L^{**} < 1$.

When there is no bargaining friction and assets are homogeneous ($k = 0$), the model reduces to Duffie et al. (2007). Since assets are identical, there is no bargaining delay and all assets are accepted for trade. In this case, search delays are random and do not vary across assets. For $k = 0$, as $\lambda \to \infty$ all potential trades are executed without delay. E.g. when buyers are relatively abundant ($\frac{y_u}{y_u + y_d} a > 1$), sellers are able to immediately liquidate their positions. When sellers are relatively abundant ($\frac{y_u}{y_u + y_d} a < 1$), some sellers are rationed and have to hold their positions.

When the bargaining friction is present and assets are heterogeneous ($k > 0$), even when the search friction vanishes, the outcome need not be efficient. Differences in endogenous bargaining delays translate into differences in liquidity $x(\cdot)$ which persist even as the search friction vanishes. Because of this, even when buyers are relatively abundant ($\frac{y_u}{y_u + y_d} a > 1$), the bargaining takes time ($\overline{x}^{**} < 1$) and trades are not executed immediately. The buyers are willing to tolerate bargaining delays because there is an endogenous shortage of sellers, so buyers spend in expectation $\frac{1}{\Lambda_b}$ to get to trade with the seller of a sufficiently liquid asset (in $\Theta_I^{**}$). Interestingly, despite the congestion on the buyer’s side, buyers may reject some assets when $\frac{y_u}{y_u + y_d} a < 1 + \frac{y_d}{\rho} \left( \frac{x}{k} \left( e^{\frac{k}{\overline{L}}} - 1 \right) - 1 \right)$. For rejected assets, buyers prefer to search further for more quickly negotiated assets. When $\frac{y_u}{y_u + y_d} a < 1$
and sellers are relatively abundant, buyers can quickly search for the most liquid asset ($\Lambda^*_b = \infty$), and thus, they accept only assets associated with virtually no bargaining delay ($x^{**} = 1$). In this case, when $k$ is sufficiently large ($k > y_d\xi$), there is a range of assets that buyers reject in their search.

By looking at two frictions in isolation, Proposition 5 shows that bargaining delays fundamentally vary from search delays in several respects. First, bargaining delays create an endogenous heterogeneity in liquidity and lead to the extensive trade margin, not present in models with only random search delays. Second, the extensive trade margin leads to an endogenous length of search for buyers. In their search process, buyers can reject several assets until they find a sufficiently liquid asset. Third, on the intensive margin, bargaining delays are deterministic in our model conditional on the quality of the asset.

6 Transparency and Flights-to-Liquidity

This section shows that the substitutability between asset classes leads to flights-to-liquidity during periods of market uncertainty and adverse liquidity effects of the gradual transparency policies.

We extend the baseline model to two asset classes $i = 1, 2$ with each class in unit fixed supply. The total mass of investors is $a > 2$. Asset class $i$ is characterized by the asset heterogeneity $k_i$. The mass $a_i \geq 1$ of investors trading assets in each class $i$ is determined in equilibrium so that $a_1 + a_2 = a$. Other than that, parameters of the model are the same for each asset class. The equilibrium of this two-class extension is defined next. (Subscripts indicate equilibrium quantities for the corresponding asset class).

Definition 3. A tuple $(\sigma_i, M_i, a_i)_{i=1,2}$ is a two-class equilibrium if $(\sigma_i, M_i)$ is the equilibrium of the baseline model with mass of investors $a_i$ and the following conditions hold

$$
\begin{cases}
x_1 = x_2, & \text{if } a - 1 > a_1 > 1, \\
x_1 \leq x_2, & \text{if } a_1 = 1, \\
x_1 \geq x_2, & \text{if } a_1 = a - 1.
\end{cases}
$$

(6.1)

Condition (6.1) reflects the mobility of buyers. If trading assets in one of the classes is more profitable, buyers will migrate into trading this asset class. To see this, recall that the buyers’ utility of trading each asset class is proportional to strategy thresholds $x_1$ and
$x_2$ (cf. Lemma 2), and buyers would migrate into the asset class with lower $x_i$. In the Appendix, we show that the equilibrium of the two-class model exists and is unique. We next show that flights-to-liquidity occur after an increase in the bargaining friction in the asset class or a decrease in the bargaining friction in the substitute asset class.

**Proposition 6.** Suppose $a - 1 > a_1 > 1$ and consider an increase in $k_1$ such that in the baseline model with only asset class 1 (and mass of investors $a_1$), it results in a decrease in $L_1$. Then $L_1$ and $a_1$ decrease, and the decrease in $L_1$ is larger in the two-class model. The same conclusion holds if instead $\lambda_2$ increases or $k_2$ decreases such that in the baseline model with only asset class 2 (and mass of investors $a_2$), $L_2$ decreases.

First, when the bargaining friction increases in asset class 1, investors migrate to trading assets in class 2, for which the bargaining friction is lower. This flight-to-liquidity exacerbates the drop in $L_1$ caused by the increase in $k_1$, as buyers’ migration improves the ratio of buyers to sellers in the market for asset class 1, and thus, makes them more selective in their choice of assets for trade. OTC markets are known to be prone to flights-to-liquidity episodes when, due to increased market uncertainty, investors shift their portfolio preferences to safer and more liquid assets. (see Friewald et al. 2012, Dick-Nielsen et al. 2012). Flights-to-liquidity are associated with dried-up liquidity, which is consistent with the prediction of our model. An important observation is that the level of payoffs $d$ in each asset class does not affect the liquidity implications of our model. Thus, we stress that the flights are flights-to-liquidity rather than flights-to-quality, which is consistent with the empirical evidence that default risk plays a smaller role than liquidity in flights (see Beber et al. 2009).

Next, consider the effect of a decrease in the bargaining friction in class 2 on the class 1 liquidity. Because of the lower bargaining delays, the substitute asset class 2 becomes more attractive. In order to maintain the indifference of buyers between two classes, some buyers migrate into trading class 2, which improves the ratio of buyers to sellers in the class 1, and hence, the profit of buyers from trading class 1 assets. The effect on $L_1$ is similar to the effect of the reduction in the search friction in the baseline model. As class 2 becomes more attractive, buyers become more selective in assets they accept for trade in class 1 and so, they reduce the range of actively traded class-1 assets. In fact, the reduction in the search friction in class 2 would have the same effect on the liquidity of asset class 1.

This result shows that a gradualism in introducing transparency or improving the quality of public information can hurt the market liquidity. E.g. the post-trade trans-
Transparency in the corporate bonds market was introduced starting in 2002 in several phases with early phases requiring disclosure only for larger issues of investment grade bonds, and later phases expanding the requirement to high-yield bonds and other assets, such as agency-backed securities. We have already discussed that the reduction in the search friction can reduce the market liquidity $L$ of the asset class affected. Proposition 6 shows that it also has a negative effect on the liquidity of substitute asset classes. Asquith et al. (2013) demonstrates that the introduction of post-trade transparency in corporate bonds market decreased the trading activity in high-yield bonds and this decrease started before the post-trade transparency was actually implemented for such bonds.

7 Conclusion

This paper develops a tractable model of decentralized asset markets with both search and endogenous bargaining delays. The key to the tractability is the application of the novel screening bargaining solution that captures bargaining delays arising from the coarse public, but precise two-sided private information about the asset quality. We microfound the screening bargaining solution in the standard alternating-offer bargaining game with the global games information structure and show that it leads to a distinct U-shaped liquidity in the OTC markets, which contrasts the decreasing prediction in adverse selection models. We show that the prevalent approach to OTC liquidity viewing search delays as a reduced form for all types of delay is with a loss. Search and bargaining delays operate quite differently on the extensive margin: The search friction increases, while the bargaining friction decreases the range of actively traded assets. Our analysis sheds light on the effect of market uncertainty on OTC liquidity, as well as the design and implementation of transparency policies in OTC markets.

There are several directions for future research. First, in our model the extensive margin arises through an endogenous variation in bargaining delays among assets. There are many other forms of post-search delays that naturally vary across assets and operate similarly to bargaining delays, e.g. time to gather information and do the evaluation, time to process the deal, time to raise financing for investment in a particular asset. Building a general model that tackles this form of heterogeneity is important, as it allows us to better match important features of existing markets. Second, our focus is on the limit when investors infinitely precise private signals about the quality, but lack common knowledge.

\footnote{In fact, Lemma 1 is a first step in this direction, as it derives a steady state distribution of asset holding for an arbitrary specification of post-search delays.}
This approach allows us to identify the distinct liquidity pattern in the case when both sides have private information. Exploring the case when private signals are noisy is an exciting and challenging direction for future research.\footnote{As a useful step towards this agenda, in the proof of Theorem 1, we construct BNEs of the auxiliary continuous-time bargaining game $G(p^s, p^b)$ for any affiliated distribution of signals $F$, and approximate them with continuous-time PBE limits of the alternating-offer bargaining game with the distribution of types $F$.} Finally, in the analysis we abstract from market makers. Incorporating them into the model would require a better understanding of how market makers can bargain with multiple counter-parties and is left for future research.

## A Appendix

This appendix contains the analysis of the OTC model. Online Appendix A contains proofs for the micro-foundation of the SBS and Online Appendix B contains auxiliary results.

### Equilibrium Characterization

**Step 1: Proof of Lemma 1.** We first derive the steady-state distribution of times spent in the match. For $\theta \in \Theta_L \cup \Theta_M$ and $u \in [0, t(\theta)]$, let $G(\theta, u)$ be the mass of sellers that have spent time $u$ negotiating the price of an asset of quality $\theta$. During the time interval $du$, a fraction $(y_u + y_d)du$ of matches is destroyed due to the switching of intrinsic types, and for an asset of quality $\theta$, a mass $\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)du$ of investors enters the bargaining stage. Hence, the change in the mass of sellers that have spent in the match less than $u$ is $(1 - (y_u + y_d))G(\theta, u) du$, and an asset of quality $\theta$, a mass $\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)du - G(\theta, u)$, which equals 0 in the steady-state. Thus,

$$\frac{\partial}{\partial u} G(\theta, u) = -(y_u + y_d)G(\theta, u) + \lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta),$$  \hspace{1cm} (A.1)

which together with $G(\theta, 0) = 0$ gives $G(\theta, u) = \frac{1 - e^{-(y_u + y_d)u}}{y_u + y_d}\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta)$. The total mass of sellers in the bargaining stage for asset $\theta$ is equal to $\mu_{sm}(\theta)$ which translates into $G(\theta, t(\theta)) = \mu_{sm}(\theta)$ or equivalently

$$\mu_{sm}(\theta) = \frac{1 - e^{-(y_u + y_d)t(\theta)}}{y_u + y_d}\lambda M_{bu}(\phi)\mu_{su}(\theta)\sigma(\theta).$$  \hspace{1cm} (A.2)

Let $w(\theta)$ be the intensity with which investors leave the match. Note that this is also the instantaneous trade volume for the asset of quality $\theta$. During the time interval $du$, sellers that have already spent time $[t(\theta) - du, t(\theta)]$ in the bargaining stage complete their trades. Thus,
The rank of the system is five and we eliminate the last two equations to guarantee the full rank.

\[
\lambda M_{Bu}(\phi)\mu_{su}(\theta)e^{-(y_u+y_d)t} \sigma(\theta). \quad (A.3)
\]

Now, we derive the distribution \( M \). For \( \theta \in \Theta_L \), \( \mu_{su}(\theta) = \frac{y_d}{y_u+y_d}, \mu_{bu}(\theta) = \frac{y_u}{y_u+y_d}, \mu_{sm}(\theta) = \mu_{bm}(\theta) = 0 \) and we only consider \( \theta \in \Theta_L \cup \Theta_M \). In the steady state, \( \mu_{su}(\theta), \mu_{bu}(\theta), M_{Bu}(\phi), M_{Su}(\phi) \) stay constant over time and so,

\[
\begin{aligned}
&y_d \mu_{sm}(\theta) + y_d \mu_{bu}(\theta) = y_u \mu_{su}(\theta) + \lambda M_{Bu}(\phi) \mu_{su}(\theta) \sigma(\theta), \\
y_u \mu_{sm}(\theta) + y_u \mu_{su}(\theta) + w(\theta) = y_d \mu_{bu}(\theta), \\
y_u M_{bm}(\Theta_L \cup \Theta_M) + y_u M_{su}(\phi) = y_d M_{Bu}(\phi) + \lambda M_{Bu}(\phi) \left( \int_0^1 \mu_{su}(\theta) \sigma(\theta) d\theta \right), \\
y_d M_{bm}(\Theta_L \cup \Theta_M) + y_d M_{Bu}(\phi) + \int_0^1 w(\theta) d\theta = y_u M_{Su}(\phi),
\end{aligned}
\]  

where the left-hand sides are the inflows into and the right-hand sides are the outflows from \( \mu_{su}(\theta), \mu_{bu}(\theta), M_{Bu}(\phi), M_{Su}(\phi) \), respectively. Combining the system (A.4) with the balance conditions (A.2), (A.1), (A.3), and (A.2) – (A.3), we get:

\[
\begin{aligned}
&y_d \mu_{sm}(\theta) + y_d \mu_{bu}(\theta) - y_u \mu_{su}(\theta) - \lambda M_{Bu}(\phi) \mu_{su}(\theta) \sigma(\theta) = 0, \\
y_u \mu_{sm}(\theta) + y_u \mu_{su}(\theta) - y_d \mu_{bu}(\theta) + \lambda M_{Bu}(\phi) \mu_{su}(\theta) e^{-(y_u+y_d)t(\theta)} \sigma(\theta) = 0, \\
\mu_{su}(\theta) + \mu_{bu}(\theta) + \mu_{sm}(\theta) = 1, \\
y_u M_{sm}(\Theta_L \cup \Theta_M) + y_u M_{su}(\phi) - y_d M_{Bu}(\phi) - \lambda M_{Bu}(\phi) \left( \int_0^1 \mu_{su}(\theta) \sigma(\theta) d\theta \right) = 0, \\
M_{su}(\phi) + M_{bu}(\phi) + M_{sm}(\Theta_L \cup \Theta_M) = a - 1, \\
(y_u + y_d) \mu_{sm}(\theta) - (1 - e^{-(y_u+y_d)t(\theta)}) \lambda M_{Bu}(\phi) \mu_{su}(\theta) \sigma(\theta) = 0, \\
y_d M_{sm}(\Theta_L \cup \Theta_M) + y_d M_{Bu}(\phi) - y_u M_{Su}(\phi) + \lambda M_{Bu}(\phi) \int_0^1 \mu_{su}(\theta) e^{-(y_u+y_d)t(\theta)} \sigma(\theta) d\theta = 0. 
\end{aligned}
\]  

(A.5)

The rank of the system is five and we eliminate the last two equations to guarantee the full rank. Using \( \Lambda_s = \lambda M_{Bu}(\phi) \) we get from the first three equations in (A.5)

\[
\begin{aligned}
\mu_{su}(\theta) &= \frac{yd}{yu+yd+\Lambda_s \sigma(\theta)}, \\
\mu_{bm}(\theta) &= \frac{\Lambda_s \sigma(\theta)(1 - e^{-(yu+yd)t(\theta)}) y_d}{(yu+yd)(yu+yd+\Lambda_s \sigma(\theta))}, \\
\mu_{bu}(\theta) &= \frac{yu(yu+yd) + \Lambda_s \sigma(\theta)(yu + y_d e^{-(yu+yd)t(\theta)})}{(yu+yd)(yu+yd+\Lambda_s \sigma(\theta))}.
\end{aligned}
\]  

(A.6)

In system (A.5), subtracting the forth equation from the fifth multiplied by \( y_u \) and using \( \Lambda_s = \lambda M_{Bu}(\phi) \), we get

\[
(y_u + y_d) \frac{\Lambda_s}{\Lambda} = (a - 1)y_u - \Lambda_s \left( \int_0^1 \mu_{su}(\theta) \sigma(\theta) d\theta \right),
\]
Lemma 4. For all 
\[ \frac{y_u}{y_u + y_d} (a - 1) > 0. \] 
The left-hand side of (4.4) is strictly increasing in \( \Lambda_s \) and converges to infinity as \( \Lambda_s \to \infty \) and the right-hand side is strictly decreasing in \( \Lambda_s \) unless \( \sigma(\theta) = 0 \) for almost all \( \theta \) which does not hold in equilibrium. At \( \Lambda_s = 0 \), the left-hand side is zero and the right-hand side equals \( \frac{y_u}{y_u + y_d} (a - 1) > 0. \) Thus, equation (4.4) has a unique positive solution.

Let us summarize. Quantities \( \mu_{su}(\theta), \mu_{bu}(\theta), \mu_{bm}(\theta), \mu_{sm}(\theta) \) are given by (A.6), \( M_{bu}(\phi) = \frac{\Lambda_s}{\Lambda} \) and \( M_{su}(\phi) \) can be found from the fifth equation in (A.5). Thus, the distribution \( M \) is characterized by \((\sigma(\cdot), x(\cdot), \Lambda_s)\). Using the expression for \( \mu_{su}(\theta) \) in the first line of (A.6), we find that \( \Lambda_b \) is given by (4.5). \( F_L(\theta) = \frac{\int_{[0,\theta]\cap\Theta_L} \mu_{su}(\theta)d\theta}{\mu_{su}(\Theta_L)} = \frac{|[0,\theta]\cap\Theta_L|}{L} \) and so, \( F_L \) is uniform on \( \theta \in \Theta_L \).

If \( \sigma(\cdot) \) increases weakly pointwise, then the right-hand side of (4.4) decreases and so, it is necessary for equation (4.4) to hold that \( \Lambda_s \) decreases which by the equation (4.5) and the fact that \( L \) (weakly) increases when \( \sigma(\cdot) \) (weakly) increases leads to an increase in \( \Lambda_b \).

Step 2: At this step, we want to express value functions for every \( \theta \) through \( \Lambda_s, \Lambda_b, \sigma(\theta) \) and \( x(\theta) \). Denote by \( U_s(\theta) \) the utility of the seller who owns an asset of quality \( \theta \) and does not participate in the search market. Setting \( \sigma(\theta) = 0 \) in equations (4.8) and (4.10), we get
\[
U_s(\theta) = \frac{1}{r} \left( d + k\theta - \frac{r + y_d}{r + y_u + y_d} \right), \tag{A.7}
\]
For \( \theta \in \Theta_i, V_{su}(\theta) = U_s(\theta) \). The next lemma simplifies equations (4.7), (4.8), (4.9), (4.10) and expresses \( V_{bu}(\theta) \) and \( V_{su}(\phi) \) through \( V_{bu}(\phi) \) and \( V_{su}(\theta) \) (the proof is straightforward and omitted).

Lemma 4. For all \( \theta \in [0,1] \),
\[
V_{bu}(\theta) = \frac{d + k\theta + y_d V_{su}(\theta)}{r + y_d}, \tag{A.8}
\]
\[
V_{su}(\phi) = \frac{y_u V_{bu}(\phi)}{r + y_u}, \tag{A.9}
\]
\[
V_{bu}(\phi) = \Lambda_b \left( \frac{r + y_u}{r} \right) \left( \mathbb{E}[V_{sm}(\theta)|\theta \in \Theta_L] - V_{bu}(\phi) \right), \tag{A.10}
\]
\[
V_{su}(\theta) = U_s(\theta) + \sigma(\theta) \Lambda_s \frac{r + y_d}{r} (V_{sm}(\theta) - V_{su}(\theta)). \tag{A.11}
\]

Recall that we define the status quo \((\hat{v}, \hat{c}(\theta)) \) as the outcome that gives investors the stream payoffs they receive during the negotiation process. Thus, \( \hat{c}(\theta) \) is given by the Bellman equation
\[
r \hat{c}(\theta) = d + k\theta - 1 + y_u (V_{bu}(\theta) - \hat{c}(\theta)) + y_d (V_{su}(\theta) - \hat{c}(\theta)),
\]
and so, using (A.7) and (A.8), we get
\[ \hat{c}(\theta) = \frac{1}{\rho}(d + k\theta - 1 + y_u V_{bu}(\theta) + y_d V_{su}(\theta)) = \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta). \]

Analogously, \( \hat{v} \) is given by the Bellman equation

\[ r\hat{v} = y_u (V_{bu}(\phi) - \hat{v}) + y_d (V_{su}(\phi) - \hat{v}), \]

and so, using (A.9),

\[ \hat{v} = \frac{1}{\rho}(y_u V_{bu}(\phi) + y_d V_{su}(\phi)) = \frac{y_u}{r + y_u} V_{bu}(\phi). \]

Functions \( v \) and \( c \) introduced in (2.4) and (2.5) are given by

\[ c(\theta) = \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi), \quad (A.12) \]
\[ v(\theta) = \frac{d + k\theta}{r + y_d} + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi). \quad (A.13) \]

Observe that \( v(\theta) - c(\theta) = \frac{1}{\rho} \equiv \xi \). Thus, the price of trade is given by

\[ p(\theta) = (1 - \alpha) \frac{d + k\theta}{r + y_d} + \alpha \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi) \]
\[ = \frac{d + k\theta}{r + y_d} + \frac{y_d}{r + y_d} V_{su}(\theta) - \frac{y_u}{r + y_u} V_{bu}(\phi) - \alpha \xi. \quad (A.14) \]

The next lemma expresses the value functions of matched investors through \( x(\theta) \), \( V_{su}(\theta) \) and \( V_{bu}(\phi) \).

**Lemma 5.** For any \( \theta \in [0, 1] \),

\[ V_{bm}(\theta) = \alpha \xi x(\theta) + \frac{y_u}{r + y_u} V_{bu}(\phi), \quad (A.15) \]
\[ V_{sm}(\theta) = (1 - \alpha) \xi x(\theta) + \frac{r}{r + y_d} U_s(\theta) + \frac{y_d}{r + y_d} V_{su}(\theta). \quad (A.16) \]

**Proof.** We solve the differential equation (4.11) with the terminal condition \( V_{bm}(t(\theta), \theta) = V_{bu}(\theta) - p(\theta) \) to get

\[ V_{bm}(t, \theta) = (V_{bu}(\theta) - p(\theta)) e^{-\rho(t(\theta) - t)} + \frac{y_u V_{bu}(\phi)}{r + y_u} \left( 1 - e^{-\rho(t(\theta) - t)} \right). \]

Using (A.8), (A.14), and \( V_{bm}(0, \theta) = V_{bm}(\theta) \), we get (A.15). The derivation of (A.16) is symmetric.

**Proof of Lemma 2.** Combining (A.8) and (A.15), we get (4.14). The buyer prefers to trade with the seller of asset \( \theta \) if and only if \( V_{bm}(\theta) \geq V_{bu}(\phi) \), or combining (A.15) and (4.14), we get
the condition (4.13).

To complete the derivation of value functions, we need to find \( V_{su}(\theta) \). It follows from (A.11) and (A.16) that for \( \theta \in \Theta_L \cup \Theta_M \) function \( V_{su} \) is given by

\[
V_{su}(\theta) = U_s(\theta) + (1 - \alpha) \frac{r + yd}{r} \frac{\sigma(\theta) \Delta_s}{\rho + \sigma(\theta) \Delta_s} \xi x(\theta) \tag{A.17}
\]

Equation (A.17) implies that \( V_{su}(\theta) > U_s(\theta) \) whenever \( x(\theta) > 0 \) and so, sellers always prefer to trade.

**Step 3:** We will derive the liquidity profile \( x \) for given \( v(\cdot) \) and \( c(\cdot) \). Consider asset qualities \( \theta \in (\theta^*, 1] \) for which the bargaining delay is pinned down by (2.2). We can rewrite (2.2) as

\[
\theta \in \arg \max_{\theta' \in [\theta^*, 1]} x(\theta')(v(\theta) - p(\theta')). \tag{A.18}
\]

By the single-crossing condition, \( x(\cdot) \) is weakly increasing on \((\theta^*, 1]\).

By the envelope theorem (Milgrom and Segal (2002)), function \( x(\theta)(v(\theta) - p(\theta)) \) is absolutely continuous and so, since \( v(\cdot) \) and \( p(\cdot) \) are continuous on \( \Theta_L \cup \Theta_M \), \( x(\cdot) \) is absolutely continuous on \((\theta^*, 1] \cap (\Theta_L \cup \Theta_M) \). Thus, at the differentiability points of \( x(\cdot) \) the first-order condition for (A.18) is

\[
x'(\theta)(v(\theta) - p(\theta)) - x(\theta)p'(\theta) = 0, \tag{A.19}
\]

which we can rewrite as

\[
\frac{x'(\theta)}{x(\theta)} = \frac{v'(\theta)}{\alpha \xi}. \tag{A.20}
\]

Since \( v'(\cdot) \geq 0 \) almost everywhere, \( x'(\theta) \geq 0 \) almost everywhere on \((\theta^*, 1] \cap (\Theta_L \cup \Theta_M) \).

By the analogous argument for \( \theta < \theta^* \), we have that \( x(\cdot) \) is weakly decreasing on \([0, \theta^*) \) (thus, \( x(\cdot) \) has a U shape on \([0, 1] \)), and for \([0, \theta^*) \cap (\Theta_L \cup \Theta_M) \), \( x(\theta) \) satisfies

\[
\frac{x'(\theta)}{x(\theta)} = - \frac{c'(\theta)}{(1 - \alpha) \xi}. \tag{A.21}
\]

The following lemma follows immediately from (A.20) and (A.21).

**Lemma 6.** For differentiability points \( \theta \in (\theta^*, 1] \cap (\Theta_L \cup \Theta_M) \) of \( v(\cdot) \), \( x'(\theta) = 0 \) if and only if \( v'(\theta) = 0 \), and for differentiability points \( \theta \in [0, \theta^*) \cap (\Theta_L \cup \Theta_M) \) of \( c(\cdot) \), \( x'(\theta) = 0 \) if and only if \( c'(\theta) = 0 \).

**Proof of Lemma 3.** The analysis proceeds in a series of claims.

**Claim 1.** If \( x(\theta) = \bar{x} \) on an interval \((\theta', \theta'')\), then \( \sigma(\theta) \in (0, 1) \) for almost every \( \theta \in (\theta', \theta'') \).

**Proof.** Suppose that \( x(\theta) = \bar{x} \), but \( \sigma(\theta) = 0 \) for \( \theta \in (\theta', \theta'') \) (the argument is identical for \( \sigma(\theta) = 1 \)). By (A.17), \( V_{su} \) is strictly increasing on \((\theta', \theta'')\) and so, by (A.12) and (A.13), \( v \) and
c are strictly increasing, which contradicts Lemma 6. \textit{q.e.d.}

\textbf{Claim 2.} There exist \( \hat{\theta} \leq \hat{\theta} < \theta^* \leq \hat{\theta} \leq \hat{\theta} \) such that \( \Theta_L = [0, \hat{\theta}] \cup [\hat{\theta}, 1] \), \( \Theta_M = (\hat{\theta}, \theta] \cup [\bar{\theta}, \hat{\theta}) \), and \( \Theta_I = (\theta, \bar{\theta}) \).

\textit{Proof.} By Lemma 2, buyers accept only asset qualities with \( x(\theta) \geq x \). Since \( x(\cdot) \) has a U shape, there exist \( \hat{\theta} \leq \theta \leq \theta^* \leq \hat{\theta} \leq \hat{\theta} \) such that \( x(\theta) \geq x \) on \([0, \theta] \cup [\bar{\theta}, 1]\) and \( x(\theta) > x \) on \([0, \hat{\theta}] \cup [\hat{\theta}, 1]\), which combined with Claim 1 gives the result. \textit{q.e.d.}

\textbf{Claim 3.} \( \bar{\theta} = \hat{\theta} \).

\textit{Proof.} Suppose to contradiction \( \bar{\theta} < \hat{\theta} \). Then there exist a decreasing sequence \((\theta_i')_{i=1}^{\infty} \subset [\hat{\theta}, 1]\) and an increasing sequence \((\theta_i'')_{i=1}^{\infty} \subset [\bar{\theta}, \hat{\theta})\) both converging to \( \hat{\theta} \). Lemma 6 implies that \( v(\theta_i'') \) is constant for all \( i \), and so from (A.13) and (A.17), \( c(\theta_i'') \) is decreasing in \( \theta \). On the other hand, \( \sigma(\theta_i') = 1 \) for all \( i \). Since \( x(\theta) = x \), we get the contradiction to the continuity of \( v(\cdot) \) at \( \hat{\theta} \). \textit{q.e.d.}

\textbf{Claim 4.} \( \hat{\theta} < \hat{\theta} \) implies \( \hat{\theta} < \hat{\theta} \).

\textit{Proof.} Suppose to contradiction that \( \hat{\theta} = \hat{\theta} < \hat{\theta} \). Then there exist an increasing sequence \((\theta_i')_{i=1}^{\infty} \subset [0, \hat{\theta}]\) and a decreasing sequence \((\theta_i'')_{i=1}^{\infty} \subset \Theta_I\) both converging to \( \hat{\theta} \). We have \( x(\theta_i'') = 0 \) and \( x(\theta_i') > x > 0 \). From (A.12) and (A.17), this implies that \( c(\theta_i') > c(\theta_i'') \), which contradicts the monotonicity of \( c(\cdot) \) as \( \theta_i' < \theta_i'' \). \textit{q.e.d.}

It follows from Claims 1-4 that the only possibilities are: a) \( \hat{\theta} = \theta = \bar{\theta} = \hat{\theta} \), b) \( \hat{\theta} < \theta = \bar{\theta} = \hat{\theta} \), c) \( \hat{\theta} < \theta < \bar{\theta} = \hat{\theta} \).

Finally, we obtain equation (4.15) by solving (A.20) and (A.21) with the initial condition \( x(1) = x(0) = 1 \) (recall, by the definition of the SBS \( t(1) = t(0) = 0 \)). \hfill \Box

\textbf{Combining the steps:} We now combine all the steps to reduce the problem of finding equilibria to the problem of finding \( \Lambda_s \) and \( L \).

\textbf{Lemma 7.}

\[ \theta = 1 + \frac{r}{k} \alpha \xi \ln x(\theta) + \frac{yd}{k} (1 - \alpha) \xi \frac{\Lambda_s}{\rho + \Lambda_s} (1 - x(\theta)), \quad \text{for } \theta \geq \hat{\theta}, \] \hfill (A.22)

\[ \theta = -\frac{r}{k} (1 - \alpha) \xi \ln x(\theta) + \frac{yd}{k} (1 - \alpha) \xi \frac{\Lambda_s}{\rho + \Lambda_s} (1 - x(\theta)), \quad \text{for } \theta \leq \hat{\theta}. \] \hfill (A.23)

Moreover, \( v(\cdot) \) and \( c(\cdot) \) are strictly increasing on \( \Theta_L \).

\textit{Proof.} For almost every \( \theta > \hat{\theta} \), plugging \( v'(\theta) \) from (A.13) into (A.20), we get

\[ \frac{x'(\theta)}{x(\theta)} = \frac{k + yd V_{su}(\theta)}{\alpha \xi (r + yd)}. \quad \text{(A.24)} \]

By (A.17),

\[ V_{su}'(\theta) = \frac{k}{r} + (1 - \alpha) \frac{r + yd}{r} \frac{\Lambda_s}{\rho + \Lambda_s} \xi x'(\theta). \quad \text{(A.25)} \]
Since \( x' (\theta) \geq 0 \) for \( \theta > \hat{\theta} \), it follows from (A.13) and (A.25) that \( v' (\theta) > 0 \). Further, combining (A.24) and (A.25), we get

\[
x' (\theta) = \frac{k}{\xi r \left( \frac{\alpha}{x(\theta)} - \frac{yd}{r} \frac{\Lambda_s}{\rho + \Lambda_s} (1 - \alpha) \right)},
\]

which together with \( x(1) = 1 \) gives (A.22).

Analogously, plugging in \( c' (\theta) \) from (A.12) into (A.21),

\[
\frac{x' (\theta)}{x(\theta)} = \frac{r U'_s (\theta) + y_d V'_{su} (\theta)}{(1 - \alpha) \xi (r + y_d)},
\]

and using (A.25) for \( V'_{su} (\theta) \), we get that \( c' (\cdot) \) is strictly decreasing for \( \theta < \check{\theta} \) and that

\[
x' (\theta) = -\frac{k}{\xi r (1 - \alpha) \left( \frac{1}{x(\theta)} + \frac{yd}{r} \frac{\Lambda_s}{\rho + \Lambda_s} \right)},
\]

which together with \( x(0) = 1 \) gives (A.23).

We next express thresholds \( \hat{\theta}, \check{\theta}, \theta, \) and \( L \) through \( x \). By Lemma 7 and the fact that \( \underline{x} = x(\hat{\theta}) = x(\check{\theta}) \):

\[
\hat{\theta} = 1 + \frac{r}{k} \alpha \xi \ln \underline{x} + \frac{yd}{k} (1 - \alpha) \xi \frac{\Lambda_s}{\rho + \Lambda_s} (1 - \underline{x}),
\]

\[
\check{\theta} = -\frac{r}{k} (1 - \alpha) \xi \ln \underline{x} + \frac{yd}{k} (1 - \alpha) \xi \frac{\Lambda_s}{\rho + \Lambda_s} (1 - \underline{x}).
\]

It follows from (A.28) and (A.29) that

\[
L = 1 - \hat{\theta} + \check{\theta} = -\frac{r \xi}{k} \ln \underline{x}.
\]

For each \( \theta \in \Theta_M \), \( x(\theta) = \underline{x} \) and so, \( c(\theta) = c(\check{\theta}) \) by Lemma 6. Therefore, by (A.12),

\[
V_{su} (\theta) = V_{su} (\check{\theta}) - \frac{r}{yd} (U_s (\theta) - U_s (\check{\theta})).
\]

Using (A.17) and \( x(\check{\theta}) = \underline{x} \),

\[
V_{su} (\theta) - U_s (\theta) = \frac{r + yd}{r} \left( \frac{k}{yd} (\check{\theta} - \theta) + (1 - \alpha) \frac{\Lambda_s}{\rho + \Lambda_s} \xi \underline{x} \right).
\]

Threshold \( \theta \) is determined as the minimum of \( \check{\theta} \) and the solution to the equation \( U_s (\theta) = V_{su} (\theta) \)
and so, from (A.32),
\[ \theta = \min \left\{ \hat{\theta}, \tilde{\theta} + (1-\alpha) \frac{yd}{k} \frac{\Lambda_s}{\rho+\Lambda_s} \xi_x \right\}. \]  
(A.33)

This completes the description of \( x(\cdot) \) for given \( \Lambda_s \) and \( \bar{x} \). The next lemma determines \( \sigma(\cdot) \).

**Lemma 8.** For given \( \Lambda_s \) and \( \bar{x} \),
\[ \sigma(\theta) = \begin{cases} 
1, & \text{if } \theta \in [0,\hat{\theta}] \cup [\tilde{\theta},1], \\
0, & \text{if } \theta \in [\hat{\theta},\tilde{\theta}], \\
\frac{\rho(1-\alpha) \Lambda_s \xi_x - \frac{k}{yd}(\theta - \tilde{\theta})}{\Lambda_s (1-\alpha) \rho - \Lambda_s \xi_x + \frac{k}{yd}(\theta - \tilde{\theta})}, & \text{if } \theta \in (\tilde{\theta},\hat{\theta}). 
\end{cases} \]  
(A.34)

**Proof.** We only need to determine \( \sigma(\theta) \) for \( \theta \in \Theta_M \). It follows from (A.17), (A.32) and \( x(\theta) = \bar{x} \):
\[ \sigma(\theta) = \rho \frac{r}{r+y}(V_{su}(\theta) - U_s(\theta)) \]  
(A.35)

Proof of Theorem 2. For given \( \Lambda_s \) and \( \bar{x} \), we can determine equilibrium strategy \( \sigma(\cdot) \) from Lemma 8 and \( x(\cdot) \) from Lemma 7 where \( \hat{\theta}, \tilde{\theta} \), and \( \hat{\theta} \) are expressed through \( \Lambda_s \) and \( \bar{x} \) from (A.28), (A.29), and (A.33). Lemma 1 describes the steady-state distribution for given \( \sigma(\cdot) \) and \( x(\cdot) \). Moreover, there is a one-to-one mapping between \( \bar{x} \) and \( L \) given by (A.30) whenever \( L < 1 \). Thus, equilibrium is pinned down by \( \Lambda_s \) and \( L \). We next characterize these quantities. Lemma 21 in the Online Appendix B shows that equation (4.4) implies equation (4.17). Next, we derive (4.16). Combining (4.5) and (4.13), we get
\[ \rho \geq \frac{\lambda y_d}{y_u + y_d + \Lambda_s} \int_{x(\theta) \geq \bar{x}} \left( \frac{x(\theta)}{\bar{x}} - 1 \right) d\theta, \]  
(A.36)

which holds as equality whenever \( L < 1 \). Given (A.22) and (A.23), we can explicitly calculate
\[ X = \int_{x(\theta) \geq \bar{x}} x(\theta) d\theta = \int_{\hat{\theta}}^{1} x(\theta) d\theta + \int_{0}^{\hat{\theta}} x(\theta) d\theta = \int_{\bar{x}}^{1} x \frac{d\theta(x)}{dx} dx - \int_{\bar{x}}^{1} x \frac{d\theta(x)}{dx} dx = \frac{r\xi}{k} (1-x) \]  
(A.37)

and combined with (A.30) and (A.35) this gives (4.16). Therefore, \( \Lambda_s \) and \( L \) are determined by the system (4.16) and (4.17). Let function \( \Lambda_s(L) \) and \( \Lambda_s^2(L) \) be implicit functions given by equations (4.16) and (4.17), resp. At \( L = 1 \), any \( \Lambda_s \) above \( \frac{\lambda y_d}{p} \left( \frac{\xi_r}{k} \left( e^{\frac{k}{r}} - 1 \right) - y_u - y_d \right) \) satisfies (4.16). Moreover, \( \Lambda_s^2(L) > 0 \) for any \( L \) (to see this, note that there is no \( L \in [0,1] \) that solves (4.17) with \( \Lambda_s = 0 \), while \( \Lambda_s(0) = -y_u - y_d < 0 \). Therefore, the solution to (4.16) and
(4.17) always exists. In the Online Appendix B, Lemma 23 shows that \( \Lambda_s^1 \) is strictly increasing and \( \Lambda_s^2 \) is strictly decreasing and so, the uniqueness obtains.

Condition \( \alpha \geq \frac{y_d}{r+y_d} \) guarantees that functions \( v \) and \( c \) are indeed monotone (as conjectured). From (A.27), \( x'(\theta) < 0 \) for \( \theta < \hat{\theta} \) and so (A.21) implies \( c'(\theta) < 0 \) for \( \theta < \hat{\theta} \). Whenever \( \alpha \geq \frac{y_d}{r+y_d} \), \( \frac{\alpha}{x(\theta)} \geq \alpha \geq \frac{y_d}{r}(1-\alpha) \geq \frac{y_d}{r} \Lambda_s^{-} (1-\alpha) \) and so from (A.26), \( x'(\theta) > 0 \) for \( \theta \geq \hat{\theta} \). Hence, it follows from (A.21) that \( v'(\theta) > 0 \) for \( \theta > \hat{\theta} \). By Lemma 6, \( c(\theta) \) is constant on \([\hat{\theta}, \theta] \). On \([\theta, \bar{\theta}] \), \( c(\theta) = U_s(\theta) - \frac{y_u}{r+y_u} V_{bu}(\phi) \). Finally, \( \lim_{\theta \downarrow \hat{\theta}} v(\theta) < \lim_{\theta \downarrow \bar{\theta}} v(\theta) \), as qualities just below \( \hat{\theta} \) are not traded, while qualities just above \( \bar{\theta} \) are accepted. Therefore, since \( v'(\theta) = c'(\theta) \), \( v \) and \( c \) are increasing on \([0, 1] \).

Finally, note that \( d \) does not enter in the expressions characterizing \( M, x(\cdot), \sigma(\cdot) \).

**Lemma 9.** If \( \Theta_I \neq \emptyset \), then

\[
I(L, \Lambda_s) = \frac{\Lambda_s L}{y_u + y_d + \Lambda_s} + (1-\alpha) \frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k L}{\rho s}} \int_0^1 \frac{1-s}{1 + \frac{y_u+y_d}{\Lambda_s} - \frac{1}{\rho s}} ds. \tag{A.37}
\]

**Proof.** If \( \Theta_I \neq \emptyset \), then \( \hat{\theta} > \bar{\theta} \) and so, by (A.28) and (A.33), \( 1-L > (1-\alpha) \frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k L}{\rho s}} \) which implies (A.37).

**Bargaining Delays and Liquidity**

**Proof of Proposition 1.** To get (5.2), plug functions \( c \) and \( v \) from (A.12) and (A.13) into (2.1) and then substitute \( V_{bu}(\phi) \) and \( V_{su}(\theta) \) from (4.14) and (4.17). By the definition of the SBS, the first offers are accepted by the sellers of the lowest quality and buyers of the highest quality, thus, \( ba = p(1) - p(0) = \frac{k}{r} \).

**Proof of Proposition 2.** Follows directly from Lemma 3 and (5.1).

**Proof of Proposition 3.** By moving all terms to the right-hand side, we can rewrite the system (4.16) and (4.17) as

\[
\begin{align*}
0 &= g_1(\Lambda_s, L, k), \\
0 &= g_2(\Lambda_s, L, k),
\end{align*}
\tag{A.38}
\]

where the first line corresponds to (4.16) and the second line to (4.17). From Lemma 23, \( \frac{\partial g_1}{\partial \Lambda_s} < 0, \frac{\partial g_2}{\partial \Lambda_s} < 0, \frac{\partial g_1}{\partial L} > 0, \frac{\partial g_2}{\partial L} < 0 \). Moreover, we can see that \( \frac{\partial g_2}{\partial L} < \frac{1}{k} \) and \( \frac{\partial g_1}{\partial \Lambda_s} = -1 \). Further, \( \frac{\partial g_1}{\partial k} = \frac{\Lambda_s \xi y_u}{k^2} \left(1 + e^{\frac{k L}{\rho s}} \left( \frac{k}{\xi L} - 1 \right) \right) > 0 \) and \( \frac{\partial g_2}{\partial k} > (1-\alpha)C \) for some constant \( C \). By the implicit function theorem,

\[
\frac{dL}{dk} = \frac{-\frac{\partial g_1}{\partial \Lambda_s} \frac{\partial g_2}{\partial L} + \frac{\partial g_2}{\partial \Lambda_s} \frac{\partial g_1}{\partial L}}{\frac{\partial g_1}{\partial L} \frac{\partial g_2}{\partial \Lambda_s} - \frac{\partial g_2}{\partial L} \frac{\partial g_1}{\partial \Lambda_s}} \leq \frac{(1-\alpha)C - \frac{\xi y_u r^2}{k^3} \left(1 + e^{\frac{k L}{\rho s}} \left( \frac{k}{\xi L} - 1 \right) \right)}{\frac{\partial g_1}{\partial \Lambda_s} \frac{\partial g_2}{\partial L} - \frac{\partial g_2}{\partial L} \frac{\partial g_1}{\partial \Lambda_s}}.
\]
The denominator of this upper bound on \( \frac{dt}{dk} \) is positive and the numerator is negative for sufficiently large \( \alpha \). Thus, if \( \alpha > 1 - \frac{\xi y d r^2}{\xi \kappa r^2 \rho} \left( 1 + e^{\frac{L}{\xi r}} \left( \frac{k}{\xi r} - 1 \right) \right) \), then for all equilibria with \( L > \tilde{L}, \frac{dt}{dk} < 0 \). Further, \( \frac{d\Lambda_s}{dk} = -\frac{\partial \phi_2}{\partial L} \frac{\partial \Lambda_s}{\partial k} > 0 \).

To show that \( \frac{dx}{dk} < 0 \), we use (4.30) to express (4.16) and (4.17) in terms of \((\Lambda_s, x)\):

\[
\begin{align*}
0 &= \frac{\xi \rho y u}{k} \left( \frac{1}{x} - 1 + \ln x \right) - (y_u + y_d) - \Lambda_s, \\
0 &= \frac{y_s (a-1)}{y_u + y_d} - \Lambda_s \frac{1}{x} + \frac{y_d}{y_u + y_d} \left( \frac{\Lambda_s}{y_u + y_d + \Lambda_s} r \xi \ln x - (1 - \alpha) y_d \xi \rho} \right) \frac{\Lambda_s}{1 + \frac{y_u + y_d - \Lambda_s}{\rho} ds}.
\end{align*}
\]

(A.39)

We write \( f_1 \) for the first equation and \( f_2 \) for the second. By the implicit function theorem, \( \frac{dx}{dk} = \frac{\partial f_1}{\partial \Lambda_s} \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial \Lambda_s} \). From Lemma 23, \( \frac{\partial f_1}{\partial \Lambda_s} < 0, \frac{\partial f_2}{\partial x} < 0, \frac{\partial f_1}{\partial y} < 0, \frac{\partial f_2}{\partial \Lambda_s} > 0 \). Moreover, \( \frac{\partial f_2}{\partial x} > 0, \frac{\partial f_1}{\partial \Lambda_s} < 0 \) and so, \( \frac{dx}{dk} < 0 \). From (A.30) and (A.36), \( \bar{\tau} = \frac{1-x}{\ln x} \) is increasing in \( \bar{\tau} \), and so, \( \frac{d \bar{\tau}}{dk} < 0 \). Since \( X = \bar{\tau} L, \frac{dX}{dk} < 0 \).

To derive the comparative statics in \( \lambda \), we express equilibrium conditions (4.16) and (4.17) in terms of variables \( L \) and \( M_{u} (\phi) \) (which we shortly denote by \( M_{\phi} \) below) as follows\(^{40}\)

\[
\begin{align*}
M_{\phi} &= \frac{y_s}{\rho} \left( \frac{\xi r}{\rho} \left( e^{\frac{k}{\xi r} L} - 1 \right) \right) - \frac{y_u + y_d}{x} \\
M_{\phi} &= \frac{y_s (a-1)}{y_u + y_d} - \frac{y_d}{y_u + y_d} \left( \frac{\lambda M_{\phi} L}{y_u + y_d + \lambda M_{\phi}} + (1 - \alpha) y_d \xi \rho} \right) \frac{\lambda M_{\phi} e^{-\frac{k}{\lambda M_{\phi}} L} \int_0^1 - \frac{y_u + y_d - \Lambda_s}{\rho} ds}.
\end{align*}
\]

(A.40)

Again, we write \( h_1 \) for the first equation and \( h_2 \) for the second. By the implicit function theorem, \( \frac{dL}{dx} = -\frac{\partial h_1}{\partial M_{\phi}} \frac{\partial h_2}{\partial M_{\phi}} \frac{\partial h_1}{\partial x} \frac{\partial h_2}{\partial x} \). From Lemma 23, \( \frac{\partial h_1}{\partial M_{\phi}} = -1, \frac{\partial h_2}{\partial M_{\phi}} < 0, \frac{\partial h_1}{\partial M_{\phi}} < 0 \) and so, the denominator is positive. Moreover, \( \frac{\partial h_2}{\partial x} < 0 \) and \( \frac{\partial h_1}{\partial M_{\phi}} = \frac{y_u + y_d}{x} > 0 \) and so, \( -\frac{\partial h_2}{\partial M_{\phi}} \frac{\partial h_1}{\partial x} + \frac{\partial h_2}{\partial x} \frac{\partial h_1}{\partial M_{\phi}} < 0 \). Thus, \( \frac{dL}{dx} < 0 \). By (A.30) and \( \bar{\tau} = \frac{1-x}{\ln x} \), \( \frac{dx}{dk} > 0 \) and \( \frac{d \bar{\tau}}{dx} > 0 \). Further, by applying the implicit function theorem to (A.38), \( \frac{d \lambda_s}{dk} = -\frac{\partial \phi_1}{\partial \lambda_s} \frac{\partial \phi_2}{\partial \lambda_s} + \frac{\partial \phi_2}{\partial \lambda_s} \frac{\partial \phi_1}{\partial \lambda_s} \). From Lemma 23, \( \frac{\partial \phi_1}{\partial \lambda_s} < 0, \frac{\partial \phi_2}{\partial \lambda_s} < 0, \frac{\partial \phi_1}{\partial x} > 0, \frac{\partial \phi_2}{\partial x} < 0 \) and so, the denominator is negative. Moreover, \( \frac{\partial \phi_s}{\partial \lambda_s} > 0, \frac{\partial \phi_s}{\partial x} > 0 \) and so, \( \frac{d \lambda_s}{dx} > 0 \).

\(^{40}\) Again we consider only cases where before and after an increase in \( \lambda \), \( L < 1 \), as other cases are straightforward to show from (4.4).
Proof of Proposition 5. 1) As \( k \to 0 \), the right-hand side of (4.16) converges to \(-y_u - y_d\), thus, \( L^* = 1 \). It follows from equation (A.30), than \( \varepsilon^* = 1 \). By setting \( L = 1 \) in equation (4.17),

\[
\frac{\Lambda_s^*}{\lambda} = \frac{y_u(a - 1)}{y_u + y_d} - \frac{\Lambda_s^*}{y_u + y_d y_u + y_d + \Lambda_s^*}.
\]

(A.41)

Suppose \( \Lambda_s^* \to \infty \) as \( \lambda \to \infty \). From (A.41), \( \lim_{\lambda \to \infty} \frac{\Lambda_s^*}{\lambda} = \frac{y_u}{y_u + y_d} a - 1 \) and so, \( 1 < \frac{y_u}{y_u + y_d} a \). Now, suppose \( \Lambda_s^* \to C \in (0, \infty) \) as \( \lambda \to \infty \). From (A.41), \( \frac{y_u}{y_u + y_d}(a - 1) = \frac{y_d}{y_u + y_d y_u + y_d + C} \) and so, \( 1 \geq \frac{y_u}{y_u + y_d} a \).

2a) First observe that if \( \Lambda_s \to \Lambda_s^* < \infty \), then \( L^* = 0 \) by (4.16). Thus, whenever \( L^* > 0 \), \( \Lambda_s \to \infty \) and from (4.5), \( \Lambda_b^* < \infty \).

Taking the limit of (4.16) and (4.17) as \( \lambda \to \infty \) when \( L^* \in (0, 1] \) and \( \Lambda_s \to \infty \), we get

\[
\lim_{\lambda \to \infty} \frac{\Lambda_s}{\lambda} = \frac{y_u(a - 1)}{y_u + y_d} - \frac{y_d}{y_u + y_d} L^* - \frac{(1 - \alpha) y_d^2 \xi}{(y_u + y_d) k} - \frac{e^{-L^*}}{\rho} \int_0^{\min\{1-L^*, (1-\alpha) y_d^2 \xi e^{-\frac{L^*}{k}}\}} \frac{\rho(1-s) ds}{\rho - rs}.
\]

(A.43)

Further, it follows from (A.30) and (A.36) that \( \varepsilon^* < 1, \varpi^* < 1 \) whenever \( L^* > 0 \). If \( L^* = 1 \), then from (A.42) and (A.43), \( \frac{y_u}{y_u + y_d} a - 1 \geq \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\varepsilon^*} - 1} \right) - 1 \right) > 0 \). If \( L^* \in (0, 1) \), then from (A.42) and (A.43),

\[
\frac{y_u}{y_u + y_d} a - 1
\]

\[
= \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\varepsilon^*} - 1} \right) - 1 \right) - \frac{y_d}{y_u + y_d} (1 - L^*) - \frac{y_d}{y_u + y_d} \left( 1 - L^* - \int_0^{\min\{1-L^*, (1-\alpha) y_d^2 \xi e^{-\frac{L^*}{k}}\}} \frac{\rho(1-s) ds}{\rho - rs} \right)
\]

\[
\leq \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\varepsilon^*} - 1} \right) - 1 \right) - \frac{y_d}{y_u + y_d} (1 - L^* - \int_0^{\min\{1-L^*, (1-\alpha) y_d^2 \xi e^{-\frac{L^*}{k}}\}} \frac{\rho(1-s) ds}{\rho - rs} \right)
\]

\[
\leq \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\varepsilon^*} - 1} \right) - 1 \right) - \frac{y_d}{y_u + y_d} \left( 1 - L^* - \int_0^{1-L^*} \frac{(1-\alpha) y_d^2 \xi e^{-\frac{L^*}{k}} e^{-\frac{L^*}{k}} - \varpi^*}{\rho - \varpi^*} dz \right)
\]

\[
\leq \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\varepsilon^*} - 1} \right) - 1 \right) - \frac{y_d}{y_u + y_d} \left( 1 - L^* - \int_0^{1-L^*} \frac{(1-\alpha) y_d^2 \xi e^{-\frac{L^*}{k}} e^{-\frac{L^*}{k}} - \varpi^*}{\rho - \varpi^*} dz \right)
\]

\[
\leq \frac{y_d}{\rho} \left( \frac{\xi r}{k} \left( e^{\frac{k}{\varepsilon^*} - 1} \right) - 1 \right).
\]
2b) Taking the limit of (4.17) as $\lambda \to \infty$ when $L^{**} = 0$ and $\Lambda^{**} < \infty$,

$$
y_u(a - 1) = \frac{(1 - \alpha) y_d^2 \xi}{k} \frac{\Lambda^{**}_s}{\rho + \Lambda^{**}_s} \int_0^{\min \left\{\frac{\rho + \Lambda^{**}_s}{\rho + \Lambda^{**}_s}, \frac{1 - \alpha}{\rho + \Lambda^{**}_s}\right\}} \frac{\Lambda^{**}_s(1 - s)}{y_u + y_d + \Lambda^{**}_s \left(1 - \frac{r}{\rho}s\right)} ds. \tag{A.44}
$$

It follows from (A.30) and (A.36) that $\bar{a}^{**} = \bar{a}^{**} = 1$. It follows from (4.5) and (4.16) that

$$
\lim_{\lambda \to \infty} \Lambda_b = \lim_{\lambda \to \infty} \frac{k \rho L}{e^{k \rho \lambda} - 1 - \frac{k}{k}L} = \frac{k \rho L}{e^{k \rho \lambda} - 1} = \infty,
$$

where we used l'Hospital rule and $\lim_{\lambda \to \infty} L = 0$. Note that

$$
y_d \int_0^{\min \left\{\frac{\rho + \Lambda^{**}_s}{\rho + \Lambda^{**}_s}, \frac{1 - \alpha}{\rho + \Lambda^{**}_s}\right\}} \frac{\Lambda^{**}_s(1 - s)}{y_u + y_d + \Lambda^{**}_s \left(1 - \frac{r}{\rho}s\right)} ds < y_d,
$$

and so, if (A.44) has a solution, then $\frac{y_u}{y_u + y_d} a < 1$ and (A.44) does not have a solution when $\frac{y_u}{y_u + y_d} a \geq 1$. It follows from Lemma 3 that $|\Theta_I| = 1 - |\Theta_L| - |\Theta_M| = 1 - L - (\bar{a} - \bar{a}) \to 1 - \bar{a}^{**}$ as $\lambda \to \infty$ and by (A.33), $\bar{a}^{**} = \min \{1, (1 - \alpha) \frac{\mu + \xi}{k} \frac{\Lambda^{**}_s}{\rho + \Lambda^{**}_s}\} < \min \{1, \frac{\mu + \xi}{k} \}$ which proves $|\Theta_I| > 1 - \frac{\mu + \xi}{k}$.

2c) The last statement follows from the fact that $L$ is decreasing in $\lambda$ proven in Proposition 3.

The following lemma shows that the expression for $I$ in (4.18) is greatly simplified when $\Theta_I \neq \emptyset$, which allows for a clean derivation of the comparative statics that follows.

**Model with Two Asset Classes**

**Lemma 10.** An increase in $a$ leads to an increase in $L, X$ and $\Lambda_s$ and a decrease in $\bar{a}$.

**Proof.** An increase in $a$ leads to an upward shift of $\Lambda^*_s$ and so, an increase in $\Lambda_s$ and $L$. By (A.30) and (A.36), it leads to a decrease in $\bar{a}$ and increase in $X$. \hfill \Box

**Lemma 11.** There exists a unique two-class equilibrium.

**Proof.** Equilibrium quantities $(\Lambda_{s,1}, \bar{a}_1)$ and $(\Lambda_{s,2}, \bar{a}_2)$ in two classes are determined by the unique solution to the system (A.39) with $a = a_1$ and $a = a_2$, respectively. Denote by $\bar{a}(a)$ the equilibrium threshold of the buyer’s strategy given that the mass of investors is $a$. Equations in the system (A.39) are continuous in parameters and so, the solution $(\Lambda_s, \bar{a})$ varies continuously with $a$ and an increase in $a$ leads to a decrease in $\bar{a}$ by Lemma 10. Thus, $\bar{a}(\cdot)$ is continuous and decreasing in $a$. By (6.1), $a_1$ is determined by $\bar{a}(a_1) = \bar{a}(a - a_1)$ which has a unique solution. \hfill \Box

**Proof of Proposition 6.** Suppose $k_1$ increases. We show that as a result $a_1$ decreases and $a_2$ increases. Suppose to contradiction that $a_1$ increases and $a_2$ decreases. By Propositions 3 and
Lemma 10, $x_1$ decreases and $x_2$ increases which contradicts the indifference of buyers (6.1). By Lemma 10, the drop in $L_1$ resulting from a decrease in $k_1$ is larger because of the additional effect of the increase in $a_1$. The argument for an increase in $\lambda_2$ and a decrease in $k_2$ is analogous.

References


Online Appendix A: Proofs for Microfoundation
(Not for Publication)

Example of \( F \)

We provide an example of a sequence of \( F \) approximating \( F^* \) that satisfies our assumptions. Fix \( \gamma > 0 \). Suppose \( \theta \) normally distributed with zero mean and variance \( \gamma^2 - \frac{1}{\gamma} \), and \( \varepsilon_b \) and \( \varepsilon_s \) are independent normals with zero mean and variance \( \frac{1}{\gamma} \). Let \( F \) be the distribution of \( (\theta + \varepsilon_s, \theta + \varepsilon_b) \) conditional on \( (\theta + \varepsilon_s, \theta + \varepsilon_b) \in [0, 1] \).

**Proposition 7.**

1. \( F \) is affiliated;
2. \( F \xrightarrow{p} F^* \) as \( \gamma \to \infty \);
3. for any \( \varepsilon > 0 \), there is \( \gamma \) so that for all \( \gamma > \gamma \),

\[
\sup_{(\theta^s, \theta^b): |\theta^s - \theta^b| > \varepsilon} \max\{f(\theta^b|\theta^s), f(\theta^s|\theta^b)\} < \varepsilon. \quad (45)
\]

**Proof.**

1) By definition, \( F \) is a bivariate normal distribution with zero mean and covariance matrix \( \Sigma = \begin{pmatrix} \gamma^2 & \gamma^2 - \frac{1}{\gamma} \\ \gamma^2 - \frac{1}{\gamma} & \gamma^2 \end{pmatrix} \) conditional on \( (\theta^s, \theta^b) \in [0, 1]^2 \). Since the density of the positively correlated bivariate normal distribution is log-supermodular so is \( f \). Thus, \( F \) is affiliated.

2) The density \( f \) is given by

\[
f(\theta^s, \theta^b) = \frac{\exp\left(-\frac{(\theta^s)^2 + (\theta^b)^2 - 2(1 - \gamma^{-3})\theta^s\theta^b}{2\gamma^2(1 - (1 - \gamma^{-3})^2)}\right)}{\int_0^1 \int_0^1 \exp\left(-\frac{(\theta^s)^2 + (\theta^b)^2 - 2(1 - \gamma^{-3})\theta^s\theta^b}{2\gamma^2(1 - (1 - \gamma^{-3})^2)}\right) d\theta^s d\theta^b}.
\]

After the change of variables \( x = \theta^s - \theta^b, y = \theta^s\theta^b \), we have

\[
f(x, y) = \frac{\exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2 - \gamma^{-3})}\right) dx dy}{2\int_0^1 \int_0^{1-x} \exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2 - \gamma^{-3})}\right) dx dy}.
\]

We next construct upper and lower bounds on the nominator and the denominator of \( f \). For the nominator, the bounds are

\[
\exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2 - \gamma^{-3})}\right) \leq \exp\left(-\frac{\gamma x^2 + 2\gamma^{-2}y}{2(2 - \gamma^{-3})}\right) \leq \exp\left(-\frac{\gamma x^2}{2}\right).
\]
For the denominator, for any \( \varepsilon_0 \in (0, \frac{1}{2}) \) the upper bound is

\[
\int_0^1 \int_0^{1-x} \exp \left( -\frac{\gamma x^2 + 2 \gamma^{-2} y}{2(2 - \gamma^{-3})} \right) \frac{dy}{\sqrt{x^2 + 4y}} \leq \int_0^1 \int_0^{1-x} \exp \left( -\frac{\gamma x^2}{5} \right) \frac{dy}{\sqrt{x^2 + 4y}} \\
= \int_0^1 \exp \left( -\frac{1}{5} \gamma x^2 \right) (1 - x) \, dx \\
\leq \int_0^1 \exp \left( -\frac{1}{5} \gamma x^2 \right) \, dx \\
\leq \exp \left( -\frac{1}{5} \gamma^{2\varepsilon_0} \right) (1 - \gamma^{-1/2 + \varepsilon_0}) + \gamma^{-1/2 + \varepsilon_0} \\
\leq c_1 \gamma^{-1/2 + \varepsilon_0},
\]

and the lower bound is

\[
\int_0^1 \int_0^{1-x} \exp \left( -\frac{\gamma x^2 + 2 \gamma^{-2} y}{2(2 - \gamma^{-3})} \right) \frac{dy}{\sqrt{x^2 + 4y}} \geq \int_0^1 \int_0^{1-x} \exp \left( -\frac{1}{3} \gamma x^2 - \frac{2}{3} \gamma^{-2} \right) \frac{dy}{\sqrt{x^2 + 4y}} \\
= \int_0^1 \exp \left( -\frac{1}{3} \gamma x^2 - \frac{2}{3} \gamma^{-2} \right) (1 - x) \, dx \\
\geq \gamma^{-1/2} \exp \left( -\frac{2}{3} \right) (1 - \gamma^{-1/2}) \\
\geq c_2 \gamma^{-1/2}.
\]

Thus,

\[
\frac{1}{c_1} \gamma^{1/2 - \varepsilon} \exp \left( -\frac{1}{3} \gamma x^2 - \frac{2}{3} \gamma^{-2} \right) \leq f(x, y) \leq \frac{1}{c_2} \gamma^{1/2} \exp \left( -\frac{1}{5} \gamma x^2 \right),
\]

and so,

- for all \( |x| > \gamma^{-1/4} \), \( f(x, y) < \frac{1}{c_2} \gamma^{1/2} \exp \left( -\frac{1}{5} \gamma^{1/2} \right) \rightarrow 0 \)
- for all \( |x| < \gamma^{-1} \), \( f(x, y) > \frac{1}{c_1} \gamma^{1/2 - \varepsilon_0} \exp \left( -\frac{1}{3} \gamma^{-1} - \frac{2}{3} \gamma^{-2} \right) \rightarrow \infty \)
- for any \( x \),

\[
1 \leq \frac{\max_{y \in [0, |x|]} f(x, y)}{\min_{y \in [0, |x|]} f(x, y)} = \frac{\max_{y \in [0, |x|]} \exp \left( -\frac{\gamma x^2 + 2 \gamma^{-2} y}{2(2 - \gamma^{-3})} \right)}{\min_{y \in [0, |x|]} \exp \left( -\frac{\gamma x^2 + 2 \gamma^{-2} y}{2(2 - \gamma^{-3})} \right)} \leq \exp \left( \frac{\gamma^{-2}}{2 - \gamma^{-3}} \right) \rightarrow 1.
\]

This implies that \( F \overset{p}{\rightarrow} F^* \) as \( \gamma \rightarrow \infty \).
3) For $|\theta^s - \theta^b| > \gamma^{-1/4}$,

$$f(\theta^s|\theta^b) = \int_0^1 f(\theta^s, \theta^b) d\theta^s = \frac{\exp\left(-\frac{(\theta^s)^2}{2\gamma^2(1-\gamma^{-3})}\right)}{\int_0^1 \exp\left(-\frac{(\theta^s)^2}{2\gamma^2(1-\gamma^{-3})}\right) d\theta^s} \leq \frac{1}{c_2} \gamma^{1/2} \exp\left(-\frac{1}{5} \gamma (\theta^s - \theta^b)^2\right) \leq \frac{c_1}{2c_2} \gamma^{1/4+\varepsilon_0} \exp\left(-\frac{1}{5} \gamma x^2 + \frac{3}{5} \gamma^{-1/2} + \frac{2}{5} \gamma^{-2}\right) \rightarrow 0,$$

and the symmetric argument holds for $f(\theta^b|\theta^s)$. Thus, (45) obtains. \qed

**Proof of Part 1 of Theorem 1**

Denote $\overline{\alpha} = \frac{1-e^{-\rho\Delta}}{1-e^{-\rho\Delta}}, \quad \underline{\alpha} = \frac{e^{-\rho\Delta} - e^{-\rho\Delta^*}}{1-e^{-\rho\Delta}}$, $\overline{p}(\theta^s, \theta^b) = (1-\overline{\alpha})v(\theta^b) + \overline{\alpha}c(\theta^s)$, and $\underline{p}(\theta^s, \theta^b) = (1-\underline{\alpha})v(\theta^b) + \underline{\alpha}c(\theta^s)$. Let $\underline{v}$ and $\overline{v}$, resp., be the minimum and maximum on $[0,1]$ of derivatives of $v$. We first derive the following bounds on the price of trade.

**Lemma 12.** Suppose after some history, the highest remaining types of the buyer and seller equal to $\hat{\theta}^b$ and $\hat{\theta}^s$, resp., and the lowest remaining types of the buyer and seller equal to $\tilde{\theta}^b$ and $\tilde{\theta}^s$, resp. Suppose $\hat{\theta}^b > \tilde{\theta}^b$ and $\hat{\theta}^s > \tilde{\theta}^s$. Then

$$\underline{p}(\hat{\theta}^s, \hat{\theta}^b) \leq p_{F,\Delta}(\theta^s, \theta^b) \leq \overline{p}(\hat{\theta}^s, \hat{\theta}^b).$$

Moreover, any offer below $\underline{p}(\hat{\theta}^s, \hat{\theta}^b)$ and any offer above $\overline{p}(\hat{\theta}^s, \hat{\theta}^b)$ is accepted with probability 1 by the buyer and seller, resp.

**Proof.** Let $P$, resp. $Q$, be the supremum over all histories of price offers accepted by the buyer, resp. rejected by the seller, with positive probability in PBE. By the definition of $Q$, after any history the buyer’s type $\theta^b$ can guarantee himself the utility arbitrarily close to $e^{-\rho\Delta^*}(v(\theta^b) - Q)$ by making an offer marginally above $Q$ (that is guaranteed to be accepted by the seller). By the definition of $P$,

$$e^{-\rho\Delta^*}(v(\theta^b) - Q) \leq v(\theta^b) - P. \quad (47)$$

Let $U(\theta^s)$ be the supremum over all histories of the continuation utilities of the seller’s type $\theta^s$ after the rejection of the offer in the current round. If type $\theta^s$ rejects an offer, she cannot guarantee more than $\max\{e^{-\rho\Delta_b}(P-c(\theta^s)), e^{-\rho\Delta}U(\theta^s)\}$, which implies $U(\theta^s) \leq e^{-\rho\Delta^*}(P-c(\theta^s))$. 


By the definition of $Q$,
\[
Q - c(\theta^s) \leq e^{-\rho \Delta_b} (P - c(\theta^s)). \tag{.48}
\]

By (.48), $Q \leq P$. Combining (.47) and (.48), we get
\[
P \leq (1 - e^{-\rho \Delta_1}) v(\theta^b) + e^{-\rho \Delta_b} Q \\
\leq (1 - e^{-\rho \Delta_1}) v(\theta^b) + e^{-\rho \Delta_b} (1 - e^{-\rho \Delta_1}) c(\theta^s) + e^{-\rho \Delta} P,
\]
where we used the fact that the support of beliefs does not expand to put an upper bound on $v(\theta^b)$ and $c(\theta^s)$ in (.47) and (.48). By iterating the inequality (.49), we obtain the upper bound in (.46). By the definition of $Q$, any offer above $\bar{p}(\hat{\theta}^s, \hat{\theta}^b)$ is accepted with probability one by the seller. The argument for the lower bound is symmetric. \hfill \Box

Denote $D = \{ (\theta^s, \theta^b) : |\theta^s - \theta^b| < \frac{1}{4} \varepsilon^2 \}$, $\Omega = \{ (\theta^s, \theta^b) : |p_{F, \Delta}(\theta^s, \theta^b) - p(\theta^s, \theta^b)| < \frac{1}{4} \varepsilon^2 \}$, and
\[
\Theta = \Omega \cap D.
\]
Fix $\varepsilon > 0$. For $F$ sufficiently far in the sequence, $\mathbb{P}_F(D) > 1 - \frac{1}{4} \varepsilon^2$. Moreover, for any such $F$ and $\Delta$ sufficiently small, there is a PBE in $\mathcal{G}(F, \Delta)$ such that $\mathbb{P}_F(\Omega) > 1 - \frac{1}{4} \varepsilon^2$. Define $B^{1-\varepsilon, s} = \{ \theta^s : \mathbb{P}_F(\theta^s) > 1 - \varepsilon \}$ and $B^{1-\varepsilon, b} = \{ \theta^b : \mathbb{P}_F(\theta^b) > 1 - \varepsilon \}$.

**Lemma 13.** For any interval $I$ such that $|I| < \varepsilon$, $I \cap B^{1-\varepsilon, s} \neq \emptyset$ and $I \cap B^{1-\varepsilon, b} \neq \emptyset$.

**Proof.** We show that $I \cap B^{1-\varepsilon, s} \neq \emptyset$ ($I \cap B^{1-\varepsilon, b} \neq \emptyset$ is analogous). Let $F^s$ be the marginal of $F$ on $\theta^s$. First, we show that $\mathbb{P}_{F^s}(B^{1-\varepsilon, s}) > 1 - \frac{\varepsilon}{2}$. Note that
\[
\mathbb{P}_F(\Theta) = \mathbb{P}_F(\Omega) + \mathbb{P}_F(D) - \mathbb{P}_F(\Omega \cup D) \geq \mathbb{P}_F(\Omega) + \mathbb{P}_F(D) - 1 > 1 - \frac{\varepsilon^2}{2}.
\]

Now,
\[
\mathbb{P}_F(\Theta) = \int_0^1 \mathbb{P}_F(\theta^s) dF^s(\theta^s) \leq (1 - \varepsilon)(1 - \mathbb{P}_{F^s}(B^{1-\varepsilon, s})) + \mathbb{P}_{F^s}(B^{1-\varepsilon, s}) = 1 - \varepsilon + \varepsilon \mathbb{P}_{F^s}(B^{1-\varepsilon, s})
\]
and so, $\mathbb{P}_{F^s}(B^{1-\varepsilon, s}) \geq 1 - \frac{1}{\varepsilon} (1 - \mathbb{P}_F(\Theta)) > 1 - \frac{\varepsilon}{2}.$

Note that $F \overset{p}{\Rightarrow} F^* \Rightarrow F \overset{d}{\Rightarrow} F^* \Rightarrow F^s \overset{d}{\Rightarrow} F^{ss}$. Since $F^s$ and $F^{ss}$ are continuous, they converge uniformly as functions of $\theta^s$. Thus, for $F$ far enough in the sequence $|F^s(\theta^s) - F^{ss}(\theta^s)| < \frac{\varepsilon}{4}$ for all $\theta^s$. Let $I = [\hat{\theta}^s, \hat{\theta}^s]$. By the triangular inequality,
\[
|\mathbb{P}_{F^s}(I) - \mathbb{P}_{F^{ss}}(I)| \leq |F^s(\hat{\theta}^s) - F^{ss}(\hat{\theta}^s)| + |F^s(\hat{\theta}^s) - F^{ss}(\hat{\theta}^s)| \leq \frac{\varepsilon}{2},
\]
and so, $\mathbb{P}_{F^s}(I) \in [I - \frac{\varepsilon}{2}, I + \frac{\varepsilon}{2}]$. Therefore, $\mathbb{P}_{F^s}(I \cap B^{1-\varepsilon, s}) \geq \mathbb{P}_{F^s}(I) + \mathbb{P}_{F^s}(B^{1-\varepsilon, s}) - 1 \geq |I| - \varepsilon > 0$ which proves $I \cap B^{1-\varepsilon, s} \neq \emptyset$. \hfill \Box

Let $[\xi_n, \bar{\xi}_n] = [b_n, \bar{b}_n] = [0, 1]$. For $n = 1, \ldots, N$ (N to be specified below), define the nested intervals of seller’s types $[\xi^*_n, \bar{\xi}^*_n]$ as follows. For given $\xi^*_{n-1}, \bar{\xi}^*_{n-1}$, let $\mathcal{S}^*_n$ be the collection of all
intervals \([\underline{s}_{n-1}, \overline{s}_{n-1}]\) such that

- before round \(n\), seller’s types in \([\underline{s}_n, \overline{s}_n]\) pool with types in \([\underline{s}_{n-1}, \overline{s}_{n-1}]\), and in round \(n\), they pool with each other and separate from types in \([\underline{s}_{n-1}, \overline{s}_{n-1}]\) \(\setminus [\underline{s}_n, \overline{s}_n]\), and (since players use interval strategies, \([\underline{s}_n, \overline{s}_n]\) is well defined);
- \(\underline{s}_n < 1 - 2\varepsilon\).

Let \([\underline{s}_n^*, \overline{s}_n^*]\) be the set in \(S_n^*\) such that \(\underline{s}_n^* > \underline{s}_n\) for all intervals \([\underline{s}_n, \overline{s}_n]\) \(\in S_n^*\). Analogously, for \(n = 1, \ldots, N\), define the nested intervals of buyer’s types \((\underline{b}_n^*, \overline{b}_n^*)\) as follows. For given \((\underline{b}_{n-1}^*, \overline{b}_{n-1}^*)\), let \(B_n^*\) be the collection of all \((\underline{b}_n, \overline{b}_n)\) of seller’s types that satisfy

- buyer’s types in \((\underline{b}_n, \overline{b}_n]\) pool with each other in round \(n\) and pool with types in \((\underline{b}_{n-1}, \overline{b}_{n-1}]\) before round \(n\);
- \(\overline{b}_n > 2\varepsilon\).

Let \((\underline{b}_n^*, \overline{b}_n^*)\) be the set in \(B_n^*\) such that \(\overline{b}_n^* < \overline{b}_n\) for all \((\underline{b}_n, \overline{b}_n) \in B_n^*\). Let round \(N\) be the first round \(n\) in which either \(\frac{1}{3} \leq \underline{s}_n^*\) or \(\overline{b}_n^* \leq \frac{2}{3}\).

**Lemma 14.** For \(\varepsilon\) sufficiently small and any \(n < N\),

1. \(\overline{s}_n^* \geq 1 - 2\varepsilon\) and \(b_n^* \leq 2\varepsilon\);
2. \(\overline{s}_n^* - \underline{s}_n^* > \frac{1}{3}\) and \(\overline{b}_n^* - b_n^* > \frac{1}{3}\);
3. types in \((\underline{s}_n^*, \overline{s}_n^*)\) and \((\underline{b}_n^*, \overline{b}_n^*)\) reject the opponent’s offer and make some counter-offers \(p_n^{s*}\) and \(p_n^{b*}\), resp.;
4. there is a positive constant \(C\) (independent of \(n\)) such that

\[
p_b^{b*} \leq \mathbf{p}(\frac{1}{3}) + C\varepsilon; \quad (50)
\]
\[
p_s^{b*} \geq \mathbf{p}(\frac{2}{3}) - C\varepsilon. \quad (51)
\]

**Proof.** 1,2) Fix \(n < N\). If \(\overline{s}_n^* < 1 - 2\varepsilon\), then there is \([\underline{s}_n, \overline{s}_n]\) such that \(\underline{s}_n = \overline{s}_n^* > \underline{s}_n^*\), which contradicts the definition of \([\underline{s}_n^*, \overline{s}_n^*]\). Thus, \(\overline{s}_n^* \geq 1 - 2\varepsilon\), and analogously, \(b_n^* \leq 2\varepsilon\). Since \(\underline{s}_n^* < \frac{1}{3}\) by the definition of \(N\), \(\overline{s}_n^* - \underline{s}_n^* \geq \frac{2}{3} - 2\varepsilon > \frac{1}{3}\) (when \(\varepsilon < \frac{1}{6}\)), and analogously, \(\overline{b}_n^* - b_n^* > \frac{1}{3}\).

3) Suppose before round \(n\), both players pool with \([\underline{s}_n^*, \overline{s}_n^*]\) and \((\underline{b}_{n-1}^*, \overline{b}_{n-1}^*)\), resp., and suppose to contradiction that seller’s types in \([\underline{s}_n^*, \overline{s}_n^*]\) accept \(p_n^{b*}\). Let \(\hat{\theta}^s = \text{sup}\{\frac{1}{3}, \frac{2}{3} - \frac{1}{2}\varepsilon^2\} \cap B^{1-\varepsilon, s}\) and \(\hat{\theta}^s = \text{inf}\{\frac{1}{3}, \frac{2}{3} - \frac{1}{2}\varepsilon^2\} \cap B^{1-\varepsilon, s}\) (these types also accept \(p_n^{b*}\) as \([\hat{\theta}^s, \hat{\theta}^s]\) \(\subseteq [\underline{s}_n^*, \overline{s}_n^*]\)).

We first show that for some \(c_0\), \(|\mathbf{p}(\hat{\theta}^s) - p_n^{b*}\| \leq c_0\varepsilon\) for \(\hat{\theta}^s \in \{\hat{\theta}^s, \hat{\theta}^s\}\). Consider \(\hat{\theta}^s \in \{\hat{\theta}^s, \hat{\theta}^s\}\). Since \(\hat{\theta}^s \in B^{1-\varepsilon, s}\), type \(\hat{\theta}^s\) assigns probability at least \(1 - \varepsilon\) to \(\Theta\). Note that \([\hat{\theta}^s - \frac{1}{4}\varepsilon^2, \hat{\theta}^s + \frac{1}{4}\varepsilon^2]\) \(\subseteq \Theta\). Consider \(\hat{\theta}^s \in \{\hat{\theta}^s, \hat{\theta}^s\}\). Since \(\hat{\theta}^s \in B^{1-\varepsilon, s}\), type \(\hat{\theta}^s\) assigns probability at least \(1 - \varepsilon\) to \(\Theta\). Note that \([\hat{\theta}^s - \frac{1}{4}\varepsilon^2, \hat{\theta}^s + \frac{1}{4}\varepsilon^2]\) \(\subseteq \Theta\).
Proof. There are two cases depending on whether $\theta^s$ accepts $p_{n-1}^{bs}$ and $\theta^s \in B^{1-\varepsilon, s}$, it is necessary

$$p_{n-1}^{bs} \in \left[\frac{1-\varepsilon}{1-\beta}(p(\theta^s) - \frac{1}{4}\varepsilon^2) - \frac{1}{4}\varepsilon^2) + \frac{\varepsilon}{\beta}p(0), \frac{1-\varepsilon}{2}(p(\theta^s) + \frac{1}{4}\varepsilon^2) + \frac{\varepsilon}{2}\beta p(0)\right] \subseteq \left[(1-\varepsilon)p(\theta^s) - \frac{1}{4}\varepsilon^2) + \varepsilon p(0), (1-\varepsilon)p(\theta^s) + \frac{1}{4}\varepsilon^2) + \varepsilon p(1)\right] \quad (.52)$$

and so, $|p(\theta^s) - p_{n-1}^{bs}| \leq c_0\varepsilon$ where $c_0$ is some positive constant. Thus,

$$|\ell(\theta^s - \tilde{\theta}^s)\leq p(\tilde{\theta}^s) - p(\tilde{\theta}^s) \leq |p(\tilde{\theta}^s) - p_{n-1}^{bs}| + |p(\tilde{\theta}^s) - p_{n-1}^{bs}| \leq 2c_0\varepsilon$$

and so, $\tilde{\theta}^s - \tilde{\theta}^s \leq 2c_0\varepsilon$. On the other hand, by Lemma 13, $\tilde{\theta}^s - \tilde{\theta}^s \geq \frac{1}{\ell} - 3\varepsilon$ and so, for $\varepsilon < \frac{3\varepsilon}{3(3\varepsilon + 2c_0)}$ this leads to a contradiction. Therefore, types in $[s^*, s^\eta]$ reject $p_{n-1}^{bs}$ and make a counter-offer $p_{n-1}^{bs}$. The argument is analogous for the buyer.

4) Consider type $\theta^s$ defined above. By Lemma 13, $\tilde{\theta}^s < \frac{1}{\ell} + \varepsilon$. By (.52),

$$p_{n-1}^{bs} \leq (1-\varepsilon)p(\tilde{\theta}^s + \frac{1}{4}\varepsilon^2) + \varepsilon p(1) \leq (1-\varepsilon)p(\tilde{\theta}^s + \frac{1}{4}\varepsilon^2) + \varepsilon p(1) \leq \frac{1}{3} + C\varepsilon,$$

where $C$ is some constant. \hfill $\square$

Lemma 15. For sufficiently small $\varepsilon$, one of the two obtains:

- there is the seller’s type $\tilde{\theta}^b \in (0, 4\varepsilon) \cap B^{1-\varepsilon, b}$ and the seller’s strategy that guarantees the expected utility at least $\left(1 - \varepsilon\right)e^{-\rho \Delta(N+1)}(p(\frac{1}{3}, 0) - c(\tilde{\theta}^b))$ in the beginning of the game;

- there is the buyer’s type $\tilde{\theta}^b \in (1 - 4\varepsilon, 1) \cap B^{1-\varepsilon, b}$ and the buyer’s strategy that guarantees the expected utility at least $\left(1 - \varepsilon\right)e^{-\rho \Delta(N+1)}(v(\tilde{\theta}^b) - p(\frac{1}{3}, 0))$ in the beginning of the game.

Proof. There are two cases depending on whether $\frac{1}{6} \leq s^* \leq \frac{2}{3}$ and $\frac{1}{3} > \frac{2}{3}$. \hfill $\square$

**Case 1:** $\frac{1}{3} \leq s^*$. There are two possibilities:

1. First, suppose that there is $\eta \leq N$ and an interval of seller’s types $[s^\eta, \eta^\eta] \subset [s^*_n, s^\eta_{n-1}]$ with $s^*_N \leq \eta$ that reject $p_{\eta-1}^{bs}$ and make a counter-offer $\tilde{\theta} > p(s^*_N, 0)$ in round $\eta$. By Lemma 12, if such a counter-offer in round $\eta$ is rejected, then the seller can guarantee to trade at price $p(s^\eta, 0)$ in round $\eta + 1$. Consider the following strategy of the seller:

- as long as the buyer pools with types in $[b^*_{n-1}, b^*_{n-1}]$, the seller pools with types $[s^*, s^\eta]$ for $n \leq \eta$, pools with $[s^\eta, s^\eta]$ in round $\eta$ and, if in round $n$, offer $\tilde{\theta}$ is rejected, offers $p(s^\eta, 0)$ in round $\eta + 1$;

- otherwise, the seller rejects all offers and makes unacceptable offers above $p(1)$. 


Type $\tilde{\theta} \in (2\varepsilon + \frac{1}{4}\varepsilon^2, 4\varepsilon) \cap B^{1-\varepsilon,s}$ (this set is non-empty by Lemma 13) assigns probability at least $1 - \varepsilon$ to the buyer’s type belonging in $[\theta^{*}_{\eta-1}, \tilde{\theta}^{*}_{\eta-1})$, as $[\theta^{*} - \frac{1}{4}\varepsilon^2, \tilde{\theta}^{*} + \frac{1}{4}\varepsilon^2] \subseteq [2\varepsilon, \frac{2}{3}] \subseteq [\theta^{*}_{\eta-1}, \tilde{\theta}^{*}_{\eta-1})$. Therefore, by deviating to the described strategy, type $\tilde{\theta}^{s}$ can guarantee utility

$$(1 - \varepsilon)e^{-\rho \Delta(N+1)}(v(\tilde{\theta}^{s}) - c(\tilde{\theta}^{s})) \geq (1 - \varepsilon)e^{-\rho \Delta(N+1)}(1 - \varepsilon) \geq (1 - \varepsilon)e^{-\rho \Delta(N+1)}(1 - \varepsilon)(\frac{1}{2}, 0) - c(\tilde{\theta}^{s})).$$

2. Now, suppose that for any $\eta \leq N$ and any interval of seller’s types $[s, \bar{s}] \subseteq [s^{*}_{\eta-1}, \bar{s}^{*}_{\eta-1})$ with $s^{*}_{\eta} \leq \bar{s}$ that pool with each other and separate from other types in $[s^{*}_{\eta-1}, \bar{s}^{*}_{\eta-1})$, they either accept $p^{b}_{\eta-1}$ or make a counter-offer below $p(\tilde{s}^{*}_{\eta}, 0)$ in round $\eta$. Consider the following strategy of the buyer:

- for $\eta \leq N$, as long as the seller pools with types in $[s^{*}_{\eta}, \bar{s}^{*}_{\eta})$, the buyer pools with types in $[\theta_{\eta}, \tilde{\theta}_{\eta})$, unless the seller makes an offer below $p(\tilde{s}^{*}_{\eta}, 0)$, in which case the buyer accepts it;
- otherwise, the buyer rejects all offers and makes unacceptable offers below $p(0)$.

By Lemma 13, there exists type $\tilde{\theta}^{b} \in (1 - 2\varepsilon + \frac{1}{4}\varepsilon^2, 1 - \frac{1}{4}\varepsilon^2] \cap B^{1-\varepsilon,b}$ that assigns probability at least $1 - \varepsilon$ to $\theta^{s} > 1 - 2\varepsilon$, as $[\tilde{\theta}^{b} - \frac{1}{4}\varepsilon^2, \tilde{\theta}^{b} + \frac{1}{4}\varepsilon^2] \subseteq [1 - 2\varepsilon, 1] \subseteq [s^{*}, 1]$. Therefore, by deviating to the described strategy, type $\tilde{\theta}^{b}$ can guarantee utility

$$(1 - \varepsilon)e^{-\rho \Delta(N+1)}(v(\tilde{\theta}^{b}) - \max\{p^{b}_{1}, \ldots, p^{b}_{N-1}, p(\tilde{s}^{*}_{\eta}, 0)\}) \geq (1 - \varepsilon)e^{-\rho \Delta(N+1)}(v(\tilde{\theta}^{b}) - \max\{p(\frac{1}{2}) + C\varepsilon, p(\tilde{s}^{*}_{\eta}, 0)\}),$$

where the inequality follows from Lemma 14.

Case 2: $\tilde{\theta}^{*}_{\eta} \leq \frac{2}{3}$, but $\frac{1}{3} > \tilde{s}^{*}_{\eta}$. By the symmetric argument as in case 1, one of the following holds:

- there is the seller’s type $\tilde{\theta}^{s} \in (0, 4\varepsilon) \cap B^{1-\varepsilon,s}$ and the seller’s strategy that guarantees the expected utility at least $(1 - \varepsilon)e^{-\rho \Delta(N+1)}(\min\{p(\frac{1}{2}) - C\varepsilon, p(1, \tilde{\theta})\} - c(\tilde{\theta}^{s}))$ in the beginning of the game;
- there is the buyer’s type $\tilde{\theta}^{b} \in (1 - 4\varepsilon, 1) \cap B^{1-\varepsilon,b}$ and the buyer’s strategy that guarantees the expected utility at least $(1 - \varepsilon)e^{-\rho \Delta(N+1)}(v(\tilde{\theta}^{b}) - p(1, \frac{1}{2}))$ in the beginning of the game.

This completes the proof of the lemma.
We can now prove part 1 of Theorem 1. Suppose in Lemma 15 the first case is realized (the argument is symmetric for the second case): there is a type $\tilde{\theta}^s \in (0, 4\varepsilon) \cap B^{1-\varepsilon, s}$ who is guaranteed utility $(1 - \varepsilon)e^{-r\Delta(N+1)}(p(\frac{1}{3}, 0) - c(\tilde{\theta}^s))$ in the beginning of the game. Since $\tilde{\theta}^s \in B^{1-\varepsilon, s}$, it is necessary that

$$(1 - \varepsilon)e^{-r\Delta(N+1)}(p(\frac{1}{3}, 0) - c(\tilde{\theta}^s)) \leq (1 - \varepsilon)(p(\tilde{\theta}^s + \frac{1}{4}\varepsilon^2) - c(\tilde{\theta}^s)) + \varepsilon v(1)$$

and so,

$$e^{-r\Delta N} \leq \frac{(1 - \varepsilon)(p(\tilde{\theta}^s + \frac{1}{4}\varepsilon^2) - c(\tilde{\theta}^s)) + \varepsilon v(1)}{(1 - \varepsilon)(p(\frac{1}{3}, 0) - c(\tilde{\theta}^s))} \to \frac{p(0) - c(0)}{p(\frac{1}{3}, 0) - c(0)} < 1.$$  

Therefore, when $\varepsilon$ is small, there is $\overline{T} > 0$ such that $\Delta N > \overline{T}$ and so, types in $(\frac{1}{3}, \frac{2}{3})^2$ trade with a delay at least $\overline{T}$. The probability of such types approaches $\frac{1}{2}$ as $F \rightarrow F^*$, and so there is $c_1 > 0$ such that when $\varepsilon < \frac{1}{6c_1}$,

$$\mathbb{E}_F[e^{-r\rho F, \Delta}] \leq (\frac{1}{2} + c_1\varepsilon)e^{-r\overline{T}} \leq \frac{1}{2}e^{-r\overline{T}} \equiv x_l,$$

for all $F$ sufficiently far in the sequence and all $\Delta$ sufficiently small. This proves (3.1).

To show (3.2), observe that every seller type can trade at price $p(0)$. Thus, for any $\theta^s \in B^{1-\varepsilon, s}$,

$$p(0) - c(\theta^s) \leq (1 - \varepsilon)\mathbb{E}_F[e^{-r\rho F, \Delta} | \theta^s, \Theta](p(\theta^s + \frac{1}{4}\varepsilon^2) + \frac{1}{4}\varepsilon^2 - c(\theta^s)) + \varepsilon v(1)$$

$$\leq \mathbb{E}_F[e^{-r\rho F, \Delta} | \theta^s, \Theta](1 - \alpha)\xi + c_2\varepsilon,$$

for some constant $c_2 > 0$. This implies that for any $\overline{\theta}^s$

$$\mathbb{E}_F[e^{-r\rho F, \Delta} | \theta^s, \Theta] \geq \frac{p(0) - c(\theta^s) - c_2\varepsilon}{(1 - \alpha)\xi} > \frac{p(0) - c(\theta^s) - c_2\varepsilon}{(1 - \alpha)\xi}$$

for all $\theta^s \in B^{1-\varepsilon, s}$ below $\theta^s$. By choosing $\theta^s$ and $\varepsilon$ sufficiently close to zero, we can guarantee that there is $\tilde{x} > x_l$ such that for all $\Delta$ sufficiently small, $\mathbb{E}_F[e^{-r\rho F, \Delta} | \theta^s, \Theta] \geq \tilde{x}$. Moreover,

$$\mathbb{E}_F[e^{-r\rho F, \Delta} | \theta^s < \theta^s] = \int_0^{\theta^s} \mathbb{E}_F[e^{-r\rho F, \Delta} | \theta^s] dF^s(\theta^s)$$

$$\geq \mathbb{E}_F[e^{-r\rho F, \Delta} | \theta^s, \Theta] \mathbb{P}_F(\Theta | \theta^s) dF^s(\theta^s)$$

$$\geq \tilde{x} \int_0^{\theta^s} \mathbb{P}_F(\Theta | \theta^s) dF^s(\theta^s)$$

$$\geq (1 - \varepsilon)\tilde{x} \mathbb{P}_F(s) (B_1^{1-\varepsilon, s}) \geq (1 - \varepsilon)(1 - \varepsilon)^2 \tilde{x}. $$
Thus, there is $x_h > x_l$ such that for $F$ sufficiently close to $F^*$ and all $\Delta$ sufficiently small,
\[ \mathbb{E}_F[e^{-\rho t_F \Delta} | \theta^s < \theta^b}] \geq x_h. \]

The analogous argument applied to buyer’s types close to 1 gives that there is $\theta^b > \theta^s$ such that $\mathbb{E}_F[e^{-\rho t_F \Delta} | \theta^b < \theta^b}] \geq x_h$. Observing that for fixed $\theta^b$, $|\mathbb{E}_F[e^{-\rho t_F \Delta} | \theta^b < \theta^b}] - \mathbb{E}_F[e^{-\rho t_F \Delta} | \theta^s < \theta^b}]| \to 0$ as $F \overset{p}{\to} F^*$, we get the inequality (3.2).

Proof of Part 2 of Theorem 1

The proof proceeds as follows. We first introduce and analyze the continuous-time bargaining game $\mathcal{G}(p^s, p^b|F)$ which is a generalization of the game $\mathcal{G}(p^s, p^b)$ in Section 2 to affiliated distributions of types $F$ (thus, $\mathcal{G}(p^s, p^b) = \mathcal{G}(p^s, p^b|F^*)$). We then proceed with a series of approximations. First, we approximate the SBS outcome with the BNE outcome in $\mathcal{G}(|F^*)$. Second we approximate each BNE outcome of $\mathcal{G}(|F^*)$ with BNE outcomes of $\mathcal{G}(|F), F \overset{p}{\to} F^*$. Finally, we approximate each BNE outcome of $\mathcal{G}(p^s, p^b|F)$ with PBE outcome in the discrete-time bargaining game $\mathcal{G}(F, \Delta), \Delta \to 0$. Thus, we construct a sequence of PBE frequent-offer limits (indexed by $F \overset{p}{\to} F^*$) that approximates the SBS outcome, and hence prove the theorem.

Continuous-Time Bargaining Game $\mathcal{G}(|F)$ Consider a strictly decreasing function $p^s$ and a strictly increasing function $p^b$ and the following continuous-time bargaining game $\mathcal{G}(p^s, p^b|F)$. The buyer continuously makes offers $p^b_t$ and the seller continuously makes offers $p^s_t$. Players can choose only the time when they accept the price offer of the opponent, and the trade happens once the first acceptance happens (at the accepted price). The difference from $\mathcal{G}(p^s, p^b)$ is that players’ types are distributed according to a general affiliated distribution $F$, not $F^*$. We consider BNEs in threshold strategies. Let $T$ be the first time when $p^s_t = p^b_t$. For any $t \in [0, T]$, let $\theta^s_t$ and $\theta^b_t$ be, resp., strictly increasing and strictly decreasing functions such that $\theta^s_0 = 0$ and $\theta^b_0 = 1$. At time $t$, all types of the seller below $\theta^s_t$ (resp., all types of the buyer above $\theta^b_t$) accept the offer $p^b_t$ (resp., $p^s_t$).

Lemma 16. Suppose that a tuple $(p^s, p^b, \theta^s, \theta^b)$ satisfies the system of differential equations

\[
\begin{align*}
 r(v(\theta^b_t) - p^s_t) + \dot{p}^s_t &= (p^s_t - p^b_t)\frac{f(\theta^s_t | \theta^b_t)}{1 - F(\theta^s_t | \theta^b_t)}, \\
 -r(p^b_t - c(\theta^s_t)) + \dot{p}^b_t &= (p^b_t - p^b_t)\frac{f(\theta^b_t | \theta^s_t)}{F(\theta^b_t | \theta^s_t)},
\end{align*}
\]

with initial conditions $\theta^s_0 = 0$ and $\theta^b_0 = 1$, and $\theta^s_T < 1$ and $\theta^b_T > 0$. Then threshold strategies $(\theta^s, \theta^b)$ constitute a BNE in $\mathcal{G}(p^s, p^b|F)$.

Proof. We show that if $\theta^b$ satisfies the first equation in the system (53), then it is a best response to the threshold strategy $\theta^s$. Buyer’s type $\theta^b$ chooses the acceptance time $t$ to maximize $u(\theta^b, t)$.
given by

\[ u(\theta^b, t) = \int_0^t e^{-rt}(v(\theta^b) - p_u^b) dF(\theta_u^*|\theta^b) + (1 - F(\theta_u^*|\theta^b))e^{-rt}(v(\theta^b) - p_t^b). \]

The first-order condition for this problem is

\[ (p_t^s - p_t^b)f(\theta_t^*(\theta^b))\dot{\theta}_t^b = (1 - F(\theta_t^*(\theta^b))(r(v(\theta^b) - p_t^b) + \dot{p}_t^b). \]

From the first-order condition,

\[ u(1, t(1)) - u(\tilde{\theta}^b, t(\tilde{\theta}^b)) = \int_{\tilde{\theta}^b}^1 \left( \frac{\partial}{\partial \theta^b} u(\theta^b, t(\theta^b)) + \frac{\partial}{\partial t} u(\theta^b, t(\theta^b)) t'(\theta^b) \right) d\theta^b \]

\[ = \int_{\tilde{\theta}^b}^1 \frac{\partial}{\partial \theta^b} u(\theta^b, t(\theta^b)) d\theta^b, \tag{54} \]

where \( t(\theta^b) \) is the inverse of \( \theta^b \). In Claim 5 below, we show that \( u(\theta^b, t) \) satisfies the smooth single crossing difference (SSCD) condition in \((\theta^b, -t)\). Together with the envelope formula (54), this verifies the conditions of Theorem 4.2 in Milgrom (2004) and proves that \( \hat{\theta}^b \) is a best response to \( \theta^b \). Therefore, \((\theta^s, \hat{\theta}^b)\) constitute a BNE of \( G(F) \).

**Claim 5.** \( u(\theta^b, t) \) satisfies the SSCD condition in \((\theta^b, -t)\).

**Proof:** We will show the following conditions are satisfied which imply the SSCD.

1. \( u(\theta^b, t) \) satisfies the (strict) single crossing difference condition in \((\theta^b, -t)\), i.e. for all \( \tilde{t} > t \) and \( \tilde{\theta}^b > \theta^b \),

\[ u(\theta^b, t) - u(\theta^b, \tilde{t}) \geq 0 \implies u(\tilde{\theta}^b, t) - u(\tilde{\theta}^b, \tilde{t}) > 0. \]

2. for all \( t \), if \( \frac{\partial}{\partial t} u(\theta^b, t) = 0 \), then for all \( \delta > 0 \), \( \frac{\partial}{\partial t} u(\theta^b, t - \delta) \geq 0 \) and \( \frac{\partial}{\partial t} u(\theta^b, t + \delta) \leq 0 \).

Let us start with the single crossing difference condition. Consider \( \theta^b < \tilde{\theta}^b \) and \( t < \tilde{t} \leq T \) and suppose that

\[ u(\theta^b, t) \geq u(\theta^b, \tilde{t}). \tag{55} \]

We will show that \( u(\tilde{\theta}^b, t) > u(\tilde{\theta}^b, \tilde{t}) \). Define function

\[ g(u|\theta^b, t) = e^{-ru}(v(\theta^b) - p_u^b)1\{u < t\} + e^{-rt}(v(\theta^b) - p_t^s)1\{u \geq t\}. \]

Then

\[ \int_0^T g(u|\theta^b, t)dF(\theta_u^*|\theta^b) \geq \int_0^T g(u|\theta^b, \tilde{t})dF(\theta_u^*|\theta^b) \geq \int_0^T g(u|\theta^b, t)dF(\theta_u^*|\tilde{\theta}^b), \]

where the first inequality follows from (55), the second inequality follows from the fact that
\( g(\cdot | \theta^b, \tilde{t}) \) is decreasing and \( F(\cdot | \tilde{\theta}^b) \) first-order stochastically dominates \( F(\cdot | \theta^b) \) (as \( f \) is affiliated). This implies that

\[
\begin{align*}
 u(\theta^b, t) &= \int_0^t e^{-ru} (v(\theta^b) - p_u^b) dF(\theta_u^s | \theta^b) + (1 - F(\theta_u^s | \theta^b)) e^{-rt} (v(\theta^b) - p^*_t) \\
 &\geq \int_0^t e^{-ru} (v(\theta^b) - p_u^b) dF(\theta_u^s | \theta^b) + (1 - F(\theta_u^s | \theta^b)) e^{-rt} (v(\theta^b) - p^*_t),
\end{align*}
\]

or equivalently,

\[
\begin{align*}
 v(\theta^b) \left( \int_0^t e^{-ru} dF(\theta_u^s | \theta^b) + (1 - F(\theta_u^s | \theta^b)) e^{-rt} - \int_0^{\tilde{t}} e^{-ru} dF(\theta_u^s | \tilde{\theta}^b) - (1 - F(\theta_u^s | \tilde{\theta}^b)) e^{-r\tilde{t}} \right) \\
 &\geq p^*_t - \int_0^t e^{-ru} p_u^B dF(\theta_u^s | \tilde{\theta}^b) - (1 - F(\theta_u^s | \tilde{\theta}^b)) e^{-r\tilde{t}} p^*_t. \tag{56}
\end{align*}
\]

We will show that the left-hand side of (56) is positive and so, the left-hand side would increase if we substitute \( v(\tilde{\theta}^b) \) instead of \( v(\theta^b) \). This in turn implies that \( u(\tilde{\theta}^b, t) > u(\tilde{\theta}^b, \tilde{t}) \) and completes the proof of the strict single crossing difference. Let \( h(u|t) = e^{-ru} 1 \{ u < t \} + e^{-rt} 1 \{ u \geq t \} \). Then the left-hand side of (56) is equal to

\[
\begin{align*}
 v(\theta^b) \left( \int_0^T h(u|t) dF(\theta_u^s | \theta^b) - \int_0^{\tilde{t}} h(u|\tilde{\theta}) dF(\theta_u^s | \tilde{\theta}^b) \right) \\
 &\geq v(\theta^b) \left( \int_0^T h(u|t) dF(\theta_u^s | \theta^b) - \int_0^{\tilde{t}} h(u|\tilde{\theta}) dF(\theta_u^s | \tilde{\theta}^b) \right) \\
 &= v(\theta^b) \int_0^T (h(u|t) - h(u|\tilde{\theta})) dF(\theta_u^s | \tilde{\theta}^b) > 0,
\end{align*}
\]

where the first inequality follows from \( F(\cdot | \tilde{\theta}^b) \) first-order stochastically dominates \( F(\cdot | \theta^b) \) and \( h(\cdot | t) \) decreasing, and the last term is strictly positive by \( t < \tilde{t} \).

Now, let us show the second requirement of the SSCD condition. Suppose \( \frac{\partial}{\partial t} u(\theta^b, t) = 0 \). By taking the partial derivative

\[
e^{rt} \frac{\partial}{\partial t} u(\theta^b, t) = (p_t^s - p_t^b) f(\theta_t^s | \theta^b) \dot{\theta}_t^s - (1 - F(\theta_t^s | \theta^b))(r(v(\theta^b) - p_t^s) + \dot{p}_t^s),
\]

we get that

\[
e^{rt} \frac{\partial}{\partial t} u(\theta^b - \delta, t) = \\
(p_t^s - p_t^b) f(\theta_t^s | \theta^b - \delta) \dot{\theta}_t^s - (1 - F(\theta_t^s | \theta^b - \delta))(r(v(\theta^b - \delta) - p_t^s) + \dot{p}_t^s) = \\
(1 - F(\theta_t^s | \theta^b - \delta)) \left( \frac{f(\theta_t^s | \theta^b - \delta)}{1 - F(\theta_t^s | \theta^b - \delta)} \dot{\theta}_t^s - (r(v(\theta^b - \delta) - p_t^s) + \dot{p}_t^s) \right).
\]
Since \( v(\theta^b - \delta) \leq v(\theta^b) \) and \( \frac{f(\theta^b; \theta^b - \delta)}{1 - F(\theta^b; \theta^b - \delta)} \geq \frac{f(\theta^b; \theta^b)}{1 - F(\theta^b; \theta^b)} \) (by the affiliation of \( f \)), it follows that \( \frac{\partial}{\partial t} u(\theta^b - \delta, t) \geq 0 \). Showing that \( \frac{\partial}{\partial t} u(\theta^b + \delta, t) \leq 0 \) is analogous. \( \quad \square \)

**Approximate the SBS with BNEs in \( \mathcal{G}(\cdot | F^s) \)**  The next lemma constructs price-offer paths \( p^{s, \varepsilon} \) and \( p^{b, \varepsilon} \) and BNEs in \( \mathcal{G}(p^{s, \varepsilon}, p^{b, \varepsilon} | F^s) \) that approximate the SBS outcome.

**Lemma 17.** For any \( \varepsilon \geq 0 \). Let

\[
\begin{align*}
p^{s, \varepsilon}_t &= p(0) + (1 - \alpha)\varepsilon + (1 - \alpha)r(\xi + \varepsilon)t, \\
p^{b, \varepsilon}_t &= p(1) - \alpha\varepsilon - \alpha r(\xi + \varepsilon)t, \\
\theta^{s, \varepsilon}_t &= v^{-1}(p(1) + \alpha\xi - \alpha r(\xi + \varepsilon)t), \\
\theta^{b, \varepsilon}_t &= c^{-1}(p(0) - (1 - \alpha)\xi + (1 - \alpha)r(\xi + \varepsilon)t); 
\end{align*}
\]

for \( t \leq T^\varepsilon = \frac{p(1) - p(0) - \varepsilon}{r(\xi + \varepsilon)} \). Then

1. \( (\theta^{s, \varepsilon}, \theta^{b, \varepsilon}) \) constitutes a BNE in \( \mathcal{G}(p^{s, \varepsilon}, p^{b, \varepsilon} | F^s) \);
2. the outcome of \( (\theta^{s, 0}, \theta^{b, 0}) \) in \( \mathcal{G}(p^{s, 0}, p^{b, 0} | F^s) \) coincides with the SBS outcome;
3. \( \theta^{b, \varepsilon}_T \geq \theta^{b, 0}_T = \varepsilon/\tau \);
4. the outcome of \( (\theta^{s, \varepsilon}, \theta^{b, \varepsilon}, p^{s, \varepsilon}, p^{b, \varepsilon}) \) converge uniformly to \( (\theta^{s, 0}, \theta^{b, 0}, p^{s, 0}, p^{b, 0}) \) as \( \varepsilon \to 0 \).

**Proof.** 1) \( (\theta^{s, \varepsilon}, \theta^{b, \varepsilon}, p^{s, \varepsilon}, p^{b, \varepsilon}) \) satisfy the premise of Lemma 16 and so, \( (\theta^{s, \varepsilon}, \theta^{b, \varepsilon}) \) constitutes a BNE in \( \mathcal{G}(p^{s, \varepsilon}, p^{b, \varepsilon} | F^s) \).

2) We can verify using (57) that \( \theta^{s, 0}_T = \theta^{b, 0}_T \) and

\[
\begin{align*}
p^{s, 0}_t &= v(\theta^{b, 0}_t) - \alpha \xi = p(\theta^{b, 0}_t), \\
p^{b, 0}_t &= c(\theta^{b, 0}_t) + (1 - \alpha)\xi = p(\theta^{s, 0}_t).
\end{align*}
\]

Since types \( \theta^{s, 0}_t \) and \( \theta^{b, 0}_t \) accept offers \( p^{b, 0}_t \) and \( p^{s, 0}_t \), resp., (2.1) obtains. Since threshold types \( \theta^{s, 0}_T \) and \( \theta^{b, 0}_T \) prefer to accept at time \( t \) rather than any other type \( t \leq T^0 \), (2.2) and (2.3) obtain where \( \theta^* = \theta^{s, 0}_T = \theta^{b, 0}_T \). Thus, the outcome of \( (\theta^{s, 0}, \theta^{b, 0}) \) in \( \mathcal{G}(p^{s, 0}, p^{b, 0} | F^s) \) coincides with the SBS outcome.

3) \( v(\theta^{b, \varepsilon}_T) - c(\theta^{s, \varepsilon}_T) = \xi + \varepsilon \) and so, \( v(\theta^{b, \varepsilon}_T) = v(\theta^{s, \varepsilon}_T) + \varepsilon \geq v(\theta^{b, \varepsilon}_T) - \ell(\theta^{b, \varepsilon}_T - \theta^{s, \varepsilon}_T) + \varepsilon \) which implies \( \theta^{b, \varepsilon}_T > \theta^{s, \varepsilon}_T + \varepsilon/\tau \).

4) From (57), \( (\theta^{s, \varepsilon}, \theta^{b, \varepsilon}, p^{s, \varepsilon}, p^{b, \varepsilon}) \) converge pointwise to \( (\theta^{s, 0}, \theta^{b, 0}, p^{s, 0}, p^{b, 0}) \) as \( \varepsilon \to 0 \) on a compact \( [0, T^0] \), and by the continuity of \( (\theta^{s, \varepsilon}, \theta^{b, \varepsilon}, p^{s, \varepsilon}, p^{b, \varepsilon}) \) and \( (\theta^{s, 0}, \theta^{b, 0}, p^{s, 0}, p^{b, 0}) \), the convergence is also uniform. \( \quad \square \)
Approximate the BNEs in $\mathcal{G}(\cdot|F^*)$ with BNEs in $\mathcal{G}(\cdot|F)$ For each $(\theta^{s,\varepsilon}, \theta^{b,\varepsilon})$ in $\mathcal{G}(p^{s,\varepsilon}, p^{b,\varepsilon}|F^*)$, we construct an approximating sequence of BNEs $(\theta^s, \theta^b)$ in $\mathcal{G}(p^{s,\varepsilon}, p^{b,\varepsilon}|F), F \xrightarrow{p} F^*$.\footnote{We do not explicitly index the sequence by corresponding $F$ and $\varepsilon$ and simply write $(\theta^s, \theta^b)$.}

Lemma 18. Let $T = T^s, \theta^s = \theta^{s,\varepsilon}, \theta^b = \theta^{b,\varepsilon}$ and $p^s_t, p^b_t$ be given by the differential equations (.53) with the terminal condition $p^s_T = p^s_t$ and the initial condition $p^s_0 = p^{s,\varepsilon}_0$. Then $(\theta^s, \theta^b)$ constitute BNE in $\mathcal{G}(p^s_t, p^b_t|F)$ and $(\theta^{s,\varepsilon}, \theta^{b,\varepsilon}, p^s_t, p^b_t)$ converge uniformly to $(\theta^s, \theta^b, p^s_t, p^b_t)$ as $F \xrightarrow{p} F^*$.

Proof. To prove the convergence, we show that $p^s_t, p^b_t$ converge pointwise to $p^{s,\varepsilon}_t, p^{b,\varepsilon}_t$ as $F \xrightarrow{p} F^*$.

Denote

$$
\psi_1(t) = \left(1 - \alpha \right) r(\xi + \varepsilon) \cdot \frac{f(\theta^{s,\varepsilon}_t | \theta^{b,\varepsilon}_t)}{c'(\theta^{s,\varepsilon}_t)} \cdot \frac{1}{1 - F(\theta^{s,\varepsilon}_t | \theta^{b,\varepsilon}_t)},
$$

$$
\psi_2(t) = \frac{\alpha r(\xi + \varepsilon)}{v'(\theta^{b,\varepsilon}_t)} \cdot \frac{f(\theta^{b,\varepsilon}_t | \theta^{s,\varepsilon}_t)}{F(\theta^{b,\varepsilon}_t | \theta^{s,\varepsilon}_t)}.
$$

Using (.57) to compute $\theta^{s,\varepsilon}_t$ and $\theta^{b,\varepsilon}_t$, we rewrite the system (.53) as

$$
p^s_t = p^s_t(\psi_1(t) + r) - p^b_t \psi_1(t) - r v(\theta^{b,\varepsilon}_t),
$$

$$
p^b_t = p^b_t(\psi_2(t) + r) - p^s_t \psi_2(t) - r c(\theta^{s,\varepsilon}_t),
$$

By subtracting the second equation from the first and denoting $\Delta p_t = p^s_t - p^b_t$, we get

$$
\Delta \dot{p}_t = \Delta p_t(\psi_1(t) + \psi_2(t) + r) - r (v(\theta^{b,\varepsilon}_t) - c(\theta^{s,\varepsilon}_t))
$$

with the terminal condition $\Delta p_T = 0$, which we can solve to get

$$
\Delta p_t = r \int_t^T \left( v(\theta_u^{b,\varepsilon}) - c(\theta_u^{s,\varepsilon}) \right) e^{- \int_t^u (\psi_1(s) + \psi_2(s) + r) ds} du.
$$

We can now solve for individual price-offer paths. We have

$$
\dot{p}^s_t = r p^s_t + \Delta p_t \psi_1(t) - r v(\theta^{b,\varepsilon}_t),
$$

from which we get

$$
p^s_t = e^{rt}(p(1) - \alpha \varepsilon) + \int_0^t \left( \Delta p_u \psi_1(u) - r v(\theta_u^{b,\varepsilon}) \right) e^{r(t-u)} du,
$$

$$
p^b_t = p^s_t - \Delta p_t.
$$

By Lemma 17, $\theta^b_T = \theta^{b,\varepsilon}_T > \theta^{s,\varepsilon}_T + \varepsilon \tau = \theta^s_T + \varepsilon \tau$ for all $t \leq T$, and so, for $F$ sufficiently far in the sequence, for all $t \leq T$, $\frac{f(\theta^{s,\varepsilon}_t | \theta^{b,\varepsilon}_t)}{1 - F(\theta^{b,\varepsilon}_t | \theta^{s,\varepsilon}_t)}$ and $\frac{f(\theta^{b,\varepsilon}_t | \theta^{s,\varepsilon}_t)}{F(\theta^{b,\varepsilon}_t | \theta^{s,\varepsilon}_t)}$ are bounded from above by $c_0 \varepsilon$ for some
implies that $|\psi_1(t)|$ and $|\psi_2(t)|$ converge to zero pointwise on $[0,T]$ as $F \xrightarrow{p} F^*$. Therefore, price-offer paths and their derivatives converge pointwise on $[0,T]$ as $F \xrightarrow{p} F^*$ and so, by the continuity of $p^s, p^b, p_{s,e}^s, p_{b,e}^b$ and their derivatives on the compact $[0,T]$, the convergence is also uniform.

The derivatives of $p_{s,e}^s$ and $p_{b,e}^b$ are bounded away from zero (from above and below, resp.) and so for $F$ sufficiently far in the sequence, $p^s$ and $p^b$ are strictly decreasing and increasing, resp. This together with the construction of $(\theta^s, \theta^b, p^s, p^b)$, implies that $(\theta^s, \theta^b, p^s, p^b)$ satisfies the conditions of Lemma 16 and so, $(\theta^s, \theta^b)$ constitutes a BNE in $G(p^s, p^b|F)$. □

Approximate BNEs in $G(\cdot|F)$ with PBEs in $G(F, \Delta)$ We use grim trigger strategies to construct PBEs in $G(F, \Delta)$ that approximate each BNE $(\theta^s, \theta^b)$ in $G(p^s, p^b|F)$. On the equilibrium path, the seller makes decreasing offers $q^s_n$ and the buyer makes increasing offers $q^b_n$. Offers do not depend on the type, but the acceptance of the opponent’s offer does. Specifically, players follow threshold strategies on-path: in round $n$, all types of the seller below $s_n$ accept $q^s_{n-1}$ and types above $s_n$ reject it and counter-offer $q^s_n$; all types of the buyer above $b_n$ accept $q^b_n$ and types below $b_n$ reject it and counter-offer $q^b_n$. If there is a deviation from the equilibrium path, the play switches to the punishing path (described below) that punishes the deviator.

Construction of the Equilibrium Path: We first construct on-path strategies and show that no type wants to mimic another type in the acceptance decision. We construct the discrete-time approximation of $\theta^s$ and $\theta^b$ (defined in the previous step) using the Euler method: $s_{N+1} = 1, s_N = \theta^s_T, b_N = \theta^b_T$, and for $n < N = \lceil \frac{T}{\Delta} \rceil$, $s_n = s_{n+1} - \theta^s_{(n+1)\Delta}$ and $b_n = b_{n+1} - \theta^b_{(n+1)\Delta}$. We construct price-offer paths $q^s$ and $q^b$ backwards in time starting from $N$ and $q^s_{N+1} = q^b_N = q^s_T$ as follows: for $n \leq N$,

\[
\begin{align*}
v(b_n) - q^s_n &= e^{-r\Delta_n} \frac{F(s_{n+1}|b_n) - F(s_n|b_n)}{1 - F(s_n|b_n)} (v(b_n) - q^b_n) + e^{-r\Delta_n} \frac{1 - F(s_{n+1}|b_n)}{1 - F(s_n|b_n)} (v(b_n) - q^s_{n+1}), \\
q^b_{n-1} - c(s_n) &= e^{-r\Delta_n} \frac{F(b_n|s_n) - F(b_{n-1}|s_n)}{F(b_{n-1}|s_n)} (q^s_n - c(s_n)) + e^{-r\Delta_n} \frac{F(b_n|s_n)}{F(b_{n-1}|s_n)} (q^b_n - c(s_n)).
\end{align*}
\]

Denote by $(\bar{s}, \bar{b}, \bar{q}^s, \bar{q}^b)$ the linear extrapolation of $(s, b, q^s, q^b)$ to the continuous time on $[0,T]$.

Lemma 19. $(\bar{s}, \bar{b}, \bar{q}^s, \bar{q}^b)$ converges uniformly to $(\theta^s, \theta^b, p^s, p^b)$ as $F \xrightarrow{p} F^*$. When both players use threshold strategies, no player wants to deviate to a different acceptance strategy.

Proof. To prove the first part, the convergence of $\bar{s}$ and $\bar{b}$ is by construction. Next, rewrite equation (58) as follows

\[
\frac{1 - e^{-r\Delta_n}}{\Delta_n} (v(b_n) - q^s_n) - e^{-r\Delta_n} \frac{q^s_n - q^s_{n+1}}{\Delta_n} = \frac{F(s_{n+1}|b_n) - F(s_n|b_n)}{(1 - F(s_n|b_n))} (e^{-r\Delta_n} (v(b_n) - q^b_n) - e^{-r\Delta_n} (v(b_n) - q^s_{n+1})).
\]
Since $\dot{\theta}^s$ is positive and bounded uniformly on $[0, T)$, by the construction of $s_n$, there is an upper bound $c_0$ on $\frac{1}{\Delta}|s_{n+1} - s_n|$ for $n < N$. Choose $F$ sufficiently close to $F^*$ so that $\sup(\theta^b, \theta^q; |b_c - \theta| > 0) f(\theta^q | \theta^b) < \varepsilon / \ell$. Then since $s_n \leq s_N = \theta_T^b < \theta_T^b - \varepsilon / \ell = b_N - \varepsilon / \ell \leq b_n - \varepsilon / \ell$, we have

$$0 \leq \frac{F(s_{n+1}|b_n) - F(s_n|b_n)}{(1 - F(s_n|b_n)) \Delta} \leq \frac{|s_{n+1} - s_n|\varepsilon}{\Delta(1 - \varepsilon / \ell)} \leq \frac{c_0 \varepsilon}{\ell (1 - \varepsilon / \ell)} \equiv C \varepsilon,$$

for $n < N$. Thus,

$$1 - e^{-r\Delta} \frac{\Delta}{\Delta} (v(b_n) - q_n^s) - e^{-r\Delta} \frac{q_n^s - q_{n+1}}{\Delta} = C \varepsilon$$

where $C(\varepsilon)$ is some function that is bounded in absolute value by $C \varepsilon$. This implies that $\overline{q}^s$ converges pointwise to $p^s$ as $F \xrightarrow{0} F^*$, and also uniformly by the continuity of $\overline{q}^s$ and $p^s$ on a compact $[0, T]$. Moreover, the left-derivative of $\overline{q}^s$ equals to $\frac{q_{n+1}^s - q_n^s}{\Delta}$ for some $n$, and from equation (60), it converges uniformly to $\dot{p}^s$ as $F \xrightarrow{0} F^*$. This implies that $\overline{q}^s$ is strictly decreasing for $F$ sufficiently far in the sequence. The uniform convergence of $\overline{q}^s$ and its left-derivative is proven analogously.

To prove the second part, first observe that we can follows the same line of argument as the proof of the single crossing difference condition in Lemma 16 (Claim 5) to prove the following claim.

**Claim 6.** Buyer’s types $\theta^b > b_n$ and seller’s types $\theta^s < s_n$, strictly prefer to accept in round $n - 1$ to accepting in round $n$. Buyer’s types $\theta^b < b_n$ and seller’s types $\theta^s > s_n$, strictly prefer to accept in round $n + 1$ to accepting in round $n$.

Claim 6 implies that no player wants to deviate from the threshold strategy in the acceptance decision. \(\square\)

**Construction of the Punishing Path:** We now construct punishing path for deviations to offers that are off the equilibrium path and show that such are not profitable. Suppose that the seller deviates from the price-offer path $q^s$ in round $n$ (the construction of the punishing path for the buyer is symmetric). Then specify that the buyer assigns probability 1 to the lowest remaining seller’s type, $s_{n-1}$. The next lemma states that there is a continuation equilibrium that is efficient in deterring deviations to off-path offers.

**Lemma 20 (Coasian Property).** Let $\underline{s} < \theta_T^s$ and $\overline{b} > \theta_T^b$. Suppose after some history, the buyer assigns probability 1 to type $\underline{s}$ and the seller’s beliefs are $F(\theta^b | \theta^s, \theta^b < \overline{b})$. Then for any $\varepsilon_0 > 0$ there is $\Delta$ (that does not depend on $\underline{s}$ and $\overline{b}$) such that for all $\Delta < \Delta$, there is a continuation PBE strategies in which the seller’s initial offer is below $p(\underline{s}, 0) + \varepsilon_0$.

**Proof.** We construct the continuation PBE in which the buyer’s types pool on the unacceptable offer above $\overline{p}(1)$ and the seller (with commonly known type $\underline{s}$) makes offers to screen the buyer’s
type. We can follow the step in the proof of Proposition 1 in Fudenberg et al. (1985) to construct the screening path of the seller in which the last seller’s offer equals \( p(s, 0) \). By the Uniform Coase Conjecture in Ausubel and Deneckere (1989), there is \( \Delta \) (that does not depend on \( s \) and \( b \)) such that for all \( \Delta < \Delta \), there is a PBE in which the seller’s initial offer is below \( p(s, 0) + \varepsilon \).

To guarantee that the buyer does not have incentives to make acceptable offers, we specify that if the buyer deviates from making the unacceptable offer, the seller assigns probability 1 to type \( b \). Specify that after histories in which the seller assigns probability 1 to a certain type of the buyer (there are two possibilities: either 0 or \( b \)), the continuation play is as in the complete information game.\(^{42}\) Thus, if the buyer deviates, he trades at price at best \( p(s, \bar{b}) \) which is strictly higher than \( p(s, 0) + \varepsilon \) when \( \bar{b} > \theta_T > s \) and \( \varepsilon \) is small. Therefore, such a deviation is not profitable for sufficiently small \( \Delta \).

We now show that players do not have incentives to deviate to off-path price offers. By Lemma 20, if such a type deviates in round \( n \), she trades at a price at best \( p(s_n, 0) + \varepsilon_0 \). Thus, the price of the punishing path converges uniformly to \( p(\theta_T^0) \) as \( \Delta \to 0 \) by Lemma 19. On the other hand, on the equilibrium path the buyer’s offers \( p_n^b \) converge uniformly to \( p^s = p(\theta_T^0) + (1 - \alpha)\varepsilon > p(\theta_T^0) \) by Lemmas 17 and 18. Since \( c(1) - c(0) < \min\{\frac{1 - \alpha}{\alpha}, \frac{\alpha}{1 - \alpha}\} \xi \),

\[
p^s_T \geq p^s_T = p(1) - \alpha \varepsilon - \alpha \xi (\xi + \varepsilon) T \varepsilon \\
= p(1) - \alpha (p(1) - p(0)) \\
= c(1) + (1 - \alpha) \xi - \alpha (c(1) - c(0)) > c(1).
\]

Therefore, any deviation from the on-path offers is not profitable when \( \Delta \) is sufficiently small. This completes the construction of the PBEs in \( G(F, \Delta) \) and completes the proof of part 2 of Theorem 1.

### Online Appendix B: Auxiliary Results

(Not for Publication)

#### Flow Payoff Specification for Fixed-Income Assets

Consider the infinite-maturity bond with the coupon 1 paid at a constant rate \( c_0 \). The issuer defaults on the bond with a constant rate \( c_1(1 - \theta) \). In the case of default, the bond-holder incurs costs \( c_2 \), and the bond is immediately reissued to the same holder after the default. Thus, the flow payoff from holding this security is \( c_0 - c_1 c_2 (1 - \theta) = c_0 - c_1 c_2 + c_1 c_2 \theta \) which gives the

\(^{42}\) This is the only place where we use the weakening of the support restriction on beliefs.
flow payoff specification in this paper.\textsuperscript{43}

**Auxiliary Steps in the Proof of Theorem 2**

**Lemma 21.** Equation (4.4) implies equation (4.17).

**Proof.** Using (A.34) to substitute \( \sigma(\theta) \) in equation (4.4), we get

\[
\frac{\Lambda_s}{\lambda} = \frac{y_u}{y_u + y_d} (a - 1) - \frac{y_d}{y_u + y_d} \left( \frac{\Lambda_s L}{y_u + y_d + \Lambda_s} + \int_{\theta}^{y_d} \frac{\Lambda_s \sigma(\theta)}{y_u + y_d + \Lambda_s \sigma(\theta)} d\theta \right).
\]

To prove (4.17), we show that

\[
\int_{\theta}^{\bar{\theta}} \frac{\Lambda_s \sigma(\theta)}{y_u + y_d + \Lambda_s \sigma(\theta)} d\theta = (1 - \alpha) \frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k}{\rho L} \theta} + \int_{0}^{\bar{\theta} + \min \left\{ \frac{1 + \frac{\rho}{\lambda} \ln x}{1 - \alpha \frac{y_u + y_d}{\Lambda_s}}, 1 \right\}} \frac{(1 - \alpha) \frac{\Lambda_s}{\rho + \Lambda_s} \xi x - \frac{k}{\rho} \sigma(\theta - \bar{\theta})}{1 + \frac{y_u + y_d}{\Lambda_s} - \frac{\xi}{\rho} \sigma(\theta - \bar{\theta})} d\theta.
\]

Expressing \( \sigma(\theta) \) from (A.34), \( \bar{\theta} \) from (A.29), and \( \bar{\theta} \) from (A.33), we get

\[
\int_{\theta}^{\bar{\theta} + \min \left\{ \frac{1 + \frac{\rho}{\lambda} \ln x}{1 - \alpha \frac{y_u + y_d}{\Lambda_s}}, 1 \right\}} \frac{(1 - \alpha) \frac{\Lambda_s}{\rho + \Lambda_s} \xi x - \frac{k}{\rho} \sigma(\theta - \bar{\theta})}{1 + \frac{y_u + y_d}{\Lambda_s} - \frac{\xi}{\rho} \sigma(\theta - \bar{\theta})} d\theta = (1 - \alpha) \frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k}{\rho L} \theta} \int_{\theta}^{\bar{\theta} + \min \left\{ \frac{1 + \frac{\rho}{\lambda} \ln x}{1 - \alpha \frac{y_u + y_d}{\Lambda_s}}, 1 \right\}} \frac{(1 - \alpha) \frac{\Lambda_s}{\rho + \Lambda_s} \xi x - \frac{k}{\rho} \sigma(\theta - \bar{\theta})}{1 + \frac{y_u + y_d}{\Lambda_s} - \frac{\xi}{\rho} \sigma(\theta - \bar{\theta})} d\theta.
\]

where in the second line we make a change of variables \( s = \frac{\theta - \bar{\theta}}{(1 - \alpha) \frac{y_u + y_d}{\Lambda_s}} \). After expressing \( x \) from (A.30) we get equation (A.61). \( \square \)

**Lemma 22.** \( \Lambda_s^2(\cdot) \) is strictly decreasing.

**Proof.** \( \Lambda_s^2(L) \) is given implicitly by the equation (4.17). It is convenient to rewrite function \( I \) as follows

\[
I(L, \Lambda_s) = \frac{\Lambda_s L}{y_u + y_d + \Lambda_s} + \int_{\theta - \bar{\theta}}^{\theta} \frac{(1 - \alpha) \frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k}{\rho L} \theta} - s}{\Lambda_s + y_u + y_d \frac{(1 - \alpha) \frac{y_d \xi}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k}{\rho L} \theta} - s}{1 + \frac{y_u + y_d}{\Lambda_s} - \frac{\xi}{\rho} \sigma(\theta - \bar{\theta})}} ds.
\]

\( \textsuperscript{43} \)Recall that the shift by the constant of flow payoffs will change only asset prices, but not affect the results about the asset liquidity (see footnote 11).
By the implicit function theorem,

$$\frac{d\Lambda^2_s}{dL} = \frac{-\frac{yc}{y_a+y_d} \frac{\partial}{\partial L} I(L, \Lambda_s)}{1 \times \frac{yc}{y_a+y_d} \frac{\partial}{\partial \Lambda_s} I(L, \Lambda_s)}$$

Thus, to prove lemma it suffices to show that $\frac{\partial}{\partial \Lambda_s} I(L, \Lambda_s) > 0$ and $\frac{\partial}{\partial L} I(L, \Lambda_s) > 0$.

We first show $\frac{\partial}{\partial \Lambda_s} I(L, \Lambda_s) > 0$. The first term in (62) is strictly increasing in $\Lambda_s$. Let us consider the integrand in (62). Denote $c = (1-\alpha) \frac{yc_k}{k} e^{-\frac{L}{k}}$. Then

$$\text{sgn} \frac{\partial}{\partial \Lambda_s} \left( \frac{\Lambda_s c - s(\rho + \Lambda_s)}{(\Lambda_s + y_a + y_d)c - \frac{r}{\rho} s(\rho + \Lambda_s)} \right)$$

$$= \text{sgn} \frac{\partial}{\partial \Lambda_s} \left( \frac{\Lambda_s (c - s) - s \rho}{\Lambda_s (c - s) + (y_a + y_d)c - rs} \right)$$

$$= \text{sgn} \left( (c - s)(\Lambda_s (c - s) + (y_a + y_d)c - rs) - (c - s)(\Lambda_s (c - s) - s \rho) \right)$$

$$= \text{sgn} \left( (c - s)((y_a + y_d)c - rs) + (c - s)s \rho \right)$$

$$= \text{sgn} \left( (y_a + y_d)c^2 - (y_a + y_d)cs - rs + cs \rho \right)$$

$$= \text{sgn} \left( c^2(y_a + y_d) \right) = 1,$$

and so, the integrand is strictly increasing in $\Lambda_s$. Finally, $\hat{\theta} - \bar{\theta} = \min\{1 - L, (1-\alpha) \frac{yc_k}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{L}{k}} \}$ is weakly increasing in $\Lambda_s$. Therefore, $\frac{\partial}{\partial \Lambda_s} I(\Lambda_s, L) > 0$.

We next show $\frac{\partial}{\partial L} I(L, \Lambda_s) > 0$. Let us introduce functions

$$A(L, \Lambda_s) = (1-\alpha) \frac{yc_k}{k} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{L}{k}}$$

$$b(\Lambda_s) = \frac{\Lambda_s + y_a + y_d}{\Lambda_s}.$$

To simplify expression, we will omit arguments of $A$ and $b$ below. We can rewrite function $I$ as follows

$$I(L, \Lambda_s) = \frac{L}{b} + \int_0^{\frac{\theta - \bar{\theta}}{\theta - \hat{\theta}}} \frac{A - s}{bA - \frac{r}{\rho} s} ds.$$  (63)

Claim 7. $\frac{1}{b} + \frac{\partial A}{\partial L} \frac{\rho}{\bar{\theta} - \hat{\theta}} > 0$ implies $\frac{\partial}{\partial L} I(L, \Lambda_s) > 0$.

Proof of Claim 7: Differentiating (63),

$$\frac{\partial}{\partial L} I(L, \Lambda_s) = \frac{1}{b} + \frac{\partial (\theta - \bar{\theta})}{\partial L} \left( \frac{A - (\theta - \bar{\theta})}{bA - \frac{r}{\rho} (\theta - \bar{\theta})} \right) + \frac{\partial A}{\partial L} \int_0^{\frac{\theta - \bar{\theta}}{\theta - \hat{\theta}}} \frac{1}{bA - \frac{r}{\rho} s} ds.$$  (64)
There are two cases. First, suppose \( \theta = \hat{\theta} \) and so, \( \theta - \hat{\theta} = 1 - L \). Then

\[
\frac{\partial}{\partial L} I(L, \Lambda_s) = \frac{1}{b} - \frac{A - (1 - L)}{bA - \frac{r}{\rho}(1 - L)} + \frac{\partial A}{\partial L} \int_0^{1-L} \frac{\partial}{\partial A} \left( \frac{A - \frac{r}{\rho}s}{bA - \frac{r}{\rho}s} \right) ds
\]

\[
= \frac{(b - \frac{r}{\rho})(1 - L)}{b(bA - \frac{r}{\rho}(1 - L))} + \frac{\partial A}{\partial L} \int_0^{1-L} \frac{(b - \frac{r}{\rho})s}{(bA - \frac{r}{\rho}s)^2} ds.
\]

Using the change of variables \( z = bA - \frac{r}{\rho}s \), we can compute the integral as follows

\[
\int_0^{1-L} \frac{(b - \frac{r}{\rho})s}{(bA - \frac{r}{\rho}s)^2} ds = \left( b - \frac{r}{\rho} \right) \frac{\rho}{\rho} \int_0^{bA - \frac{r}{\rho}(1 - L)} \left( - \frac{bA}{z^2} + \frac{1}{z} \right) dz
\]

\[
= \left( b - \frac{r}{\rho} \right) \frac{\rho}{\rho} \int_{bA}^{bA - \frac{r}{\rho}(1 - L)} d \left( \frac{bA}{z} + \ln z \right)
\]

\[
= \left( b - \frac{r}{\rho} \right) \frac{\rho}{\rho} \left( \frac{bA}{bA - \frac{r}{\rho}(1 - L)} - 1 + \ln \left( \frac{bA - \frac{r}{\rho}(1 - L)}{bA} \right) \right)
\]

\[
= \left( b - \frac{r}{\rho} \right) \frac{\rho}{\rho} \left( \frac{\frac{r}{\rho}(1 - L)}{bA - \frac{r}{\rho}(1 - L)} + \ln \left( \frac{bA - \frac{r}{\rho}(1 - L)}{bA} \right) \right)
\]

Since \( \frac{\partial A}{\partial L} = -\frac{k}{\rho} A < 0 \),

\[
\frac{\partial}{\partial L} I(L, \Lambda_s) = \frac{(b - \frac{r}{\rho})(1 - L)}{b(bA - \frac{r}{\rho}(1 - L))} + \frac{\partial A}{\partial L} \left( b - \frac{r}{\rho} \right) \frac{\rho}{\rho} \left( \frac{\frac{r}{\rho}(1 - L)}{bA - \frac{r}{\rho}(1 - L)} + \ln \left( \frac{bA - \frac{r}{\rho}(1 - L)}{bA} \right) \right) \geq \]

\[
\geq \frac{(b - \frac{r}{\rho})(1 - L)}{b(bA - \frac{r}{\rho}(1 - L))} + \frac{\partial A}{\partial L} \left( b - \frac{r}{\rho} \right) \frac{\rho}{\rho} \frac{\frac{r}{\rho}(1 - L)}{bA - \frac{r}{\rho}(1 - L)} = \]

\[
= \frac{(b - \frac{r}{\rho})(1 - L)}{bA - \frac{r}{\rho}(1 - L)} \left( \frac{1}{b} + \frac{\partial A}{\partial L} \frac{\rho}{\rho} \right).
\]

Now, suppose \( \underline{\theta} < \hat{\theta} \) and so \( \underline{\theta} = \hat{\theta} + A \). Then (.64) becomes

\[
\frac{\partial}{\partial L} I(L, \Lambda_s) = \frac{1}{b} + \frac{\partial A}{\partial L} \int_0^A \frac{\partial}{\partial A} \left( \frac{A - \frac{r}{\rho}s}{bA - \frac{r}{\rho}s} \right) ds
\]

\[
= \frac{1}{b} + \frac{\partial A}{\partial L} \int_0^A \frac{(b - \frac{r}{\rho})s}{(bA - \frac{r}{\rho}s)^2} ds.
\]
Again, using the change of variables \( z = \frac{bA - r}{\rho} s \), we can compute the integral as follows

\[
\int_{0}^{A} \frac{(b - \frac{z}{\rho})}{(bA - \frac{z}{\rho} s)^2} ds = (b - \frac{r}{\rho}) \int_{0}^{A} \left( \frac{bA - \frac{z}{\rho} s}{(bA - \frac{z}{\rho} s)^2} - \frac{1}{(bA - \frac{z}{\rho} s)} \right) ds \\
= (b - \frac{r}{\rho}) \left( \frac{\rho}{r} \right)^2 \int_{bA}^{(b - \frac{z}{\rho})A} \left( -\frac{bA}{z^2} + \frac{1}{z} \right) dz \\
= (b - \frac{r}{\rho}) \left( \frac{\rho}{r} \right)^2 \int_{bA}^{(b - \frac{z}{\rho})A} d \left( \frac{bA}{z} + \ln z \right) \\
= (b - \frac{r}{\rho}) \left( \frac{\rho}{r} \right)^2 \left( \frac{b}{b - \frac{r}{\rho}} - 1 + \ln \left( \frac{b - \frac{z}{\rho}}{b} \right) \right) \\
= (b - \frac{r}{\rho}) \left( \frac{\rho}{r} \right)^2 \left( \frac{\frac{r}{\rho}}{b - \frac{r}{\rho}} + \ln \left( \frac{b - \frac{z}{\rho}}{b} \right) \right)
\]

Since \( \frac{\partial A}{\partial L} = -\frac{b}{\rho} A < 0 \),

\[
\frac{\partial}{\partial L} I(L, \Lambda_s) = \frac{1}{b} + \frac{\partial A}{\partial L} \left( b - \frac{r}{\rho} \right) \left( \frac{\rho}{r} \right)^2 \left( \frac{\frac{r}{\rho}}{b - \frac{r}{\rho}} + \ln \left( \frac{b - \frac{r}{\rho}}{b} \right) \right) \\
\geq \frac{1}{b} + \frac{\partial A}{\partial L} \frac{\rho}{r} \frac{r}{r} > 0.
\]

Claim 8. \( 1 - \alpha < \frac{r^2}{y_d(r+y_u+y_d)} \) implies \( \frac{1}{b} + \frac{\partial A}{\partial L} \frac{\rho}{r} > 0 \).

Proof of Claim 8:

\[
1 + \frac{\partial A}{\partial L} r = \Lambda_s \left( 1 - (1 - \alpha) \frac{y_d \rho}{r^2} \frac{\Lambda_s}{\rho + \Lambda_s} e^{-\frac{k}{\tau} L} \right) \\
\geq \Lambda_s \frac{\rho + \Lambda_s}{\rho + \Lambda_s} \left( 1 - (1 - \alpha) \frac{y_d \rho}{r^2} e^{-\frac{k}{\tau} L} \right) \\
\geq \Lambda_s \left( \frac{r^2 - (1 - \alpha) y_d \rho}{r^2} \right)
\]

which is positive whenever \( 1 - \alpha < \frac{r^2}{y_d(r+y_u+y_d)} \). q.e.d.

To summarize, \( 1 - \alpha < \frac{r^2}{y_d(r+y_u+y_d)} \) implies that \( \frac{\partial}{\partial L} I(L, \Lambda_s) > 0 \).

Lemma 23. If \( 1 - \alpha < \frac{r^2}{y_d(r+y_u+y_d)} \), then there is a unique solution \((\Lambda_s, L)\) to (4.16) and (4.17).

Proof. By Lemma 22, \( \Lambda_s^2(\cdot) \) solving (4.17) is strictly decreasing. The right-hand of equation (4.16) is strictly increasing in \( L \), as \( \left( \frac{\tau}{k} \left( e^{\frac{k}{\tau} L} - 1 \right) - L \right)' = e^{\frac{k}{\tau} L} - 1 > 0 \). Thus, \( \Lambda_s^1 \) is strictly increasing. Therefore, the solution to (4.16) and (4.17) is unique.