Test-Set Equilibrium with Auction Applications*

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Abstract

We introduce a new refinement of Nash equilibrium, which formalizes the idea that players contemplate only deviations from equilibrium play in which a single competitor plays a non-equilibrium best response. In games with three or more players, this refinement neither implies, nor is implied by, previous refinements. In three well-known auction games, the Nash equilibria selected by this refinement are similar to but different from those selected in the original papers using arguments based on the economic context. Our general game-theoretic argument provides a new perspective on those earlier selections.

1 Introduction

We introduce a new refinement of Nash equilibrium for non-cooperative games, which we call test-set equilibrium. The refinement incorporates a novel, yet intuitive restriction on the sorts of deviations that players regard as most likely. Each player chooses its strategy believing that the only relevant strategy profiles for competitors are those in which at most one competitor deviates from equilibrium play and any deviator plays a best response to the equilibrium profile. This refinement neither implies nor is implied by any previous trembling hand refinement. What makes this refinement especially interesting is that when it is applied to several auction game models, its selections are very similar to ones made by the diverse criteria that had previously been applied to those models. Test-set equilibrium, however, does not make selections identical to those in the original papers, and the differences deepen our understanding of the original selections.

To motivate and illustrate the test-set refinement, consider the three-player game in Figure 1. Each player has two strategies. The “Geo” player picks the payoff matrix – East or West. Geo’s payoff is always zero, regardless of what everyone does, so we omit its payoff from the matrices. For Row, the strategy Up strictly dominates Down: the former always pays one and the latter always pays zero. Column’s decision is the one of interest: its best

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1In a companion paper (Milgrom and Mollner, 2016), we introduce extended proper equilibrium, which is a new tremble-based refinement that is similar in spirit to test-set equilibrium. Extended proper equilibrium has the advantage that at least one exists in every finite game, but the disadvantage that it is undefined for games with a continuum of strategies, such as all the interesting applications analyzed below.
Table 1: A three player game†

<table>
<thead>
<tr>
<th>West</th>
<th>Left</th>
<th>Right</th>
<th>East</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up</td>
<td>1, 1</td>
<td>1, 1</td>
<td>Up</td>
<td>1, 0</td>
<td>1, 1</td>
</tr>
<tr>
<td>Down</td>
<td>0, 1</td>
<td>0, 0</td>
<td>Down</td>
<td>0, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

†Row’s payoffs are listed first. Column’s payoffs are listed second. Geo’s payoffs are constant and suppressed.

choice depends on what it believes the other two players will do. The Nash equilibria of this game include ones in Column plays Right, Row plays Up, and Geo mixes between West and East with any probabilities. All of these equilibria seem equally unobjectionable. There is also one more Nash equilibrium, in which Column plays Left and Geo plays West with probability one. Although all the Nash equilibria are also proper equilibria (Myerson, 1978), the last one, in which Column plays Left is very different. It embeds the idea that Column believes it is more likely that Row will deviate to play its strictly dominated strategy than that Geo will deviate to play its perfectly sensible alternative strategy. It is this last equilibrium that is ruled out by the new test-set refinement.

A test-set equilibrium is a Nash equilibrium in which each player’s equilibrium strategy is undominated when tested against a particular set of strategy profiles for the other players. That set comprises the equilibrium strategy profile and every profile in which exactly one player’s equilibrium strategy is replaced by some other best response. In our example, the (Up, Left, West) Nash equilibrium flunks the test-set condition. Row has no other best response besides its equilibrium strategy, but Geo has two pure best responses to the equilibrium profile. Consequently, Column’s test set consists of precisely two strategy profiles: the equilibrium profile (Up, West), and the profile (Up, East) in which Geo plays its alternative best response. Against this test set, Left is weakly dominated by Right, so test-set criterion bars Column from playing Left.

This example also highlights the reason that tremble-based refinements like proper equilibrium may lose power when applied to games with three or more players. In such games, different equilibria may implicitly entail players holding different beliefs about the relative likelihood of mistakes by two or more distinct competitors. Concepts like perfect and proper equilibrium do not restrict such relative likelihoods, and that prevents them from eliminating unappealing Nash equilibrium profiles like (Up, Left, West) in the preceding example.

In addition to implying new restrictions in games with three or more players, the test-set concept can also imply new restrictions in two-person games with incomplete information. For such games, a player may need to assess the relative likelihood of mistakes by the players. In such games, different equilibria may implicitly entail players holding different beliefs about the relative likelihood of mistakes by two or more distinct competitors. Concepts like perfect and proper equilibrium do not restrict such relative likelihoods, and that prevents them from eliminating unappealing Nash equilibrium profiles like (Up, Left, West) in the preceding example.

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\[ \sigma_{\text{row}}^k = \left( \frac{k}{k+1}, \frac{1}{k+1} \right), \quad \sigma_{\text{col}}^k = \left( \frac{k}{k+1}, \frac{1}{k+1} \right), \quad \sigma_{\text{geo}}^k = \left( \frac{k^2}{k^2+1}, \frac{1}{k^2+1} \right). \]

For all \( k \in \mathbb{N} \), \((\sigma_{\text{row}}^k, \sigma_{\text{col}}^k, \sigma_{\text{geo}}^k)\) is a \( \frac{1}{k} \)-proper equilibrium. Taking the limit as \( k \to \infty \) establishes that (Up, Left, West) is a proper equilibrium.
different types of the other player. The test-set condition can bite when it is applied to the 
agent normal form, in which each type is treated as a separate player.

We show below that in three well-known applied auction models, the test-set refinement 
eliminates most Nash equilibria. We describe and analyze these three applications inform-
ally in the introduction, and formally in the main body of the paper. Each application is 
modeled as a game; each has a continuum of Nash equilibria; and the original papers that 
introduced each application also introduced an idiosyncratic selection motivated by some 
special structure of the particular environment. In these three games, the test-set crite-
rion selects equilibria that are intriguingly close, but not identical, to the papers’ original 
selections.

The simplest of the auction applications is a two-bidder second-price auction with in-
complete information, introduced and analyzed in a paper by Abraham, Athey, Babaiof
and Grubb (2014) for an application to Internet display advertising. In such settings, some 
bidders may own private cookies, which provide them with an information advantage about 
the value of an advertising opportunity. In their model, there is a single advertising slot for 
sale that has a common value of $v$ to both bidders, where $v \in \{0,1\}$. One bidder knows 
the value; the other does not. The informed bidder has a dominant strategy, which is to 
bid the value $v$. For the uninformed bidder, however, every bid $b \geq 0$ is a best response to 
the informed bidder’s dominant strategy, and so is part of a Nash equilibrium. Abraham, 
Athey, Babaiof and Grubb (2014) find that standard equilibrium concepts are powerless to 
refine away what the authors regard as implausible equilibria. In particular, an analogue 
of trembling-hand perfection for continuous games eliminates only Nash equilibria in which 
the uninformed bidder makes a weakly dominated bid $b > 1$. Nevertheless, they argue, 
there is a unique equilibrium consistent with the economics of this setting. They isolate 
this preferred equilibrium by studying a sequence of games in which there is a small positive 
probability that an additional bidder is present who submits a random bid. They show that 
the equilibrium in which the uninformed bidder bids $b = 0$ is the unique limit of equilibria 
in such a sequence of games as the small probability approaches zero, and they dub this 
selection the tremble-robust equilibrium. According to their proposed interpretation, any 
bid $b > 0$ is not “robust” because it exposes the uninformed bidder to the possibility of 
paying the positive price set by the “noise bidder” when the actual value is zero.

In the same game, the test-set condition refines the equilibrium by restricting the unin-
formed bidder’s beliefs about which deviations the informed bidder is likely to make. 
Consider any pure Nash equilibrium in which the uninformed bidder bids an amount $b$ 
strictly between 0 and 1. For the informed bidder, when $v = 0$, the best responses are bids 
in the half-open interval $[0,b)$; when $v = 1$, they are the bids in $(b,\infty)$. If the uninformed 
bidder evaluates a proposed bid of $b \in (0,1)$ considering only those possible deviations for 
the informed bidder, it concludes that $b$ is weakly dominated both by a bid of 0 and by 
a bid of 1. Thus, the test-set condition eliminates all pure Nash equilibria in which the 
uninformed bidder bids $b \in (0,1)$, but it eliminates neither of the pure Nash equilibria with 
$b = 0$ or $b = 1$. The $b = 0$ equilibrium is the tremble-robust equilibrium, selected by the 
original paper. The $b = 1$ test-set equilibrium could be similarly selected by a robustness 
argument in which, with vanishingly small probability, the informed bidder has a random

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3The auction-game refinements were called “tremble-robust equilibrium,” “locally envy-free equilibrium,” 
and “truthful equilibrium.”
budget that constrains its bid.

A second application is to a game known as the generalized second price auction (hereafter called the GSP auction) which, together with its variants, have been used by Google, Microsoft, Yahoo!, and others to sell advertising on search pages. Higher positions on a search page tend to be clicked more frequently by searchers, and so are more valuable to advertisers.

Edelman, Ostrovsky and Schwarz (2007) and Varian (2007) both introduced similar models of the GSP auction. Adopting those models, let us denote the numbers of clicks associated with the $I$ positions by $\kappa_1 > \cdots > \kappa_I > 0$. There are $N$ bidders, and bidder $n$ values clicks at $v_n$ apiece. A bid, $b_n$, specifies a maximum price per-click. The highest bid wins the first position; the second highest wins the second position; and so on. Using the usual notation for order statistics, the winner of position $i$ pays a per-click price of $b^{(i+1)}$, so its total payment is $\kappa_i b^{(i+1)}$. With $I = 1$, this reduces to a second-price auction, and that special case is the reason for the GSP name.

The GSP auction game has a continuum of Nash equilibrium bid profiles with a variety of outcomes, including both efficient and inefficient assignments of positions and with various levels of total payment. Given a Nash equilibrium bid profile $b$ with no ties, let us number the bidders so that the equilibrium bids satisfy $b_1 > \cdots > b_N$. It is easy to see that, in equilibrium, no bidder $n = 1, \ldots, N$ can prefer the price and allocation of any bidder $m > n$ in a lower position on the page, for the bidder could obtain that outcome by reducing its bid to $b_m - \varepsilon$ for some $\varepsilon > 0$. Reasoning about the economics of the game, both Edelman, Ostrovsky and Schwarz (2007) and Varian (2007) propose a refinement in which no bidder $n$ can prefer the price and allocation of bidder $n - 1$, who has a higher position on the page. In the formalization, an equilibrium is “locally envy-free” if for all $n \in \{2, \ldots, N\}$, $\kappa_n(v_n - b_{n+1}) \geq \kappa_{n-1}(v_n - b_n)$. Any equilibrium that violates this inequality also violates the test-set condition, for the following reason.

Suppose that $b$ is a pure Nash equilibrium of the GSP auction game that is not locally envy free. There is then some highest index $n$ such that bidder $n$ envies bidder $n - 1$, so that $\kappa_n(v_n - b_{n+1}) < \kappa_{n-1}(v_n - b_n)$. In such a case, the GSP auction has the property that no bidder $m > n$ has a best response that is greater than or equal to $b_n$. With that “lemma” in hand, suppose bidder $n$ considers raising its bid from $b_n$ to $b_n + \varepsilon$, for some small $\varepsilon > 0$. It may reason as follows:

I have several best responses, so I should compare these based on their robustness in case one of my competitors deviates from the play that I expect. For that evaluation, I should emphasize the most likely alternative strategy profiles, and those are the ones in which just one competitor deviates but is still playing a best response. Raising my bid slightly from $b_n$ to $b_n + \varepsilon$ changes my outcome only if one of my competitors deviates within $[b_n, b_n + \varepsilon]$. The only strategy profiles in my test set with this property (here comes the “lemma”) are those in which it is a higher bidder who deviates within that interval. Against such strategy profiles, my equilibrium bid $b_n$ would result in my winning position $n$ at the per-click price $b_{n+1}$. However, $b_n + \varepsilon$ would result in my winning position $n - 1$ at a per-click price of about $b_n$. That change would be profitable for me, because $\kappa_n(v_n - b_{n+1}) < \kappa_{n-1}(v_n - b_n)$. So, a bid of $b_n + \varepsilon$ weakly dominates the bid of $b$ on the test set.
Since test-set equilibrium allows no such dominance, it follows that every test-set equilibrium is locally envy-free.

The preceding analysis highlights an alternative logic for describing locally envy-free equilibria; they are precisely the ones selected when each bidder believes that the most likely deviations are those by a single, higher bidder to an alternative best response. In comparison, test-set equilibria are selected when each bidder also contemplates the possibility of single deviations to an alternative best response by a lower bidder. We show in the main text that test-set equilibria are a subset of the locally envy-free equilibria.

Our third example is the menu auction game of Bernheim and Whinston (1986), which has been applied to study both competition among lobbyists and bidding in combinatorial auctions. The two applications are actually somewhat different. In this introduction, we focus on the lobbying application, in which the decision may concern legislation or a public good and the offers may be bribes or less direct forms of compensation, the payment of which will be conditional on the decision.

We denote the utility that bidder/lobbyist \( n \) receives from decisions by \( v_n : X \to \mathbb{R} \). Similarly, we denote the utility of the auctioneer by \( v_0 : X \to \mathbb{R} \). Each bidder/lobbyist \( n \) makes a vector of bids \( b_n : X \to \mathbb{R}_+ \) and the auctioneer then chooses a decision \( x^* \in \arg \max_{x \in X} \left[ v_0(x) + \sum_{n=1}^N b_n(x) \right] \). Bidder \( n \)'s payoff is \( v_n(x^*) - b_n(x^*) \).

The menu auction game typically has a continuum of Nash equilibria, some of which may involve inefficient decisions, or low total payments, or both. To refine away what they regard as the implausible Nash equilibria, Bernheim and Whinston consider two approaches. The first restricts attention to equilibria in which bidders play “truthful” strategies – strategies that result in the same payoff for a bidder if the auctioneer selects any decision \( x \) for which the bidder has made a positive bid. Their second approach introduces a new solution concept – coalition-proof equilibrium, which selects only Nash equilibria that are immune to certain coalitional deviations. These restrictions lead to the conclusion that the equilibrium outcome involves a payoff vector in the core of the economy, and, among such core vectors, involves payoffs such that no alternative core payoff is weakly better for all bidders (and worse for the seller).

Truthful strategies specify exactly the amounts that bidders can offer for decisions that are losing in equilibrium. In the menu auction, test-set equilibrium implies a similar, but weaker, restriction on losing bids; it requires that each bidder makes sufficiently high bids on each losing decision. To see how the bid bounds arise from the test-set condition, let \( b \) be a pure test-set equilibrium of the menu auction game. Let \( x^* \) denote the “winning” decision in equilibrium. Denote each bidder \( n \)'s equilibrium payoff by \( \pi_n = v_n(x^*) - b_n(x^*) \). Let \( x \neq x^* \) and consider bidder \( n \)'s reasoning as it thinks about whether to raise its bid for decision \( x \) from \( b_n(x) \) to \( b_n(x) + \varepsilon \), for some \( \varepsilon > 0 \), leaving all its other bids unchanged:

I have several best responses, so I should compare these based on their robustness in case one of my competitors deviates from the play that I expect. For that evaluation, I should emphasize the most likely alternative strategy profiles, and those are the ones in which just one competitor deviates but is still playing a best response. For strategy profiles in that test set, if I continue to play my equilibrium strategy, the decision will always be \( x^* \) and my payoff will still be \( \pi_n \). If I were to raise my bid for some decision \( x \) and that were to change...
the decision, that must mean that the new decision would be $x$ and my new
payoff would be $v_n(x) - b_n(x) - \varepsilon$. That change is profitable for me if and only
if $v_n(x) - b_n(x) - \varepsilon > \pi_n$. If that inequality holds, then the new bid weakly
dominate the equilibrium bid on the test set.

Since test-set equilibrium allows no such dominance, it follows that for all bidders $n$ and
decisions $x$, there is an implied lower bound for all losing bids:

$$b_n(x) \geq v_n(x) - \pi_n.$$  \hspace{1cm} (1)

Since truthful equilibria are those in which for all bidders $n$ and all decisions $x$, $b_n(x) = \max\{v_n(x) - \pi_n, 0\}$, it follows from (1) that for the menu auction game, test-set equilibrium
is less restrictive than truthful equilibrium. Since coalition-proof equilibrium and truthful
equilibrium implement the same payoff vectors, it also follows that test-set equilibrium is
less restrictive than coalition-proof equilibrium in terms of payoffs. In particular, although
test-set equilibrium implements payoff vectors in the core – as a routine consequence of the
lower bounds on losing bids given by (1) – these payoffs are not necessarily bidder-optimal
in the core, as are the payoffs of the Bernheim-Whinston concepts.

A casual reader might have thought that the core outcomes found by Bernheim and
Whinston were somehow the result of the level of coordination implied by the definition of
coalition-proof equilibrium, or the result of the severe restriction on candidate equilibrium
strategies implied by truthfulness. However, the test-set analysis suggests that neither
of these is required to obtain core outcomes. Instead, the extra restrictions assumed by
Bernheim and Whinston are found to imply the stronger conclusion that the payoff outcome
is bidder-optimal in the core – a conclusion that is not implied by test-set equilibrium alone.

In the main text, we also consider a general package-auction variation of the Bernheim-
Whinston model, in which test-set equilibrium outcomes may lie outside the core, but
coalition-proof equilibrium and truthful equilibrium still select core outcomes. The analysis
of that model highlights how the structure of the game interacts with the different solution
concepts.

Each of the papers that we have reviewed in this introduction selects equilibria by
referencing the context of the particular application. Like us, the authors of those papers
find no compelling reason to assume that a Nash equilibrium refinement that is useful for
studying a particular auction game must also be useful for very different applications, such
as ones in evolutionary biology or in the study of self-enforcing contracts. Still, discipline
and insight can be lost when analysts introduce and apply a different solution concept for
each different economic model. What we hope to show is that test-set equilibrium offers a
coherent and workable alternative, at least for an interesting class of auction models, and
provides a valuable perspective for understanding and critiquing the refinements that other
researchers have applied.

2 Test-Set Equilibrium

Informally, a Nash equilibrium is a test-set equilibrium if no player uses a strategy that is
weakly dominated against its test set, which consists of the equilibrium strategy profiles of
the other players and the related profiles in which exactly one player deviates to a different

best response. For games with three or more players, test-set equilibrium neither implies nor is implied by any trembles-based refinement. We also show by example that there are games for which no test-set equilibrium exists.

2.1 Definition

An $N$-person game in normal form is a $2N$-tuple $G = [(S_n)_{n=1}^N, (u_n)_{n=1}^N]$, where for each player $n$, $S_n$ is a nonempty set of pure strategies, and $u_n : \times_{n=1}^N S_n \to \mathbb{R}$ is a utility function. A mixed strategy profile is denoted $\sigma = (\sigma_1, \ldots, \sigma_N) \in \times_{n=1}^N \Delta(S_n)$. We embed $S_n$ in $\times_{m \neq n} \Delta(S_m)$ and extend the utility functions $u_n$ to the domain $\times_{n=1}^N \Delta(S_n)$ in the usual way. We use $\sigma_{-n}$ for a typical element of $\times_{m \neq n} \Delta(S_m)$ and $BR_n(\sigma_{-n})$ for the set of best responses.

**Definition 1.** Let

$$T_n(\sigma) = \bigcup_{m \neq n} \{(s_m, \sigma_{-m}) : s_m \in BR_m(\sigma_{-m})\}.$$ 

A mixed-strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ is a test-set equilibrium if and only if it is a Nash equilibrium and for all $n$, there is no $\hat{\sigma}_n \in \Delta(S_n)$ such that $u_n(\hat{\sigma}_n, \sigma_{-n}) \geq u_n(\sigma_n, \sigma_{-n})$ for all $\sigma_{-n} \in T_n(\sigma)$, with $u_n(\hat{\sigma}_n, \sigma_{-n}) > u_n(\sigma_n, \sigma_{-n})$ for some $\sigma_{-n} \in T_n(\sigma)$.

We refer to the set $T_n(\sigma)$ in the above definition as the test set of player $n$. Notice that the test set consists of profiles of strategies for the players besides $n$ in which exactly one competitor $m \neq n$ plays a best response while all other competitors play their equilibrium strategies. This formalizes the two conditions we had described in the introduction: that only one player is thought to deviate and that the deviation is to some best response to equilibrium play.

2.2 Relationship to Other Equilibrium Concepts

The game depicted in Figure 1 in the introduction shows that not every proper equilibrium is a test-set equilibrium. In the game in Figure 2, it is easily checked that $(Up, Left)$ is a test-set equilibrium of this game but is not a trembling-hand perfect equilibrium and, indeed, it has Row playing a weakly dominated strategy. Since Row’s test-set contains only $Left$, its strategy of $Up$ – although weakly dominated – is not weakly dominated against its test-set. Thus, in games with three or more players, the strongest tremble-based refinement does not imply test-set equilibrium, and even in games with just two players, test-set equilibrium does not imply the weakest tremble-based refinements. For such games, the tremble-based and test-set refinements are logically independent.

For two-player games, the situation is more complex. According to the following result, proper equilibrium implies test-set equilibrium with two players. However, even with two

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4Defining a (countably additive) mixed strategy on an infinite strategy space generally requires specifying a sigma-algebra on strategies and computing expected payoffs. This involves no new issues, so we omit the details here.
players, trembling hand perfect equilibrium is not strong enough to imply test-set equilibrium.

**Proposition 1.** In any finite, two-player game, every proper equilibrium is a test-set equilibrium.

The fact that proper equilibrium implies test-set equilibrium in two player games is perhaps reassuring. The motivation for test-set equilibrium comes from questions about how one player ought to form beliefs about the relative likelihood of deviations from its several different competitors. With two players, each player has only a single competitor, and there are no cross-player comparisons to form beliefs about. In those settings, test-set equilibrium then reduces to insisting that each player believe its competitor is more likely to make a costless deviation than a costly one, which is a restriction already captured by proper equilibrium.

### 2.3 Existence

There are finite games for which test-set equilibria do not exist. Figure 3 presents an example. It can be shown that \((Up, Left, West)\) is the unique Nash equilibrium of this game. However, it is not a test-set equilibrium because \(East\) weakly dominates \(West\) against Geo’s test set: \{\((Up, Left), (Up, Center), (Up, Right), (Down, Left)\}\．

This example also clarifies the relationship to other solution concepts not explicitly

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5The game depicted in Figure 2 of Myerson (1978) can be used as a counterexample for illustrating the latter fact. As he shows, \((α₂, β₂)\) is a perfect equilibrium of this game. However, it is not a test-set equilibrium.

6We thank Michael Ostrovsky, Markus Baldauf, and Bernhard von Stengel for helpful suggestions that led to the construction of this example.
treated here. It implies that there is no solution concept or equilibrium refinement that makes some selection in every finite game and always selects a test-set equilibrium.

Despite this non-existence possibility, there are classes of games in which test-set equilibrium always exists. The following result states three conditions, each of which is sufficient to guarantee the existence of a test-set equilibrium. One sufficient condition is for the game to have two players: proper equilibria exist and by Proposition 1 are also test-set equilibria in such games. A second sufficient condition is for the game to have three players, each of whom has two pure strategies. Together, these two results imply that the game in Figure 3 is the smallest possible game in which a test-set equilibrium may fail to exist. A third sufficient condition is for the game to be a potential game (Monderer and Shapley, 1996): any strategy profile that maximizes the potential function is a test-set equilibrium.

**Proposition 2.** A finite game has at least one test-set equilibrium if it also satisfies at least one of the following conditions:

(i) it is a two-player game,

(ii) it is a three-player game in which each player has two pure strategies, or

(iii) it is a potential game.

### 3 Second Price Auctions with Common Values and Incomplete Information

Abraham, Athey, Babaioff and Grubb (2014) study a model of second price auctions with common values and incomplete information, motivated by Internet advertising auctions. They narrow the set of equilibria by introducing an idiosyncratic concept: tremble robust equilibrium.

They begin their analysis with a two-bidder example. One bidder is informed, receiving a private signal that is either low or high, and the other uninformed. The paper also treats richer settings, but it suffices for our purposes to consider their simple example. We show that by formulating this in the agent normal form, the test-set condition eliminates all but two equilibria, one of which is the unique tremble robust equilibrium.

#### 3.1 Environment

Two bidders participate in a second price auction. The object being auctioned has a common value of \( v \) to both bidders, which is either 0 or 1, each with equal probability. One bidder is informed and learns the value of \( v \). The other bidder is uninformed. Allowable bids are the nonnegative reals, \( \mathbb{R}_+ \).

This game possesses many equilibria, yet the standard refinements do little to focus the set of predictions. Standard refinements do make a focused prediction for the informed bidder: that it will use the dominant strategy of always bidding the value \( v \). However, refinements do little to discipline the uninformed bidder’s strategy.

Trembling hand perfect equilibrium (Selten, 1975) is one example of a standard refinement that proves unhelpful. Simon and Stinchcombe (1995) propose various ways of
extending the concept to infinite games, each of which Abraham, Athey, Babaioff and Grubb (2014) apply to the agent normal form of this example. They show that for any bid $b \in [0, 1]$, it is a perfect equilibrium for the uninformed bidder to bid $b$, for the high-type informed bidder to bid 1, and for the low-type informed bidder to bid 0. Roughly speaking, it may be optimal for the uninformed bidder to bid $b$ if a tremble by the high-type informed bidder to just below $b$ is overwhelmingly more likely than any other tremble from either of the informed bidder types.

3.2 Tremble Robust Equilibrium

To deal with this multiplicity of equilibria, Abraham, Athey, Babaioff and Grubb (2014) introduce a refinement, tremble robust equilibrium, which is tailored specifically to this game. An equilibrium is tremble robust if it is the limit of equilibria of a sequence of perturbations of the game in which with vanishingly small probability an additional bidder submits a randomly chosen bid.

A more formal definition of tremble robust equilibrium is given below. In that definition, for the purposes of this setting, a distribution is said to be standard if its support is $[0, 1]$; if it is continuous, strictly increasing, and differentiable on that interval; and if its density is continuous and positive on that interval.

**Definition 2.** A Nash equilibrium $\sigma$ is tremble robust if there exists a standard distribution $F$, a sequence of positive numbers $\{\varepsilon_j\}_{j=1}^{\infty}$ converging to zero, and a sequence of strategy profiles $\{\sigma_j\}_{j=1}^{\infty}$ converging in distribution to $\sigma$ such that for all $j$:

(i) $\sigma_j$ is a Nash equilibrium of the perturbation of the game in which with probability $\varepsilon_j$ an additional bidder arrives and submits a bid sampled from $F$, and

(ii) $\sigma_j$ does not prescribe dominated bids.

Abraham, Athey, Babaioff and Grubb (2014) show that in this example, there is a single tremble robust equilibrium. In it, the uninformed bidder always bids 0. In addition, the high-type informed bidder bids 1, and the low-type informed bidder bids 0.

3.3 Test-Set Equilibrium

For applying test-set equilibrium, we follow Abraham, Athey, Babaioff and Grubb (2014) by considering the agent normal form of the game. Despite the fact that there are only two bidders, there are three agents in the agent normal form. With three or more agents, test-set equilibrium is not implied by existing concepts, so it is a natural candidate for application to this game.

In contrast to perfect equilibrium, test-set equilibrium does give a reasonably focused prediction of behavior in this example. There are two pure test-set equilibria: the tremble robust equilibrium and one other.

**Proposition 3.** There exist two pure test-set equilibria of the auction. In both, the high-type informed bidder bids 1 and the low-type informed bidder bids 0. The uninformed bidder bids 0 in the first equilibrium and bids 1 in the second.
For any \( b \in (0, 1) \), it is not a test-set equilibrium for the uninformed bidder to bid \( b \), while the high-type informed bidder bids 1 and the low-type informed bidder bids 0. This is in contrast to perfect equilibrium, which does not eliminate these equilibria. To see why test-set equilibrium rules this out, consider the uninformed bidder’s test set, which consists of two types of strategy profiles: (i) those in which the high-type informed bidder bids 1 and the low-type informed bidder bids in \([0, b)\), and (ii) those in which the high-type informed bidder bids in \((b, \infty)\) and the low-type informed bidder bids 0. Any alternative bid \( \hat{b} \neq b \) in \([0, 1]\) weakly dominates \( b \) against this test set.

In conclusion, test-set equilibrium makes a more focused set of predictions than previous general solution concepts, such as perfect equilibrium, identifying just two possible pure equilibria: those in which the uninformed bidder bids 0 or 1. The first of these is the equilibrium selected by the original authors. Although the second test-set equilibrium is ruled out by their specification, it could also be selected by a different sort of tremble-robust concept, in which there is a small probability that the high-value informed bidder faces a budget constraint and bids less than 1.

4 Generalized Second Price Auction

Edelman, Ostrovsky and Schwarz (2007) study the generalized second price (GSP) auction. They narrow the set of equilibria by introducing a second idiosyncratic concept, which they term local envy-free equilibrium. Varian (2007) studies the same auction and makes the same equilibrium selection, referring to these equilibria as “symmetric” Nash equilibria.

In this section, we recapitulate the GSP auction model and study the properties of its test-set equilibria. We find that every pure test-set equilibrium is locally envy-free, but that the test-set criterion also implies additional restrictions. The existence of pure test-set equilibria then depends on the parameters of the game.

4.1 Environment

There are \( I \) ad positions and \( N \) bidders. The click-rate of the \( i \)th position is \( \kappa_i > 0 \). The value per click of bidder \( n \) is \( v_n > 0 \). Advertiser \( n \)'s payoff from being in position \( i \) is \( \kappa_i v_n \) minus its payments to the auctioneer.

The \( N \) bidders simultaneously submit bids. Allowable bids are the nonnegative reals, \( \mathbb{R}_+ \). Let \( b^{(i)} \) denote the \( i \)th highest bid. It is convenient to define \( b^{(N+1)} = 0 \) and \( \kappa_{I+1} = 0 \). Bidders are then sorted in order of their bids, where ties are broken uniformly at random. After ties are broken, let, for \( i \leq \min\{I, N\} \), \( g(i) \) denote the identity of the \( i \)th highest bidder. Let \( G(I + 1) \) denote the set of all other bidders. The GSP mechanism allocates position \( i \) to bidder \( g(i) \) at a per-click price of \( b^{(i+1)} \), for a total payment of \( \kappa_i b^{(i+1)} \). Members of \( G(I + 1) \) win nothing and pay nothing.

The expected payoff to bidder \( n \) under the bid profile \( b = (b_1, \ldots, b_N) \) is

\[
\pi_n(b) = \mathbb{E} \left[ \kappa_{I_n(b)} \left( v_n - b^{(I_n(b)+1)} \right) \right],
\]

where the expectation is taken over the random variable \( I_n(b) \), the position won by bidder \( n \). A GSP auction is modeled as a game \( G = \left[ (\mathbb{R}_+)^N \right]_{n=1}^N, (\pi_n(\cdot))_{n=1}^N \).
The subsequent results rely on the following assumptions about the environment. Assumption 1(i) states that no two positions have the same click rate. Likewise, Assumption 1(ii) states that no two bidders have the same value for a click. Finally, without loss of generality, we assume below that positions and bidders are labeled so click rates and bidder values are in descending order, from highest to lowest.

**Assumption 1.** We assume the following:

(i) $\kappa_1 > \cdots > \kappa_I > 0$, and 

(ii) $v_1 > \cdots > v_N$.

### 4.2 Locally Envy-Free Equilibrium

Edelman, Ostrovsky and Schwarz (2007) introduce a refinement of Nash equilibrium for this application in which no bidder is able to improve its payoff by exchanging bids with the bidder one position above.

**Definition 3.** A pure equilibrium of the GSP auction $b = (b_1, \ldots, b_N)$ is a locally envy-free equilibrium if for all $i \in \{2, \ldots, \min\{I + 1, N\}\}$,

$$\kappa_i \left[ v_{g(i)} - b^{(i+1)} \right] \geq \kappa_{i-1} \left[ v_{g(i)} - b^{(i)} \right].$$

To motivate their focus on Nash equilibria, Edelman, Ostrovsky and Schwarz (2007) argue that the one-shot, complete information game of their model is a good representation of the limit point of the real game, which is a frequently-repeated game of incomplete information. They motivate their refinement to locally envy-free equilibria in several ways. For example, they point out that in the repeated game, one deviation that a bidder could undertake would be to raise its bid, thereby harming the bidder one position above, in the hopes of forcing the latter bidder out of its higher position. They argue that the equilibria of the repeated game that are immune to this particular type of deviation correspond to the locally envy-free equilibria of the one-shot, complete information game.

### 4.3 Test-Set Equilibrium

We now state the main result of this section: that pure test-set equilibria are locally envy-free.

**Theorem 4.** Every pure test-set equilibrium of the GSP auction is a locally envy-free equilibrium.

To prove Theorem 4, we show that if some bidder, say bidder $g(i)$, envies the bidder one position above, bidder $g(i - 1)$, then bidder $g(i)$’s equilibrium bid is weakly dominated against its test set by a slightly higher alternative bid. Indeed, this alternative performs no

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7This definition is worded with some abuse of notation. First, the definition ignores the possibility of tied bids. However, this is not an issue, as Lemma 11 shows that in any pure equilibrium, there are no ties among the highest $\min\{I + 1, N\}$ bids. Second, it would be more correct to say that in the case $i = I + 1$, the inequality must hold for all $g(I + 1) \in G(I + 1)$. 

---
worse than the original bid against any element of the test set. Moreover, the alternative performs strictly better against certain downward deviations by bidder \( g(i-1) \), which are best responses for bidder \( g(i-1) \) and therefore in the test set.

As a consequence of Theorem 4, pure test-set equilibria inherit the attractive properties of locally envy-free equilibria. Specifically, Lemma 1 of Edelman, Ostrovsky and Schwarz (2007) states that the outcome of any locally envy-free equilibrium is a stable assignment.\(^8\) Additionally, stability implies that the resulting matching is assortative – that is, for all \( i \leq \min\{I,N\} \), position \( i \) is won by bidder \( i \) – which is the efficient matching in this setting.

**Corollary 5.** The outcome of every pure test-set equilibrium of the GSP auction is a stable assignment and an assortative matching.

### 4.4 No Converse

Not only is every test-set equilibrium locally envy-free, but the inclusion is strict if there are two or more bidders. We illustrate the relation with the following proposition, which completely characterizes the locally envy-free equilibria and the test-set equilibria for GSP auctions with two bidders and one position.

**Proposition 6.** Let \( I = 1 \) and \( N = 2 \). The set of locally envy-free equilibria of the GSP auction is

\[
E_{\text{LEF}} = \{(b_1, b_2) \in \mathbb{R}_+^2 \mid b_1 > b_2, b_2 \geq v_2, b_2 \leq v_1\},
\]

and the set of pure test-set equilibria is

\[
E_{\text{TS}} = \{(b_1, b_2) \in \mathbb{R}_+^2 \mid b_1 > b_2, b_2 \geq v_2, b_1 \leq v_1\}.
\]

In words, the locally envy-free equilibria are those in which bidder 1 wins the auction, and in which bidder 2’s losing bid is in the interval \([v_2, v_1]\). Thus, local envy-freeness implies no upper bound on bidder 1’s winning bid. Test-set equilibrium strengthens locally envy-free equilibrium by adding such an upper bound to bidder 1’s winning bid: bidder 1 cannot bid higher than its value, as such a bid would be weakly dominated against its test set.

A closer look shows that the relation between locally envy-free equilibrium and test-set equilibrium is that the former requires each bidder to make a bid that is undominated against a different testing set, namely, one in which the bidder who is one position higher may deviate to a different best response. Test-set equilibrium requires players to use bids that are undominated against a larger set of competitors’ bid profiles, allowing as well the possibility that lower bidders may deviate to a different best response.

### 4.5 Nonexistence

The following result illustrates that pure test-set equilibria might fail to exist. In a setting with three positions and three bidders, pure test-set equilibria fail to exist if, roughly speaking, \( \kappa_1 \) and \( \kappa_2 \) are too far apart.

\(^8\)In this setting, an assignment is stable if for every pair of positions \( i \) and \( j \), \( \kappa_i v_{g(i)} - p^{(i)} \geq \kappa_j v_{g(j)} - p^{(j)} \), where \( p^{(i)} \) is the price paid by the winner of position \( i \).
Proposition 7. Let $I = N = 3$. There exists a pure test-set equilibrium of the GSP auction if and only if

$$\frac{v_3}{v_2} \leq \frac{\kappa_2^3 - \kappa_1 \kappa_3}{\kappa_2^3 - \kappa_2 \kappa_3}.$$ 

We sketch the proof, alongside an intuitive explanation. Suppose that $b = (b_1, b_2, b_3)$ is a test-set equilibrium. By Corollary 5, the outcome must be an assortative matching, which requires $b_1 > b_2 > b_3$. Test-set equilibrium implies two additional restrictions. It requires that $b_2$ must exceed some threshold $b_2^{low}$, for otherwise it is weakly dominated against the test-set by $b_2 + \varepsilon$ for some small $\varepsilon$. Intuitively, this higher bid is better when bidder 1 deviates downwards, and otherwise is no worse. But test-set equilibrium also requires that $b_2$ must lie below some threshold $b_2^{high}$, for otherwise it is weakly dominated against the test set by $b_2 - \varepsilon$ for some small $\varepsilon$. Intuitively, this lower bid is better when bidder 3 deviates upwards, and otherwise is no worse. A test-set equilibrium exists if and only if $b_2^{high} \geq b_2^{low}$. The proposition restates that inequality in terms of the exogenous parameters.

5 First Price Menu Auctions

Bernheim and Whinston (1986) study the first price menu auction, and they propose two refinements of its equilibria: truthful equilibrium and coalition-proof equilibrium. Payoffs of an equilibrium satisfying either condition lie on the bidder-optimal frontier of the core.

In this section, we recapitulate the menu auction model, and we study the properties of its test-set equilibria. We find that every truthful equilibrium is a test-set equilibrium. Similarly, every coalition-proof equilibrium payoff vector is a test-set equilibrium payoff vector. However, test-set equilibrium implies fewer restrictions, which expands the set of payoffs that can be implemented to a certain subset of the core containing the bidder-optimal frontier.

5.1 Environment

There is one auctioneer, who selects a decision that affects himself and $N$ bidders. The possible decisions for the auctioneer are those in a finite set $X$. The gross monetary payoff that bidder $n$ receives from any decision is described by the function $v_n : X \rightarrow \mathbb{R}$. Similarly, the auctioneer receives a gross monetary payoff described by $v_0 : X \rightarrow \mathbb{R}$. We make a genericity assumption to ensure that no two decisions generate exactly the same total surplus.

Assumption 2. $\sum_{n=0}^{N} v_n(x)$ is injective.

The $N$ bidders simultaneously submit bids. A bid is a menu of payments to be made to the auctioneer, contingent on the decision chosen, which can be expressed as a function $b_n : X \rightarrow \mathbb{R}_+$. Given bids, the auctioneer chooses a decision that maximizes his payoff $v_0(x) + \sum_{n=1}^{N} b_n(x)$.

Just as for the related Bertrand model, the use of continuous bid spaces here is convenient for characterizing certain solutions, but is inconvenient for creating ties in the players’ preferences. Consistent with the Bertrand modeling tradition, we simplify the analysis by
resolving ties to the extent possible by assuming that all players have lexicographic preferences, first preferring outcomes with the highest personal payoff and secondarily preferring ones with higher total surplus. Assumption 2 ensures that no further indifferences remain after ties have been broken in this way. Applying this tie-breaking assumption to the auctioneer allows his decision to be characterized by a function \( x : (X \rightarrow \mathbb{R}^+) \rightarrow X \). For players, setting aside the lexicographic issue, the payoffs are \( \pi_n(b) = v_n(x(b)) - b_n(x(b)) \). Focusing just on the bidders, a menu auction is “almost” the game \( G = [(X \rightarrow \mathbb{R}^+) \}_{n=1}^{N}, (\pi_n(\cdot)) \}_{n=1}^{N} \), except that bidders choose their bids to break ties as described above. The tie-breaking assumption implies that for any bidder \( n \) and any pure strategy profile of the other bidders, all of bidder \( n \)’s best responses lead to the same auctioneer decision.

5.2 Truthful Equilibrium

Bernheim and Whinston (1986) introduce truthful equilibrium as an idiosyncratic refinement of Nash equilibrium for this mechanism. The formal definition is as follows:

**Definition 4.** A pure Nash equilibrium of the menu auction \( b = (b_1, \ldots, b_N) \) is a truthful equilibrium if and only if for all \( n \in \{1, \ldots, N\} \) and all \( x \in X \), letting \( x^* = x(b) \), either

(i) \( v_n(x) - b_n(x) = v_n(x^*) - b_n(x^*) \), or

(ii) \( v_n(x) - b_n(x) < v_n(x^*) - b_n(x^*) \) and \( b_n(x) = 0 \).

In words, an equilibrium is truthful if each bidder’s bid for each losing decision expresses its full net willingness to pay to switch to that decision instead (subject to nonnegativity constraint on bids). Bernheim and Whinston (1986) motivate this refinement by arguing that such bids are quite simple and may be focal.

5.3 Test-Set Equilibrium

The following lemma is used to prove the main results about test-set equilibria of menu auctions, and is also of independent interest. In words, it says that if we fixed the bids for the winning option, then the test-set equilibria of menu auctions all involve bids for losing decisions that are at least as high as the corresponding truthful bid.

**Lemma 8.** A pure Nash equilibrium of the menu auction \( b = (b_1, \ldots, b_N) \) is a test-set equilibrium if and only if for all \( n \in \{1, \ldots, N\} \) and all \( x \in X \), letting \( x^* = x(b) \),

\[ v_n(x) - b_n(x) \leq v_n(x^*) - b_n(x^*). \]

To see that the inequality is necessary, notice that if some bidder’s equilibrium bid fails this condition for some decision \( x \), then it is weakly dominated against its test set by an alternative bid, which bids slightly higher for that decision. Indeed, this alternative performs no worse than the original bid against any element of the test set, and it performs strictly better in the event that another bidder also deviates by raising its bid on \( x \) by a sufficient extent. Moreover, such deviations can be best responses and can therefore occur in the test set. Next, as observed above, all best responses lead to the same decision. Consequently, the auctioneer’s decision is \( x^* \) for every profile in each bidder’s test set. Thus, for sufficiency,
bringing about a different decision \( x \) on the test set would require the bidder to raise its bid for \( x \), which from the inequality would not be profitable.

An immediate dividend paid by this lemma is the following corollary: that the truthful equilibria are a subset of the test-set equilibria. The converse does not hold.

**Corollary 9.** *Every truthful equilibrium of the menu auction is a test-set equilibrium."

Our next goal is to characterize the payoffs that can arise in a test-set equilibrium. Toward that end, we introduce the following notation. Given a coalition \( J \subseteq \{1, \ldots, N\} \), denote its complement by \( \bar{J} = \{1, \ldots, N\} \setminus J \), and let \( x^J \) denote a decision that maximizes the payoff of the coalition consisting of \( J \) together with the auctioneer:

\[
x^J = \arg \max_{x \in X} \sum_{n \in \{0\} \cup J} v_n(x).
\]

In particular, define \( x^{opt} = x^{\{1, \ldots, N\}} \) to be the decision that maximizes total surplus, which is uniquely defined by Assumption 2. Using this notation, we define \( C \) to be the set of the core payoffs for the bidders:

\[
C = \left\{ \pi \in \mathbb{R}^N : \forall J \subseteq \{1, \ldots, N\}, \quad \sum_{n \in J} \pi_n \leq \sum_{n=0}^N v_n(x^{opt}) - \sum_{n \in \{0\} \cup \bar{J}} v_n(x^J) \right\}.
\]

We also define the following set of payoffs, which we denote \( Z \). For a payoff vector to be in \( Z \), no bidder’s payoff may be worse than what it would achieve were its least favorite decision implemented and no payment made. In addition to that, at least one of two other conditions must hold. Either (i) there are two distinct bidders and two distinct decisions such that each bidder's value for its corresponding decision does not exceed its payoff, or (ii) the sum of bidder payoffs are equal to the sum of the bidder and auctioneer values for the surplus-maximizing decision, minus the auctioneer’s value for his favorite decision.

\[
Z = \left\{ \pi \in \mathbb{R}^N : \forall n \in \{1, \ldots, N\}, \pi_n \geq \min_{x \in X} v_n(x), \quad \right. \left. \text{and either} \right.
\]

\[
\begin{align*}
& (i) \exists x' \neq x'' : \pi_k \geq v_k(x'), \pi_m \geq v_m(x'') \\
& (ii) \sum_{n=1}^N \pi_n = \sum_{n=0}^N v_n(x^{opt}) - v_0(x^0)
\end{align*}
\]

With the help of Lemma 8, we can show that the test-set equilibrium payoffs are characterized as the intersection of \( C \) and \( Z \).

**Theorem 10.** *In all test-set equilibria of the menu auction, the auctioneer implements the surplus-maximizing decision \( x^{opt} \), and the bidders receive payoffs in \( C \cap Z \). Conversely, any payoff vector \( \pi \in C \cap Z \) can be supported by a test-set equilibrium."

In any test-set equilibrium, Lemma 8 requires that bids for losing decisions must be sufficiently high. It then follows from an argument similar to that in *Bernheim and Whinston*
that the surplus-maximizing decision is implemented and that bidder payoffs are in the core, $C$. In particular, if the equilibrium decision were $x^* \neq x^{\text{opt}}$, then the lemma requires that the sum of bids for $x^{\text{opt}}$ would be so high as to contradict $x^*$ being chosen. Moreover if the inequality in the definition of $C$ were violated for some coalition $J$, then the lemma requires that the sum of bids for $x_J^*$ to be so high as to contradict $x^{\text{opt}}$ being chosen.

In any Nash equilibrium (hence any test-set equilibrium) bidder $n$’s payoff must be at least $\min_x v_n(x)$, because the bidder could guarantee itself at least that much by instead submitting a constant bid of zero. This is the first criterion for membership in $Z$. The second criterion is that either condition (i) or condition (ii) from the definition of $Z$ must hold. To prove this, we suppose that condition (i) fails. Provided there are at least two bidders, this requires existence of a decision $x^\dagger$ such that for all bidders $n$ and all decisions $x \neq x^\dagger$, $\pi_n < v_n(x)$. By Lemma 8, all bidders place positive bids on all decisions $x \neq x^\dagger$ in the test-set equilibrium. If any bidder also placed a positive bid on $x^\dagger$, that would generate a contradiction, because the bidder could profitably deviate by reducing its bid an equal amount for all decisions. Thus, in the equilibrium, all bidders must bid zero for $x^\dagger$. Moreover, in the equilibrium, the auctioneer must be indifferent between $x^\dagger$ and $x^{\text{opt}}$ at those bids, for otherwise any bidder could profitably deviate by reducing its bid an equal amount for all decisions except $x^\dagger$. If the auctioneer is indifferent between $x^\dagger$ and $x^{\text{opt}}$, despite no positive bids being placed on $x^\dagger$, then $x^\dagger = x^0$, and condition (ii) holds.

To prove the converse, for any payoff vector $\pi \in C \cap Z$, define the bids $b_n(x) = \max\{v_n(x) - \pi_n, 0\}$. If the second condition in the definition of $Z$ is satisfied, then these bids are a test-set equilibrium giving rise to the payoffs $\pi$. If instead the first condition in the definition of $Z$ is satisfied, then we alter $b_k(\cdot)$ by raising $b_k(x')$ by an amount such that the auctioneer would be indifferent between $x^{\text{opt}}$ and $x'$. Similarly, we alter $b_m(\cdot)$ by raising $b_m(x')$ by an amount such that the auctioneer would be indifferent between $x^{\text{opt}}$ and $x'$. With these modifications, these bids are a test-set equilibrium giving rise to the payoffs $\pi$.

By Corollary 9, test-set equilibrium is a more permissive concept than truthful equilibrium. However, test-set equilibrium is strictly more permissive not only in terms of the strategy profiles it encompasses, but also in terms of the payoff vectors it implements. Bernheim and Whinston (1986) show that truthful equilibria implement payoffs on the bidder-optimal frontier of the core:

$$E = \{ \pi \in \mathbb{R}^N \mid \pi \in C \text{ and } \exists \pi' \in C \text{ with } \pi' \geq \pi \}.$$  

It follows from the results of this section that $E \subseteq C \cap Z$, and the inclusion is typically strict.

Bernheim and Whinston (1986) also applied their then-new concept—coalition-proof equilibrium—to the menu auction game, showing that such equilibria also implement payoffs in $E$. This approach seems to hint that cooperation in selecting bids somehow plays a role in the emergence of core outcomes in equilibrium. The test-set equilibrium analysis highlights that this is not quite right; the individual choice criterion embodied in the test-set definition is sufficient to select a core outcome. For this game, coalition-proofness implies

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9Condition (i) also fails if there is just a single bidder (i.e. $N = 1$). In that case, similar arguments also show that condition (ii) must hold.
something stronger, namely, that the bidders can coordinate to rule out core outcomes that are unanimously less preferred than some other core outcome. Test-set equilibrium does not imply that sort of coordination.

5.4 Package Auction Model

The package auction model is a generalization of the menu auction model, in which each bidder cares about and bids for only some part of the allocation. As one example, there may be a set of goods to be allocated among bidders, with each bidder caring about and bidding for only its own allocation or “package.”

Does Theorem 10 – that test-set equilibrium leads to core payoffs in the menu auction setting – extend to the full set of package auctions? In this section, we demonstrate by example that the answer is no.\(^{10}\)

As before, there is one auctioneer, who selects a decision that affects himself and \(N\) bidders. The possible packages for bidder \(n\) are given by the set \(X_n\) and the possible choices for the auctioneer by \(X \subseteq \times_{n=1}^{N} X_n\). The gross monetary payoffs that bidder \(n\) receives are described by the function \(v_n : X_n \rightarrow \mathbb{R}\). Similarly, the auctioneer receives gross monetary payoffs described by \(v_0 : X \rightarrow \mathbb{R}\).

The \(N\) bidders simultaneously submit bids. A bid is a menu of payments to the auctioneer, contingent on the package received, which can be expressed as a function \(b_n : X_n \rightarrow \mathbb{R}_+\). Given bids, the auctioneer chooses an allocation \(x \in X\) that maximizes his payoff \(v_0(x) + \sum_{n=1}^{N} b_n(x_n)\). As before, we assume that if there are several such decisions, then the auctioneer chooses the one that maximizes the total surplus.\(^{11}\)

While Bernheim and Whinston (1986) demonstrate that truthful equilibrium and coalition-proof equilibrium lead to (bidder-optimal) core payoffs only in the menu auction setting, these results generalize to all package auctions. In contrast, Theorem 10 – that test-set equilibrium leads to core payoffs in the menu auction setting – does not generalize to all package auctions. Example 1, below, describes a package auction that possesses a test-set equilibrium with non-core payoffs.

In the example, there are six bidders, with possible packages \(X_n = \{l, w\}\) (“lose” or “win”). The set \(X\) includes six combinations of packages, describing which sets of bidders can simultaneously win. First, bidder 1 may win alone. Alternatively, bidder 2 may win together with one of bidders 3, 4, 5, or 6. Finally, bidders 3, 4, 5, and 6 may win together. The last of these possibilities maximizes total surplus. However, there exists a test-set equilibrium implementing the allocation in which bidder 1 wins alone.

Example 1. Let \(N = 6\). For all \(n\), let \(X_n = \{l, w\}\). Let

\[
X = \left\{ (w, l, l, l, l), (l, w, l, l, l), (l, w, l, w, l), (l, w, l, l, w), (l, l, w, w, w) \right\}.
\]

\(^{10}\)A further generalization is to the class of core-selecting mechanisms (Day and Milgrom, 2008). The example therefore also implies that test-set equilibrium does not necessarily lead to core payoffs in all core-selecting mechanisms.

\(^{11}\)The package auction model is equivalent to the menu auction in the special case where \(X_1 = \cdots = X_N\) and \(X\) is the diagonal subset of \(\times_{n=1}^{N} X_n\).
For all $x \in X$, let $v_0(x) = 0$. Let the payoffs of the bidders be as follows:

$$
\begin{align*}
&v_1(l) = 0, \quad v_1(w) = 29 \\
v_2(l) = 0, \quad v_2(w) = 19 \\
v_3(l) = 0, \quad v_3(w) = 9 \\
v_4(l) = 0, \quad v_4(w) = 8 \\
v_5(l) = 0, \quad v_5(w) = 7 \\
v_6(l) = 0, \quad v_6(w) = 6
\end{align*}
$$

Then the following bid profile is a test-set equilibrium, which results in the inefficient allocation $(w, l, l, l, l, l)$:

$$
\begin{align*}
&b_1(l) = 0, \quad b_1(w) = 28 \\
b_2(l) = 0, \quad b_2(w) = 19 \\
b_3(l) = 0, \quad b_3(w) = 9 \\
b_4(l) = 0, \quad b_4(w) = 0 \\
b_5(l) = 0, \quad b_5(w) = 0 \\
b_6(l) = 0, \quad b_6(w) = 0
\end{align*}
$$

This package auction example helps to sharpen our understanding of how test-set equilibrium promotes core allocations in the original Bernheim-Whinston menu auction model. Test-set equilibrium is effective there because it promotes “high enough” bids for losing decisions. It does so because each bidder $n$ believes that a deviation by a single other bidder playing a different best response might offer an opportunity for a better outcome, provided that $n$ bids high enough. In this package auction example, however, bidders 4, 5, and 6 are not bidding “high enough,” yet there is no element in any of their respective test sets that offers an opportunity for a better outcome. No single deviation can create such an opportunity; only a joint deviation by two or more others could do that. Coalition-proof equilibrium refines away this package-auction equilibrium because it considers the possibility of a cooperative joint deviation. Truthful strategies work as a refinement in this context, too, because the restriction to truthful bids is a restriction to bids that are high enough for losing decisions. The test-set refinement for these package-auction games, however, does not imply high bids for losing decisions.

### 6 Conclusion

We introduce a new refinement of Nash equilibrium for games, *test-set equilibrium*, which disciplines players’ beliefs about the most likely deviations by their competitors. Test-set equilibrium incorporates the novel condition that players believe that a costless deviation by a single competitor is infinitely more likely than any other type of deviation – either a costly deviation by the same or another competitor, or a simultaneous deviation by multiple competitors. In two-player games, this condition is implied by proper equilibrium. However, with three or more players, previous tremble-based solutions impose no restrictions on the relative likelihood of deviations by different players, so the test-set condition is not implied. Moreover, because the test-set condition can be satisfied even by a dominated strategy, it does not imply any tremble-based solution.
We have illustrated the power of this new concept using three auction models. In all three, test-set equilibrium makes selections that are similar, yet not identical, to the idiosyncratic selections—tremble-robust equilibrium, locally envy-free equilibrium, and truthful equilibrium—that had previously been applied. These findings illuminate the structure and nature of equilibrium in these three celebrated auction games.

References


A Proofs

A.1 Test-Set Equilibrium

Proof of Proposition 1. Suppose that \( \sigma = (\sigma_1, \sigma_2) \) is not a test-set equilibrium. Without loss of generality, suppose it is player 1 whose test-set condition fails. Then there exists \( \hat{\sigma}_1 \in \Delta(S_1) \) and \( \hat{\sigma}_2 \in BR_2(\sigma_1) \) such that (i) \( u_1(\hat{\sigma}_1, \hat{\sigma}_2) > u_1(\sigma_1, \hat{\sigma}_2) \), and (ii) for all \( s_2 \in BR_2(\sigma_1) \), \( u_1(\hat{\sigma}_1, s_2) \geq u_1(\sigma_1, s_2) \).

Next, suppose by way of contradiction that \( \sigma \) is a proper equilibrium. Then there exist sequences \( \{\varepsilon_k\}_{k=0}^{\infty} \) and \( \{\sigma^k\}_{k=0}^{\infty} \) such that (i) each \( \varepsilon_k > 0 \) and \( \lim_{k \to \infty} \varepsilon_k = 0 \), (ii) each \( \sigma^k \) is an \( \varepsilon_k \)-proper equilibrium, and (iii) \( \lim_{k \to \infty} \sigma^k = \sigma \). By the third criterion, we have that for all \( s_2 \in S_2 \), \( \lim_{k \to \infty} u_2(\sigma^k_1, s_2) \to u_2(\sigma_1, s_2) \). Therefore, there must exist some \( K \) such that for all \( k \geq K \) and all \( s_2 \notin BR_2(\sigma_1) \), \( u_2(\sigma^k_1, s_2) < u_2(\sigma^k_2, s_2) \). By the second criterion, this requires that for all such \( k \) and all such \( s_2 \), \( \sigma^k_2(s_2) \leq \varepsilon_k \sigma^k_2(\hat{\sigma}_2) \).

Now, let \( \delta = u_1(\hat{\sigma}_1, \hat{\sigma}_2) - u_1(\sigma_1, \hat{\sigma}_2) > 0 \). Furthermore, let \( M \) denote the difference between player 1’s maximum and minimum payoffs, which exists because the game is finite. Then for all \( k \geq K \),

\[
\begin{align*}
  u_1(\sigma_1, \sigma^k_2) - u_1(\sigma_1, \sigma^k_2) & \geq \delta \sigma^k_2(\hat{\sigma}_2) - \sum_{s_2 \notin BR_2(\sigma_1)} M \sigma^k_2(s_2) \\
  & \geq (\delta - |S_2| M \varepsilon_k) \sigma^k_2(\hat{\sigma}_2),
\end{align*}
\]

which is positive for all sufficiently small values of \( \varepsilon_k \). Thus, for sufficiently large values of \( k \), \( \sigma_1 \) is not a best response to \( \sigma^k_2 \). This contradicts \( \lim_{k \to \infty} \sigma^k_1 = \sigma_1 \).

Proof of Proposition 2. Claim (i): This follows immediately from Proposition 1 and the existence of proper equilibria in finite games.

Claim (ii): We show that for three-player games in which each player has two pure strategies, test-set equilibrium is implied by extended proper equilibrium (Milgrom and Mollner, 2016). The result will then follow from the existence of extended proper equilibrium in finite games, which we establish in that paper.

Consider a three player game with strategy sets \( S_n = \{a_n, b_n\} \). Suppose \( \sigma \) is an extended proper equilibrium of this game that fails the test-set condition. Without loss of generality, suppose it is player 1 whose test-set condition fails. Then there exists \( \hat{\sigma}_1 \in \Delta(S_1) \) that weakly dominates \( \sigma_1 \) against \( T_1(\sigma) \). Also without loss of generality, suppose that \( (a_2, \sigma_3) \) is an element of the test set against which \( \hat{\sigma}_1 \) strictly outperforms \( \sigma_1 \). Thus, \( a_2 \in BR_2(\sigma_{-2}) \), and

\[
u_1(\hat{\sigma}_1, a_2, \sigma_3) > u_1(\sigma_1, a_2, \sigma_3).
\]

Now if \( \sigma_2(a_2) = 1 \), then (2) contradicts Nash equilibrium. Therefore \( \sigma_2(a_2) < 1 \), which implies \( b_2 \in BR_2(\sigma_{-2}) \). Then by the failure of player 1’s test-set condition, we also have

\[
u_1(\hat{\sigma}_1, b_2, \sigma_3) \geq u_1(\sigma_1, b_2, \sigma_3).
\]

Now if \( \sigma_2(a_2) > 0 \), then equations (2) and (3) would together contradict Nash equilibrium. Thus, \( \sigma_2 = b_2 \). The argument now splits into two cases. In the first, \( BR_3(\sigma_{-3}) = \{a_3, b_3\} \). In the second, \( BR_3(\sigma_{-3}) \) is a singleton.
Case 1: Suppose $BR_3(\sigma_{-3}) = \{a_3, b_3\}$. Then $T_1(\sigma) = \{(a_2, \sigma_3), (b_2, \sigma_3), (b_2, a_3), (b_2, b_3)\}$. Then by the failure of player 1’s test-set condition, we also have

\begin{align*}
  u_1(\hat{\sigma}_1, b_2, a_3) &\geq u_1(\sigma_1, b_2, a_3) \\
  u_1(\hat{\sigma}_1, b_2, b_3) &\geq u_1(\sigma_1, b_2, b_3)
\end{align*}

The above inequalities imply that for all totally mixed strategy profiles $\sigma^\varepsilon$ sufficiently close to $\sigma$,

\begin{align*}
  u_1(\hat{\sigma}_1, \sigma^\varepsilon_{-1}) - u_1(\sigma_1, \sigma^\varepsilon_{-1}) \\
  = [u_1(\hat{\sigma}_1, a_2, \sigma^\varepsilon_3) - u_1(\sigma_1, a_2, \sigma^\varepsilon_3)] \sigma^\varepsilon_2(a_2) \\
  + [u_1(\hat{\sigma}_1, b_2, \sigma^\varepsilon_3) - u_1(\sigma_1, b_2, \sigma^\varepsilon_3)] \sigma^\varepsilon_2(b_2) \\
  > 0.
\end{align*}

To see the last step, (2) implies that the first term is positive for totally mixed $\sigma^\varepsilon$ sufficiently close to $\sigma$, and (4) and (5) imply that the second term is nonnegative. This contradicts $\sigma$ being a trembling-hand perfect equilibrium. Since extended proper equilibria are perfect, we therefore cannot be in this case.

Case 2: Without loss of generality, the second case is $BR_3(\sigma_{-3}) = \{a_3\}$, so that $\sigma_3 = a_3$. But then equations (2) and (3) contradict an undominatedness property that Milgrom and Mollner (2016) show is possessed by all extended proper equilibria.\footnote{In the language of the proposition we prove in that paper, let $n = 1$, $m = 2$, and $\hat{s}_m = a_2$. Then $A = (a_2, a_3)$ and $B = (b_2, a_3)$. Applying the proposition, $\hat{\sigma}_1$ dominates $\sigma_1$ against the sets $A$ and $B$ in a way that cannot occur in an extended proper equilibrium.} We therefore cannot be in this case either.

Claim (iii): We use the fact that any potential game is strategically equivalent to a game in which there exists a function $P$, referred to as the potential function of the game, which is such that for all $n$ and for all $s \in X_{n-1}^N S_n$, $u_n(s) = P(s)$. We demonstrate that any $s^* \in \arg \max_{s \in X_{n-1}^N S_n} P(s)$ is a pure test-set equilibrium. It is well-known that such a strategy profile is a Nash equilibrium, so it only remains to check the test-set condition.

Let $n \in \{1, \ldots, N\}$ and $\sigma_{-n} \in T_n(s^*)$. By definition, there exist $m \neq n$ and $\hat{s}_m \in BR_m(s^*_{-m})$ such that $\sigma_{-n} = (\hat{s}_m, s^*_{-nm})$. Let $\hat{P} = P(s^*)$ be the maximum potential. Since $\hat{s}_m \in BR_m(s^*_{-m})$, we have $P(s^*_n, \hat{s}_m, s^*_{-nm}) = P(s^*_n, s^*_m, s^*_{-nm}) = \hat{P}$. Moreover, for any $\hat{\sigma}_n \in \Delta(S_n)$, we must have $P(\hat{\sigma}_n, \hat{s}_m, s^*_{-nm}) \leq \hat{P}$. Therefore, $P(s^*_n, \sigma_{-n}) \geq P(\hat{\sigma}_n, \sigma_{-n})$, as desired. \hfill \Box

### A.2 Second Price Auctions with Common Values and Incomplete Information

**Proof of Proposition 3.** We first argue that the stated strategy profiles are indeed test-set equilibria. We then argue no other pure strategy profiles can be test-set equilibria.

**Part One:** In both purported test-set equilibria, the high-type informed bidder has a dominant strategy, which it is using. Its strategy is therefore undominated against its test-set.
Similarly, the low-type informed bidder’s strategy is also undominated against its test-set. It therefore suffices to consider the uninformed bidder.

Consider the equilibrium in which the uninformed bidder bids 0. The best responses of the low-type informed bidder are \([0, \infty)\). Fix some alternative bid \(b > 0\) for the uninformed bidder. An element of the uninformed bidder’s test set is for the high-type informed bidder to bid 1 and for the low-type informed bidder to bid \(\frac{b}{2}\). The uninformed bidder’s equilibrium bid of 0 outperforms \(b\) against this element, so \(b\) cannot dominate the equilibrium bid against the test set.

Consider the equilibrium in which the uninformed bidder bids 1. The best responses of the high-type informed bidder are \([0, 1)\). Fix some alternative bid \(b < 1\) for the uninformed bidder. An element of the uninformed bidder’s test set is for the high-type informed bidder to bid \(\frac{1+b}{2}\) and for the low-type informed bidder to bid 0. The uninformed bidder’s equilibrium bid of 1 outperforms \(b\) against this element, so \(b\) cannot dominate the equilibrium bid against the test set.

**Part Two:** Consider a pure strategy profile in which the uninformed bidder bids \(b\), the high-type informed bidder bids \(b_1\), and the low-type informed bidder bids \(b_0\). We argue that if this is a test-set equilibrium, then it must be one of the stated strategy profiles. We consider four cases:

(i) \(b = 0\). Suppose by way of contradiction that \(b_0 > 0\). Then best responses for the uninformed bidder include \([0, b_0)\). But then a bid of 0 weakly dominates \(b_0\) against the test set of the low-type informed bidder. Next, suppose by way of contradiction that \(b_1 < 1\). But then a bid of 1 would be a profitable deviation for the uninformed bidder. Finally, suppose by way of contradiction that \(b_1 > 1\). Then best responses for the uninformed bidder are \([0, b_0)\). But then a bid of 1 weakly dominates \(b_1\) against the test set of the high-type informed bidder. This leaves us with the other of the stated strategy profiles.

(ii) \(b = 1\). Suppose by way of contradiction that \(b_1 < 1\). Then best responses for the uninformed bidder include \((b_1, \infty)\). But then a bid of 1 weakly dominates \(b_1\) against the test set of the high-type informed bidder. Next, suppose by way of contradiction that \(b_0 > 0\). But then a bid of 0 would be a profitable deviation for the uninformed bidder. This leaves us with the other of the stated strategy profiles.

(iii) \(b > 1\). Best responses for both the high-type informed bidder and the low-type informed bidder are \([0, b)\). But then a bid of 1 weakly dominates \(b\) against the test set of the uninformed bidder, thereby ruling out this case.

(iv) \(b \in (0, 1)\). The low-type informed bidder can be best responding only if \(b_0 < b\). Similarly, the high-type informed bidder can be best responding only if \(b_1 > b\). Suppose by way of contradiction that \(b_0 > 0\). But then a bid of 0 would be a profitable deviation for the uninformed bidder. Similarly, suppose by way of contradiction that \(b_1 < 1\). But then a bid of 1 would be a profitable deviation for the uninformed bidder. Finally, suppose by way of contradiction that \(b_1 > 1\). Then best responses for the uninformed bidder are \([0, b_1)\). But then a bid of 1 weakly dominates \(b_1\) against the test set of the high-type informed bidder. Thus, \(b_0 = 0\) and \(b_1 = 1\), and consequently
the uninformed bidder’s test set consists of two types of strategy profiles: (i) those in which the high-type informed bidder bids 1 and the low-type informed bidder bids in [0, b), and (ii) those in which the high-type informed bidder bids in (b, 1] and the low-type informed bidder bids 0. Any alternative bid $\hat{b} \in [0, 1]$ dominates $b$ against this test set, thereby ruling out this case. □

A.3 Generalized Second Price Auction

Lemma 11 shows that there will be no ties among the highest $\min\{I, N\}$ bids in any pure Nash equilibrium. It will be helpful in proving Theorem 4, which is the main result of this section.

**Lemma 11.** If $b = (b_1, \ldots, b_N)$ is a Nash equilibrium of the GSP auction, then for all $i \in \{1, \ldots, \min\{I, N-1\}\}$, $b^{(i)} > b^{(i+1)}$.

**Proof of Lemma 11.** Suppose $i \in \{1, \ldots, \min\{I, N-1\}\}$ and $K$ are such that $b^{(i)} = \cdots = b^{(i+K)} = b^*$ is a “maximal tie.” That is to say, suppose there are exactly $K + 1$ bidders who bid $b^*$. We derive a contradiction from $K \geq 1$. First, note that any bidder with per-click value $v$ who is part of the tie earns the payoff

$$\frac{1}{K+1} \left[ \kappa_{i+K}(v - b^{(i+K + 1)}) + \sum_{k=0}^{K-1} \kappa_{i+k}(v - b^*) \right].$$

By raising its bid to just above $b^*$, the bidder would earn the payoff

$$\kappa_i(v - b^*).$$

If $b^* = 0$, then this deviation would be a profitable one. We therefore assume henceforth that $b^* > 0$. In that case, the bidder can reduce its bid to just below $b^*$, which would lead to the payoff

$$\kappa_{i+K}(v - b^{(i+K + 1)}).$$

Let $v' < v''$ be the per-click values of two of the tied bidders (Assumption 1(ii) rules out the possibility of equality). Because the bidder with value $v'$ does not find it profitable to deviate to just below $b^*$,

$$\frac{1}{K+1} \left[ \kappa_{i+K}(v' - b^{(i+K + 1)}) + \sum_{k=0}^{K-1} \kappa_{i+k}(v' - b^*) \right] \geq \kappa_{i+K}(v' - b^{(i+K + 1)})$$

$$\implies \kappa_{i+K}(v' - b^{(i+K + 1)}) + K \kappa_i(v' - b^*) \geq (K + 1) \kappa_{i+K}(v' - b^{(i+K + 1)})$$

$$\implies \kappa_i(v' - b^*) \geq \kappa_{i+K}(v' - b^{(i+K + 1)}). \quad (6)$$

Because the bidder with value $v''$ does not find it profitable to deviate to just above $b^*$,

$$\frac{1}{K+1} \left[ \kappa_{i+K}(v'' - b^{(i+K + 1)}) + \sum_{k=0}^{K-1} \kappa_{i+k}(v'' - b^*) \right] \geq \kappa_i(v'' - b^*)$$

$$\implies \kappa_{i+K}(v'' - b^{(i+K + 1)}) + K \kappa_i(v'' - b^*) \geq (K + 1) \kappa_i(v'' - b^*)$$

$$\implies \kappa_{i+K}(v'' - b^{(i+K + 1)}) \geq \kappa_i(v'' - b^*). \quad (7)$$
Adding together equations (6) and (7) and canceling like terms, we obtain \( \kappa_i v' + \kappa_{i+K} v'' \geq \kappa_{i+K} v' + \kappa_i v'' \). Equivalently,

\[
(k_i - \kappa_{i+K})(v' - v'') \geq 0.
\]

By Assumption 1(i) and the fact that \( K \geq 1 \), we have \( \kappa_i > \kappa_{i+K} \). Furthermore, we previously supposed \( v' < v'' \). This is therefore a contradiction. \( \Box \)

**Proof of Theorem 4.** Suppose that \( b \) is a pure test-set equilibrium that is not locally envy-free. By Lemma 11, there are no ties among the \( \min\{I, N\} \) highest bidders. Therefore, for all \( i \in \{1, \ldots, \min\{I, N\}\} \), \( G(i) \) is a singleton, the unique element of which we denote \( g(i) \). Note, however, that in the case where \( N > I \), \( G(I+1) \) is defined as the \( N-I \) lowest bids, and might therefore not be a singleton.

Let \( i^* \) be the largest index for which the locally envy-free inequality is violated, so that for some element of \( G(i^*) \), which we henceforth denote \( g(i^*) \),

\[
\kappa_{i^*-1} (v_{g(i^*)} - b(i^*)) > \kappa_{i^*} (v_{g(i^*)} - b(i^*+1)).
\]

We use \( b^* \) to denote the equilibrium bid of \( g(i^*) \). For the case in which \( i^* \leq I \), \( b^* = b(i^*) \). For the case in which \( i^* = I+1 \), \( b^* \in [0, b(i^*)] \).

With this established, the proof consists of three parts. First, we demonstrate that bidders who bid lower than \( g(i^*) \) are sorted by their values for clicks. We then derive the desired contradiction by demonstrating that there exists an \( \varepsilon > 0 \) such that \( b(i^*) + \varepsilon \) weakly dominates \( b^* \) on \( g(i^*) \)'s test set. The second part establishes this for the case in which \( i^* = I+1 \). The third part establishes this for the case in which \( i^* \leq I \).

**Part One:** Let \( k \in \{1, \ldots, I+1 - i^*\} \). Suppose that \( g(i^* + k) \in G(i^* + k) \). Because \( i^* \) was defined as the largest index for which the locally envy-free inequality is violated, we have

\[
\kappa_{i^*+k} (v_{g(i^*+k)} - b(i^*+k+1)) \geq \kappa_{i^*+k-1} (v_{g(i^*+k)} - b(i^*+k)).
\]

Furthermore, equilibrium requires that bidder \( g(i^* + k - 1) \) cannot profit by deviating to just below \( b(i^*+k) \). Thus,

\[
\kappa_{i^*+k-1} (v_{g(i^*+k-1)} - b(i^*+k)) \geq \kappa_{i^*+k} (v_{g(i^*+k-1)} - b(i^*+k+1)).
\]

Manipulating these inequalities yields

\[
(k_{i^*+k-1} - \kappa_{i^*+k}) (v_{g(i^*+k-1)} - v_{g(i^*+k)}) \geq 0.
\]

By Assumption 1(i), \( \kappa_{i^*+k-1} > \kappa_{i^*+k} \), and so \( v_{g(i^*+k-1)} \geq v_{g(i^*+k)} \). By Assumption 1(ii), the inequality must actually be strict. Because we can make this argument for all \( k \in \{1, \ldots, I+1 - i^*\} \), we conclude that the bidders below \( g(i^*) \) are sorted by value:

\[
v_{g(i^*)} > v_{g(i^*+1)} > \cdots > \max_{g(I+1) \in G(I+1)} v_{g(I+1)}.
\]

This observation will be useful in part three of this proof.
Part Two: Suppose that \( i^* = I + 1 \). Define
\[
\varepsilon = \frac{1}{2} \min \left\{ b^{(I)} - b^{(I+1)}, v_{g(i^*)} - b^{(I+1)} \right\}.
\]

By Lemma 11, \( b^{(I)} > b^{(I+1)} \), which implies that the first component of the minimum is positive. That the second component is positive follows from (8), the violation of the locally envy-free inequality for \( g(i^*) \). (To see this, recall that we use the convention \( \kappa_{I+1} = 0 \).) Thus, \( \varepsilon > 0 \).

We now compare whether bidder \( g(i^*) \) is better off by playing \( b^{(I+1)} + \varepsilon \) or \( b^* \) against the elements of its test set. By the definition of \( \varepsilon \), \( b^{(I+1)} + \varepsilon < b^{(I)} \). Therefore, both bids perform the same in the event that none of the competitors of \( g(i^*) \) deviate. We complete the analysis by considering three classes of deviations.

(i) First, \( b^* \) and \( b^{(I+1)} + \varepsilon \) perform equally well against all elements of the test-set in which the deviating bidder does not deviate within the interval \([b^*, b^{(I+1)} + \varepsilon]\).

(ii) Second, suppose that the deviator is \( g(i) \) for some \( i \leq I \) and that the deviation is to within the interval \([b^*, b^{(I+1)} + \varepsilon]\) (i.e. a higher bidder who deviates downward). To begin, assume that the deviation, which we denote \( \hat{b} \), is to the interior of the interval. Then the incremental payoff that \( g(i^*) \) receives from playing \( b^{(I+1)} + \varepsilon \) instead of \( b^* \) is
\[
\kappa_I \left( v_{g(i^*)} - \hat{b} \right) \geq \kappa_I \left( v_{g(i^*)} - b^{(I+1)} - \varepsilon \right) > 0.
\]

The last step in the above uses the fact that by definition, \( \varepsilon < v_{g(i^*)} - b^{(I+1)} \). Finally, if the deviation is to one of the endpoints of the interval, then the tie will be broken randomly, so the expected gains are only half of those above, yet are still positive.

(iii) Third, suppose that the deviator is another member of \( G(I+1) \) and that the deviation is to within the interval \([b^*, b^{(I+1)} + \varepsilon]\) (i.e. a lower bidder who deviates upward). In these cases, \( b^{(I+1)} + \varepsilon \) and \( b^* \) perform equally well: both result in a payoff of zero for \( g(i^*) \).

Moreover, notice that since \( b^{(I+1)} + \varepsilon \) is a best response to equilibrium for bidder \( g(I) \), there is at least one element of type (ii) in the test set of \( g(i^*) \). Therefore, \( b^{(I+1)} + \varepsilon \) weakly dominates \( b^* \) against \( T_{g(i^*)}(\hat{b}) \). Consequently, \( b \) is not a test-set equilibrium.

Part Three: Suppose that \( i^* \leq I \). Define
\[
\varepsilon = \frac{1}{2} \min \left\{ b^{(i^*-1)} - b^*, \kappa_{i^*-1} \left( v_{g(i^*)} - b^* \right) - \kappa_{i^*-1} \left( v_{g(i^*)} - b^{(i^*-1)} \right) \right\}.
\]

By Lemma 11, \( b^{(i^*-1)} > b^* \), which implies that the first component of the minimum is positive. That the second component is positive follows from (8), the violation of the locally envy-free inequality for \( g(i^*) \). Thus, \( \varepsilon > 0 \).

We now compare whether bidder \( g(i^*) \) is better off by playing \( b^* + \varepsilon \) or \( b^* \) against the elements of its test set. By the definition of \( \varepsilon \), \( b^* + \varepsilon < b^{(i^*-1)} \). Therefore, both bids perform the same in the event that none of the competitors of \( g(i^*) \) deviate. We complete the analysis by considering three classes of deviations.
(i) First, \( b^* \) and \( b^* + \varepsilon \) perform equally well against all elements of the test-set in which the deviating bidder does not deviate within the interval \([b^*, b^* + \varepsilon]\).

(ii) Second, suppose that the deviator is \( g(i) \) for some \( i < i^* \) and that the deviation is to within the interval \([b^*, b^* + \varepsilon]\) (i.e. a higher bidder who deviates downward). To begin, assume that the deviation, which we denote \( \hat{b} \), is to the interior of the interval. Then the incremental payoff that \( g(i^*) \) receives from playing \( b^* + \varepsilon \) instead of \( b^* \) is

\[
\kappa_{i^*-1} \left( v_{g(i^*)} - \hat{b} \right) - \kappa_{i^*} \left( v_{g(i^*)} - b^{(i^*)+1} \right) > \kappa_{i^*-1} \left( v_{g(i^*)} - b^* - \varepsilon \right) - \kappa_{i^*} \left( v_{g(i^*)} - b^{(i^*)+1} \right) > 0.
\]

The last step in the above uses the fact that by the definition of \( \varepsilon \),

\[
\varepsilon < \frac{\kappa_{i^*-1} \left( v_{g(i^*)} - b^* \right) - \kappa_{i^*} \left( v_{g(i^*)} - b^{(i^*)+1} \right)}{\kappa_{i^*-1}}.
\]

Finally, if the deviation is to one of the endpoints of the interval, then the tie will be broken randomly, so the expected gains are only half of those above, yet are still positive.

(iii) The third remaining possibility is that the deviator is \( g(i) \in G(i) \) for some \( i > i^* \) and that the deviation is to within the interval \([b^*, b^* + \varepsilon]\) (i.e. a lower bidder who deviates upward). We argue that there are no elements in \( g(i^*) \)'s test set of this form by showing that this cannot be a best response for \( g(i) \).

Because \( i^* \) was defined as the largest index for which the locally envy-free inequality is violated, we have that for all \( k \in [1, i - i^*] \),

\[
\kappa_{i^*+k} \left( v_{g(i^*+k)} - b^{(i^*+k)+1} \right) \geq \kappa_{i^*+k-1} \left( v_{g(i^*+k)} - b^{(i^*+k)} \right).
\]

From part one of this proof, we have that for all \( k \in [1, i - i^*] \), \( v_{g(i^*+k)} \geq v_{g(i)} \), which implies

\[
\kappa_{i^*+k} \left( v_{g(i)} - b^{(i^*+k)+1} \right) \geq \kappa_{i^*+k-1} \left( v_{g(i)} - b^{(i^*+k)} \right).
\]

Summing the above equation across all \( k \in [1, i - i^*] \), then canceling like terms, we obtain

\[
\kappa_{i} \left( v_{g(i)} - b^{(i+1)} \right) \geq \kappa_{i^*} \left( v_{g(i)} - b^{(i^*+1)} \right),
\]

which implies

\[
\kappa_{i} \left( v_{g(i)} - b^{(i+1)} \right) > \kappa_{i^*} \left( v_{g(i)} - b^* \right).
\]

This implies that the deviation is not a best response for \( g(i) \), as desired.

Moreover, notice that since \( b^* + \varepsilon \) is a best response to equilibrium for bidder \( g(i^* - 1) \), there is at least one element of type (ii) in the test set of \( g(i^*) \). Therefore, \( b^* + \varepsilon \) weakly dominates \( b^* \) against \( T_{g(i^*)}(b) \). Consequently, \( b \) is not a test-set equilibrium. \( \square \)
Proof of Proposition 6. *Locally Envy-Free Equilibrium.* Suppose \( b \) is a locally envy-free equilibrium. That \( b_1 > b_2 \) follows from the fact that in any locally envy-free equilibrium, the resulting matching is assortative (Edelman, Ostrovsky and Schwarz, 2007). Next, suppose that \( b_2 < v_2 \). Then bidder 2 envies bidder 1, since \( \kappa_1(v_2 - b_2) > 0 \), contradicting local envy-freeness. Finally, suppose that \( b_2 > v_1 \). Then bidder 1’s payoff is \( \kappa_1(v_1 - b_2) < 0 \), so deviating to \( b_1 = 0 \) would be profitable. Therefore \( b \in \mathcal{E}_{LEF} \).

Now suppose that \( b_2 \in \mathcal{E}_{LEF} \). It is easily checked that \( b \) is a Nash equilibrium. For \( b \) to be locally envy-free, we must have \( 0 \leq \kappa_1(v_2 - b_2) \), which is satisfied because \( b_2 \geq v_2 \).

**Test-Set Equilibrium.** Suppose that \( b \) is a pure test-set equilibrium. That \( b_1 > b_2 \) and \( b_2 \geq v_2 \) follow from Theorem 4 and the above characterization of locally envy-free equilibria. Finally, we cannot have \( b_1 > v_1 \), because such a bid is weakly dominated by \( \hat{b}_1 = v_1 \) against \( T_1(b) \).

For the reverse direction, it is routine to check that any \( b \in \mathcal{E}_{TS} \) is a test-set equilibrium.

\[ \square \]

Proof of Proposition 7. For ease of reference, we restate here the condition in the proposition, and we denote it \((*)\):

\[
\frac{v_3}{v_2} \leq \frac{\kappa_2^2 - \kappa_1\kappa_3}{\kappa_2^2 - \kappa_2\kappa_3}.
\]

The proof proceeds in two parts. First, we demonstrate by construction that \((*)\) is sufficient for the existence of a test-set equilibrium in the environment with three bidders and three positions. Second, we also demonstrate that \((*)\) is necessary for the existence of a test-set equilibrium in this environment. Our strategy for the latter is to use Theorem 4 to establish a lower bound on bidder 2’s bid in any test-set equilibrium. We then also establish a corresponding upper bound and demonstrate that both cannot be simultaneously satisfied if \((*)\) is violated.

**Part One ( Sufficiency):** We argue that if \((*)\) holds, then the following is a test-set equilibrium:

\[
\begin{align*}
    b_1 &= \left(1 - \frac{\kappa_2}{\kappa_1}\right)v_1 + \frac{\kappa_2 - \kappa_3}{\kappa_1}v_3 \\
    b_2 &= \left(1 - \frac{\kappa_2}{\kappa_1}\right)v_2 + \frac{\kappa_2 - \kappa_3}{\kappa_1}v_3 \\
    b_3 &= \left(1 - \frac{\kappa_3}{\kappa_2}\right)v_3
\end{align*}
\]

We can see from these expressions that \( b_1 > b_2 > b_3 \). We can also see that the best responses to equilibrium for bidder 3 are \( \hat{b}_3 \in (0, b_2) \), for bidder 2 are \( \hat{b}_2 \in (b_3, b_1) \), and for bidder 1 are \( \hat{b}_1 \in (b_2, \infty) \).

Using this, we now check that bidder 2 does not have an alternate bid that weakly dominates \( \hat{b}_2 \) against its test-set. Suppose, by way of contradiction, that such a bid, \( b_2' \), exists. Since \( b_2' \) must be a best response to equilibrium, \( b_2' \in (b_1, b_1) \). There are two cases.
(i) First, suppose $b'_2 > b_2$. Then we have a contradiction, since $b_2$ outperforms $b'_2$ when bidder 1 deviates to some $b_1 \in (b_2, b'_2)$. Indeed, bidder 2’s payoff from playing $b'_2$ against this deviation is

$$\kappa_1(v_2 - \hat{b}_1) < \kappa_1(v_2 - b_2)$$

$$= \kappa_2v_2 - \kappa_2v_3 + \kappa_3v_3.$$

On the other hand, bidder 2’s payoff from playing $b_2$ against this deviation is

$$\kappa_2(v_2 - b_3) = \kappa_2v_2 - \kappa_2v_3 + \kappa_3v_3.$$

(ii) Second, suppose $b'_2 < b_2$. Then we also have a contradiction, since $b_2$ outperforms $b'_2$ when bidder 3 deviates to some $b_3 \in (b'_2, b_2)$. Indeed, bidder 2’s payoff from playing $b'_2$ against this deviation is $\kappa_3v_2$. On the other hand, bidder 2’s payoff from playing $b_2$ against this deviation is

$$\kappa_2(v_2 - \hat{b}_3) > \kappa_2(v_2 - b_2)$$

$$= \frac{\kappa_2}{\kappa_1}v_2 - \frac{\kappa_2}{\kappa_1}v_3$$

$$\geq \frac{\kappa_2}{\kappa_1}v_2 - \frac{\kappa_2}{\kappa_1}v_3 \cdot \frac{\kappa_2 - \kappa_1 \kappa_3}{\kappa_2 - \kappa_2 \kappa_3}v_2$$

$$= \kappa_3v_2,$$

where $(\star)$ is used in the penultimate step of the above.

Similarly, suppose that bidder 3 has an alternate bid, $b'_3$, that weakly dominates $b_3$ against its test set. Since $b'_3$ must be a best response to equilibrium, $b'_3 \in [0, b_2)$. It cannot be the case that $b'_3 < b_3$, since both perform the same against all elements of bidder 3’s test set. Thus, suppose that $b'_3 > b_3$. Then we have a contradiction, since $b_3$ outperforms $b'_3$ when bidder 2 deviates to some $b_2 \in (b_3, b'_3)$. Indeed, bidder 3’s payoff from playing $b'_3$ against this deviation is

$$\kappa_2(v_3 - \hat{b}_2) < \kappa_2(v_3 - b_3)$$

$$= \kappa_3v_3.$$

On the other hand, bidder 3’s payoff from playing $b_3$ against this deviation is $\kappa_3v_3$.

Lastly, suppose that bidder 1 has an alternate bid, $b'_1$, that weakly dominates $b_1$ against its test set. Since $b'_1$ must be a best response to equilibrium, $b'_1 \in (b_2, \infty)$. It cannot be the case that $b'_1 > b_1$, since both perform the same against all elements of bidder 1’s test set. Thus, suppose that $b'_1 < b_1$. Then we have a contradiction, since $b_1$ outperforms $b'_1$ when bidder 2 deviates to some $b_2 = (b'_1, b_1)$. Indeed, bidder 1’s payoff from playing $b'_1$ against this deviation is

$$\kappa_2(v_1 - b_3) = \kappa_2v_1 - \kappa_2v_3 + \kappa_3v_3.$$

On the other hand, bidder 1’s payoff from playing $b_1$ against this deviation is

$$\kappa_1(v_1 - \hat{b}_2) > \kappa_1(v_1 - b_1)$$

$$= \kappa_2v_1 - \kappa_2v_3 + \kappa_3v_3.$$
We conclude that \((b_1, b_2, b_3)\) is, indeed, a test-set equilibrium.

**Part Two (Necessity):** Suppose that \(b = (b_1, b_2, b_3)\) is a pure test-set equilibrium. By Theorem 4, the equilibrium is locally envy-free. Locally envy-free equilibria feature assortative matching, so \(b_1 > b_2 > b_3\). Furthermore, since \(b\) is locally envy-free, bidder 3 must not envy bidder 2. That is, we must have \(\kappa_2(v_3 - b_3) \leq \kappa_3 v_3\). Equivalently,

\[
b_3 \geq \left(1 - \frac{\kappa_3}{\kappa_2}\right) v_3. \tag{9}\]

Similarly, bidder 2 must not envy bidder 1. That is, we must have \(\kappa_1(v_2 - b_2) \leq \kappa_2(v_2 - b_3)\). Equivalently,

\[
b_2 \geq v_2 - \frac{\kappa_2}{\kappa_1}(v_2 - b_3). \tag{10}\]

Substituting (9) into (10), we obtain

\[
b_2 \geq v_2 - \frac{\kappa_2}{\kappa_1}(v_2 - v_3) - \frac{\kappa_3}{\kappa_1} v_3. \tag{11}\]

Equation (11) is the desired lower bound on \(b_2\). We next establish the following upper bound on \(b_2\):

\[
b_2 \leq \left(1 - \frac{\kappa_3}{\kappa_2}\right) v_2. \tag{12}\]

To see that this must be the case, assume by way of contradiction that \(b_2 > \left(1 - \frac{\kappa_3}{\kappa_2}\right) v_2\). Then define

\[
\varepsilon = \frac{1}{2} \min \left\{b_2 - \left(1 - \frac{\kappa_3}{\kappa_2}\right) v_2, b_2 - b_3, \frac{(\kappa_1 - \kappa_2)(v_1 - v_2)}{\kappa_1}\right\}.
\]

By assumption, the first component of the minimum is positive. We also know that \(b_2 > b_3\), so the second component of the minimum is positive. That the third component is positive follows from Assumption 1. Thus, \(\varepsilon > 0\).

We then compare the performance of \(b_2 - \varepsilon\) to that of \(b_2\) against the elements of \(T_2(b)\). By the definition of \(\varepsilon\), \(b_2 - \varepsilon > b_3\). Therefore, both bids perform the same in the event that none of the competitors of bidder 2 deviate. We complete the analysis by considering three classes of deviations.

(i) First, \(b_2\) and \(b_2 - \varepsilon\) perform equally well against all elements of the test set in which the deviating bidder does not deviate within the interval \([b_2 - \varepsilon, b_2]\).

(ii) Second, suppose that the deviator is bidder 1 and that the deviation is to within the interval \([b_2 - \varepsilon, b_2]\). For this to be an element of the test set, it must be a best response to equilibrium for bidder 1. This requires \(\kappa_1(v_1 - b_2) = \kappa_2(v_1 - b_3)\). To begin, we
assume that the deviation, which we denote \( \hat{b}_1 \), is to the interior of the interval. Then the incremental payoff that bidder 2 receives from playing \( b_2 - \varepsilon \) instead of \( b_2 \) is

\[
\kappa_2(v_2 - b_3) - \kappa_1(v_2 - \hat{b}_1) > \kappa_2(v_2 - b_3) - \kappa_1(v_2 - b_2 + \varepsilon) \\
= \kappa_2(v_2 - b_3) - \kappa_1(v_2 - b_2 + \varepsilon) - \kappa_2(v_1 - b_3) + \kappa_1(v_1 - b_2) \\
= (\kappa_1 - \kappa_2)(v_1 - v_2) - \kappa_1 \varepsilon \\
> 0.
\]

The last step in the above uses the fact that by definition, \( \varepsilon < \frac{\kappa_1 - \kappa_2}{\kappa_1} (v_1 - v_2) \). Finally, if the deviation is to one of the endpoints of the interval, then the tie will be broken randomly, so the expected gains are only half of those above, yet are still positive.

(iii) The third remaining possibility is that the deviator is bidder 3 and that the deviation is to within the interval \([b_2 - \varepsilon, b_2]\). To begin, we assume that the deviation, which we denote \( \hat{b}_3 \), is to the interior of the interval. Then the incremental payoff that bidder 2 receives from playing \( b_2 - \varepsilon \) instead of \( b_2 \) is

\[
\kappa_3v_2 - \kappa_2(v_2 - \hat{b}_3) > \kappa_3v_2 - \kappa_2(v_2 - b_2 + \varepsilon) > 0.
\]

The last step in the above uses the fact that by definition, \( \varepsilon < b_2 - \left(1 - \frac{\kappa_3}{\kappa_2}\right) v_2 \).

Finally, if the deviation is to one of the endpoints of the interval, then the tie will be broken randomly, so the expected gains are only half of those above, yet are still positive.

Moreover, notice that since \( b_2 - \varepsilon \) is a best response to equilibrium for bidder 3, there is at least one element of type (iii) in the test set. Therefore, \( b_2 - \varepsilon \) weakly dominates \( b_2 \) against \( T_2(b) \). This is the desired contradiction, which establishes the necessity of (12).

Thus we must have both (12), an upper bound on \( b_2 \), and (11), a lower bound on \( b_2 \). In order for both to be simultaneously satisfied we must have that, as claimed,

\[
\frac{v_3}{v_2} \leq \frac{\kappa_2 - \kappa_1 \kappa_3}{\kappa_2 - \kappa_2 \kappa_3}.
\]

\( \square \)

### A.4 First Price Menu Auction

**Proof of Lemma 8.** Let \( b \) be a Nash equilibrium of the first-price menu auction. Let \( x^* = x(b) \). We first argue that for all bidders \( n \), it must be the case that \( \hat{b}_n \in BR_n(b_{-n}) \) if and only if both \( x(\hat{b}_n, b_{-n}) = x^* \) and \( \hat{b}_n(x^*) = b_n(x^*) \).

For sufficiency, the two conditions imply that \( (\hat{b}_n, b_{-n}) \) leads to the same outcome for bidder \( n \) as \((b_n, b_{-n})\). Since \( b \) is a Nash equilibrium, \( b_n \in BR_n(b_{-n}) \), and therefore \( b_n \in BR_n(b_{-n}) \) as well. For necessity, recall that between any two decisions that give a bidder the same payoff, we assume the bidder prefers the decision with the higher total payoff (and by Assumption 2, no two decisions have the same total payoff). Therefore, as observed in the text, all of bidder \( n \)'s best responses must lead to the same decision. Furthermore, all best responses must lead to the same payments for that decision. Since \( b \) is a Nash equilibrium, \( b_n \in BR_n(b_{-n}) \). If also \( b_n \in BR_n(b_{-n}) \), then the two conditions must hold.
**Part One (Necessity):** Suppose by way of contradiction that there exists some bidder $k$ and some $\hat{x} \in X$ for which $v_k(\hat{x}) - b_k(\hat{x}) > v_k(x^*) - b_k(x^*)$. We can therefore define $\varepsilon > 0$ to be

$$
\varepsilon = [v_k(\hat{x}) - b_k(\hat{x})] - [v_k(x^*) - b_k(x^*)].
$$

Choose some $m \neq k$ and define the following bids for bidders $k$ and $m$

$$
\hat{b}_k(x) = \begin{cases} 
  b_k(x) & \text{if } x \neq \hat{x} \\
  b_k(\hat{x}) + \frac{2\varepsilon}{3} & \text{if } x = \hat{x}
\end{cases}
$$

$$
\hat{b}_m(x) = \begin{cases} 
  b_m(x) & \text{if } x \neq \hat{x} \\
  v_0(x^*) - v_0(\hat{x}) + \sum_{n=1}^{N} b_n(x^*) - \sum_{n \neq m} b_n(\hat{x}) - \frac{\varepsilon}{3} & \text{if } x = \hat{x}
\end{cases}
$$

We conclude this direction of the proof by establishing three claims. First, that $\hat{b}_m \in BR_m(b_{-m})$. Second, that $\pi_k(\hat{b}_k, \hat{b}_m, b_{-km}) > \pi_k(b_k, \hat{b}_m, b_{-km})$. Third, for any $n \neq k$ and any $\hat{b}_m \in BR_n(b_{-n})$, that (i) $\pi_k(b_k, \hat{b}_m, b_{-km}) \geq \pi_k(b_k, \hat{b}_m, b_{-km})$, and (ii) in the case of equality, $x(\hat{b}_k, \hat{b}_m, b_{-km}) = x(b_k, \hat{b}_m, b_{-km})$ as well. These three claims constitute a contradiction to $b$ having been a test-set equilibrium.

First, we show that $\hat{b}_m \in BR_m(b_{-m})$. As observed at the beginning of this proof, it suffices to show (i) that $\hat{b}_m(x^*) = b_m(x^*)$, which since $x^* \neq \hat{x}$ follows from the definition of $\hat{b}_m$, and (ii) $x(b_m, b_{-m}) = x^*$. Because $x(b_m, b_{-m}) = x^*$ and because $\hat{b}_m$ agrees with $b_m$ for all $x \neq \hat{x}$, we must have $x(\hat{b}_m, b_{-m}) \in \{x^*, \hat{x}\}$. It therefore only remains to show $x(\hat{b}_m, b_{-m}) \neq \hat{x}$. To see this, we have

$$
v_0(\hat{x}) + \hat{b}_m(\hat{x}) + \sum_{n \neq m} b_n(\hat{x})
$$

$$
= v_0(\hat{x}) + \left[ v_0(x^*) - v_0(\hat{x}) + \sum_{n=1}^{N} b_n(x^*) - \sum_{n \neq m} b_n(\hat{x}) - \frac{\varepsilon}{3} \right] + \sum_{n \neq m} b_n(\hat{x})
$$

$$
= v_0(x^*) + b_m(x^*) + \sum_{n \neq m} b_n(x^*) - \frac{\varepsilon}{3}
$$

$$
= v_0(x^*) + \hat{b}_m(x^*) + \sum_{n \neq m} b_n(x^*) - \frac{\varepsilon}{3}.
$$

Since $\varepsilon > 0$, this indeed establishes that $x(\hat{b}_m, b_{-m}) \neq \hat{x}$ and completes the proof of this claim.

Second, we show that $\pi_k(\hat{b}_k, \hat{b}_m, b_{-km}) > \pi_k(b_k, \hat{b}_m, b_{-km})$. We have just seen that $x(b_k, \hat{b}_m, b_{-km}) = x^*$. Next, we establish that $x(\hat{b}_k, \hat{b}_m, b_{-km}) = \hat{x}$. Because $x(b_k, \hat{b}_m, b_{-km}) = x^*$ and because $\hat{b}_k$ agrees with $b_k$ for all $x \neq \hat{x}$, we must have $x(\hat{b}_k, \hat{b}_m, b_{-km}) \in \{x^*, \hat{x}\}$. It
therefore remains to show \( x(b_k, \hat{b}_m, b_{-km}) \neq x^* \). To see this, we have

\[
v_0(\hat{x}) + \hat{b}_k(\hat{x}) + \hat{b}_m(\hat{x}) + \sum_{n \neq k, m} b_n(\hat{x})
\]

\[
= v_0(\hat{x}) + \left[ b_k(\hat{x}) + \frac{2\varepsilon}{3} \right] + \left[ v_0(x^*) - v_0(\hat{x}) + \sum_{n=1}^{N} b_n(x^*) - \sum_{n \neq m} b_n(\hat{x}) - \frac{\varepsilon}{3} \right] + \sum_{n \neq k, m} b_n(\hat{x})
\]

\[
= v_0(x^*) + b_k(x^*) + b_m(x^*) + \sum_{n \neq k, m} b_n(x^*) + \frac{\varepsilon}{3}
\]

\[
= v_0(x^*) + \hat{b}_k(x^*) + \hat{b}_m(x^*) + \sum_{n \neq k, m} b_n(x^*) + \frac{\varepsilon}{3}.
\]

Since \( \varepsilon > 0 \), this indeed establishes that \( x(b_k, \hat{b}_m, b_{-km}) \neq x^* \). Armed with this, we can compare the payoffs of bidder \( k \) under the two bid profiles:

\[
\pi_k(b_k, \hat{b}_m, b_{-km}) - \pi_k(b_k, \hat{b}_m, b_{-km}) = \left[ v_k(\hat{x}) - \hat{b}_k(\hat{x}) \right] - \left[ v_k(x^*) - b_k(x^*) \right] = \left[ v_k(\hat{x}) - b_k(\hat{x}) \right] - \left[ v_k(x^*) - b_k(x^*) \right] - \frac{2\varepsilon}{3}
\]

\[
= -\frac{2\varepsilon}{3},
\]

where the penultimate step uses the definition of \( \varepsilon \) in (13). Because \( \varepsilon > 0 \), this completes the proof of this claim.

Third, we show that for any \( n \neq k \) and any \( \hat{b}_n \in BR_n(b_{-n}) \), that (i) \( \pi_k(b_k, \hat{b}_n, b_{-kn}) \geq \pi_k(b_k, \hat{b}_n, b_{-kn}) \), and (ii) in the case of equality, \( x(b_k, \hat{b}_n, b_{-kn}) = x(b_k, \hat{b}_n, b_{-kn}) \) as well. As observed at the beginning of this proof, if \( \hat{b}_n \in BR_n(b_{-n}) \), then \( x(b_k, \hat{b}_n, b_{-kn}) = x^* \). Because of this and because \( \hat{b}_k \) agrees with \( b_k \) for all \( x \neq \hat{x} \), we must have \( x(b_k, \hat{b}_n, b_{-kn}) \in \{x^*, \hat{x}\} \). In the first case, \( x(b_k, \hat{b}_n, b_{-kn}) = \hat{x} \). In this case, the argument from the previous paragraph can be used to show that \( \pi_k(b_k, \hat{b}_n, b_{-kn}) > \pi_k(b_k, \hat{b}_n, b_{-kn}) \). In the second case, \( x(b_k, \hat{b}_n, b_{-kn}) = x^* \). In this case,

\[
\pi_k(b_k, \hat{b}_n, b_{-kn}) = v_k(x^*) - \hat{b}_k(x^*) = v_k(x^*) - b_k(x^*) = \pi_k(b_k, \hat{b}_n, b_{-kn}),
\]

and in addition, \( x(b_k, \hat{b}_n, b_{-kn}) = x^* = x(b_k, \hat{b}_n, b_{-kn}) \), as required.

Part Two (Sufficiency): Suppose that \( b \) is not a test-set equilibrium. Then there exists a bidder \( k \), a bid \( \hat{b}_k \in BR_k(b_{-k}) \), a bidder \( m \neq k \), and a bid \( \hat{b}_m \in BR_m(b_{-m}) \) for which \( \pi_k(b_k, \hat{b}_m, b_{-km}) \geq \pi_k(b_k, \hat{b}_m, b_{-km}) \), with equality only if the total surplus from \( x(b_k, \hat{b}_m, b_{-km}) \) exceeds that from \( x(b_k, \hat{b}_m, b_{-km}) \).

As observed at the beginning of this proof, a necessary condition for \( \hat{b}_m \in BR_m(b_{-m}) \) is that \( x(b_k, \hat{b}_m, b_{-km}) = x^* \). We also define \( \hat{x} = x(b_k, \hat{b}_m, b_{-km}) \). Rewriting the violation of test-set equilibrium in these terms:

\[
v_k(\hat{x}) - \hat{b}_k(\hat{x}) \geq v_k(x^*) - b_k(x^*),
\]

with equality only if the total surplus from \( \hat{x} \) exceeds that from \( x^* \). We argue that \( \hat{x} \neq x^* \). In the case that (14) holds with equality, this is immediate. Next, as observed at the
beginning of this proof, a necessary condition for $\hat{b}_k \in BR_k(b_{-k})$ is that $\hat{b}_k(x^*) = b_k(x^*)$. Therefore, (14) cannot hold with strict inequality unless $\hat{x} \neq x^*$.

In either case, we have that $x(\hat{b}_k, \hat{b}_m, b_{-km}) = \hat{x} \neq x^* = x(b_k, \hat{b}_m, b_{-km})$. This is only possible if $\hat{b}_k(\hat{x}) > b_k(\hat{x})$. Therefore, (14) implies

$$v_k(\hat{x}) - b_k(\hat{x}) > v_k(x^*) - b_k(x^*),$$

as desired. \(\square\)

**Proof of Theorem 10.** First, we argue that any test-set equilibrium must satisfy the asserted properties. Second, we demonstrate that any payoff vector in $C \cap Z$ can be supported by a test-set equilibrium.\(^{13}\)

**Part One (Characterization of Test-Set Equilibria):** Suppose that $b$ is a test-set equilibrium. We argue first that $x(b) = x^{opt}$, second that $b$ induces payoffs in $C$, and third that $b$ induces payoffs in $Z$.

**Part 1A (Surplus-Maximizing Decision):** Define $x^* = x(b)$ and suppose by way of contradiction that $x^* \neq x^{opt}$. By Lemma 8, we have that for all bidders $n$,

$$b_n(x^{opt}) \geq b_n(x^*) - v_n(x^*) + v_n(x^{opt}).$$

Summing over $n$,

$$\sum_{n=1}^{N} b_n(x^{opt}) \geq \sum_{n=1}^{N} b_n(x^*) - \sum_{n=1}^{N} v_n(x^*) + \sum_{n=1}^{N} v_n(x^{opt}).$$

Equivalently,

$$v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \geq v_0(x^*) + \sum_{n=1}^{N} b_n(x^*) - \sum_{n=0}^{N} v_n(x^*) + \sum_{n=0}^{N} v_n(x^{opt}). \quad (15)$$

By the definition of $x^{opt}$, we also have

$$\sum_{n=0}^{N} v_n(x^{opt}) > \sum_{n=0}^{N} v_n(x^*), \quad (16)$$

where the strictness of the inequality is by Assumption 2. Plugging (16) into (15), we obtain

$$v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) > v_0(x^*) + \sum_{n=1}^{N} b_n(x^*),$$

which contradicts $x(b) = x^*$.

---

\(^{13}\)Portions of this proof draw the proof of Theorem 2 of Bernheim and Whinston (1986).
Part 1B (Core Payoffs): Next, we argue that the test-set equilibrium $b$ generates payoffs $\pi \in C$. Let $J \subseteq \{1, \ldots, N\}$. Because $x(b) = x^{opt}$,

$$v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \geq v_0(x^J) + \sum_{n=1}^{N} b_n(x^J).$$

Consequently,

$$v_0(x^{opt}) + \sum_{n \in J} b_n(x^{opt}) + \sum_{n \notin J} b_n(x^{opt}) \geq v_0(x^J) + \sum_{n \notin J} b_n(x^J).$$

Since $x(b) = x^{opt}$, we have from Lemma 8 that for all bidders $n$, $b_n(x^J) \geq b_n(x^{opt}) - v_n(x^{opt}) + v_n(x^J)$. Plugging this into the above,

$$v_0(x^{opt}) + \sum_{n \in J} b_n(x^{opt}) + \sum_{n \notin J} b_n(x^{opt}) \geq v_0(x^J) + \sum_{n \notin J} b_n(x^{opt}) - \sum_{n \notin J} v_n(x^{opt}) + \sum_{n \notin J} v_n(x^J).$$

Rearranging, we obtain

$$\sum_{n \in J} b_n(x^{opt}) \geq \sum_{n \notin \{0\} \cup J} v_n(x^J) - \sum_{n \notin \{0\} \cup J} v_n(x^{opt}).$$

Thus, the equilibrium payoffs for coalition $J$ are

$$\sum_{n \in J} \pi_n = \sum_{n \in J} v_n(x^{opt}) - \sum_{n \notin J} b_n(x^{opt})$$

$$\leq \sum_{n=0}^{N} v_n(x^{opt}) - \sum_{n \notin \{0\} \cup J} v_n(x^J),$$

as desired.

Part 1C (Payoffs in $Z$): Suppose that $b$ induces the payoff vector $\pi$. Any bidder $n$ can guarantee itself a payoff of at least $\min_{x \in X} v_n(x)$ by placing a constant bid of zero on all decisions. Thus, we must have that for every bidder $n$, $\pi_n \geq \min_{x \in X} v_n(x)$.

Next, suppose it is not the case that there exist $x' \neq x''$ and $k \neq m$ such that $\pi_k \geq v_k(x')$ and $\pi_m \geq v_m(x'')$. Then either $N = 1$ or there exists an $x^\dagger \in X$ such that for all $n$ and all $x \in X \setminus \{x^\dagger\}$, $\pi_n < v_n(x)$. We argue that in either case, $\sum_{n=1}^{N} \pi_n = \sum_{n=0}^{N} v_n(x^{opt}) - v_0(x^\emptyset)$.

First, suppose $N = 1$. By part (iv) of the characterization of Nash equilibrium given by Lemma 2 of Bernheim and Whinston (1986), there must exist an $x^\dagger$ such that both $b_1(x^\dagger) = 0$ and $v_0(x^\dagger) = v_0(x^{opt}) + b_1(x^{opt})$. But since $x(b) = x^{opt}$, we also have $v_0(x^{opt}) + \sum_{n \in J} b_n(x^{opt}) \geq v_0(x^J) + \sum_{n \notin J} b_n(x^J).$
\[ b_1(x^{opt}) \geq v_0(x^\theta) + b_1(x^\theta). \] Furthermore, \[ v_0(x^\theta) + b_1(x^\theta) \geq v_0(x^\theta) \geq v_0(x^\dagger). \] Thus, the sandwich inequality reveals that
\[ v_0(x^{opt}) + b_1(x^{opt}) = v_0(x^\theta). \]

Plugging in \( b_1(x^{opt}) = v_1(x^{opt}) - \pi_1 \) and rearranging, we conclude
\[ \pi_1 = v_1(x^{opt}) + v_0(x^{opt}) - v_0(x^\theta), \]
as desired.

Second, suppose that for all bidders \( n \) and all \( x \in X \setminus \{ x^\dagger \} \), \( \pi_n < v_n(x) \). Plugging in the definition of \( \pi_n \), we have that for all \( n \) and all \( x \in X \setminus \{ x^\dagger \} \),
\[ v_n(x^{opt}) - b_n(x^{opt}) < v_n(x). \]

Then by Lemma 8 and the fact that \( b \) is a test-set equilibrium, we have that for all bidders \( n \) and all \( x \in X \setminus \{ x^\dagger \} \),
\[ b_n(x) \geq b_n(x^{opt}) - v_n(x^{opt}) + v_n(x) > 0. \]

Appealing again to part (iv) of the characterization of Nash equilibrium given by Lemma 2 of Bernheim and Whinston (1986), we must have that for all \( n \), \( b_n(x^\dagger) = 0 \), and furthermore, \( v_0(x^\dagger) = v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \). But since \( x(b) = x^{opt} \), we also have \( v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \geq v_0(x^\theta) + \sum_{n=1}^{N} b_n(x^\theta) \). Furthermore, \( v_0(x^\theta) + \sum_{n=1}^{N} b_n(x^\theta) \geq v_0(x^\theta) \geq v_0(x^\dagger) \).

Thus, the sandwich inequality reveals that
\[ v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) = v_0(x^\theta). \]

Plugging in \( b_n(x^{opt}) = v_n(x^{opt}) - \pi_n \) and rearranging, we conclude
\[ \sum_{n=1}^{N} \pi_n = \sum_{n=0}^{N} v_n(x^{opt}) - v_0(x^\theta), \]
as desired.

**Part Two (Implementation):** Let \( \pi \in C \cap Z \). For all bidders \( n \), define the bid
\[ b_n(x) = \max\{v_n(x) - \pi_n, 0\}, \]
which is the \( \pi_n \)-profit-target strategy for bidder \( n \). It is useful to note that since \( \pi \in C \), for any bidder \( n \), \( \pi_n \leq v_n(x^{opt}) \), and consequently,
\[ b_n(x^{opt}) = v_n(x^{opt}) - \pi_n. \tag{17} \]

There are now two cases, depending upon how membership in \( Z \) is satisfied. For the first case, we show that \( b \) is itself a test-set equilibrium implementing the payoffs \( \pi \). For the
second case, we show how to modify $b$ to obtain a test-set equilibrium implementing the payoffs $\pi$.

**Case One:** Suppose that

$$\sum_{n=1}^{N} \pi_n = \sum_{n=0}^{N} v_n(x^{opt}) - v_0(x^0). \tag{18}$$

We show that $b$ is a Nash equilibrium resulting in the decision $x^{opt}$ by checking the conditions of Lemma 2 of Bernheim and Whinston (1986). Condition (i) is satisfied, since for all bidders $n$ and all decisions $x$, $b_n(x) \geq 0$, by construction.

To establish condition (ii), define $x^* = x(b)$ and suppose by way of contradiction that $x^* \neq x^{opt}$. Thus, we must have

$$v_0(x^*) + \sum_{n=1}^{N} b_n(x^*) > v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}). \tag{19}$$

This inequality must be strict because the auctioneer resolves indifference in favor of $x^{opt}$. Then we define the coalition $J$ to be such that $n \in J$ if and only if $b_n(x^*) = v_n(x^*) - \pi_n$. Thus, if $n \in J$, then $b_n(x^*) = 0$. We therefore have

$$\sum_{n \in \{0\} \cup J} v_n(x^*) - \sum_{n \in J} \pi_n = v_0(x^*) + \sum_{n \in J} b_n(x^*)$$

$$= v_0(x^*) + \sum_{n=1}^{N} b_n(x^*)$$

$$> v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt})$$

$$= \sum_{n=0}^{N} v_n(x^{opt}) - \sum_{n=1}^{N} \pi_n,$$

where the first step uses the definition of $J$, the second step uses $b_n(x^*) = 0$ for all $n \in J$, the third step uses (19), and the fourth step uses (17). Rearranging, we obtain

$$\sum_{n \in J} \pi_n > \sum_{n=0}^{N} v_n(x^{opt}) - \sum_{n \in \{0\} \cup J} v_n(x^*)$$

$$\geq \sum_{n=0}^{N} v_n(x^{opt}) - \sum_{n \in \{0\} \cup J} v_n(x^J),$$

which would contradict $\pi \in C$.

To establish condition (iii), choose any bidder $k$ and any element $x \in X$. Then $b_k(x) \geq v_k(x) - \pi_k$, and applying (17),

$$b_k(x) \geq b_k(x^{opt}) - v_k(x^{opt}) + v_k(x).$$
Thus,
\[ v_k(x^{opt}) - v_k(x) \geq b_k(x^{opt}) - b_k(x) \]
\[ \geq b_k(x^{opt}) - b_k(x) - \left[ v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \right] + \left[ v_0(x) + \sum_{n=1}^{N} b_n(x) \right] \]
\[ = \left[ v_0(x) + \sum_{n \neq k} b_n(x) \right] - \left[ v_0(x^{opt}) + \sum_{n \neq k} b_n(x^{opt}) \right], \]
as required.

To establish condition (iv), we combine (17) and (18) to obtain \( v_0(x^{0}) = v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \). But since \( x(b) = x^{opt} \), we also have \( v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \geq v_0(x^{0}) + \sum_{n=1}^{N} b_n(x^{0}) \). Furthermore, \( v_0(x^{0}) + \sum_{n=1}^{N} b_n(x^{0}) \geq v_0(x^{0}) \). Thus, the sandwich inequality reveals that
\[ v_0(x^{0}) + \sum_{n=1}^{N} b_n(x^{0}) = v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}). \]
Furthermore, since bids must be nonnegative, the sandwich inequality also implies that for all bidders \( n \), \( b_n(x^{0}) = 0 \).

Finally, to verify that the equilibrium is in fact a test-set equilibrium, by Lemma 8, it suffices to check that for any bidder \( n \) and any element \( x \in X \), that
\[ b_n(x) \geq b_n(x^{opt}) - v_n(x^{opt}) + v_n(x). \]
However, this was already established in the process of verifying condition (iii).

**Case Two:** Suppose there exist \( x' \neq x'' \) and \( k \neq m \) such that \( \pi_k \geq v_k(x') \) and \( \pi_m \geq v_m(x'') \). Then for some such \( x', x'', k, m \), where without loss of generality we assume \( x' \neq x^{opt} \), define
\[ \hat{b}_k(x) = \begin{cases} b_k(x) & \text{if } x \in \{x', x^{opt}\} \\ b_k(x) + \left[ v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \right] - \left[ v_0(x) + \sum_{n=1}^{N} b_n(x) \right] & \text{if } x \notin \{x', x^{opt}\} \end{cases} \]
\[ \hat{b}_m(x) = \begin{cases} b_m(x) & \text{if } x \neq x' \\ b_m(x) + \left[ v_0(x^{opt}) + \sum_{n=1}^{N} b_n(x^{opt}) \right] - \left[ v_0(x) + \sum_{n=1}^{N} b_n(x) \right] & \text{if } x = x' \end{cases} \]
For all other bidders \( n \), define \( \hat{b}_n = b_n \). We first argue that \( \hat{b} \) is a Nash equilibrium resulting in the decision \( x^{opt} \) by checking the conditions of Lemma 2 of Bernheim and Whinston (1986).

To establish condition (i), first note that, as shown in the course of the proof of Case One, \( x^{opt} \) is a maximizer of \( v_0(x) + \sum_{n=1}^{N} b_n(x) \). Thus, for any bidder \( n \) and any \( x \in X \), \( \hat{b}_n(x) \geq b_n(x) \). Moreover, by construction, \( b_n(x) \geq 0 \). Consequently, \( \hat{b}_n(x) \geq 0 \), as desired.
In order to establish condition (ii), note that by the construction of \( \hat{b}_k \) and \( \hat{b}_m \), \( v_0(x) + \sum_{n=1}^{N} b_n(x) \) is constant as a function of \( x \). Therefore, the auctioneer breaks this tie in favor of \( x^{\text{opt}} \), since it maximizes total surplus. Thus, \( x(\hat{b}) = x^{\text{opt}} \).

To establish condition (iii), choose any bidder \( l \) and any element \( x \in X \). Then \( b_l(x) \geq v_l(x) - \pi_l \), and applying (17),

\[
\hat{b}_l(x) \geq b_l(x^{\text{opt}}) - v_l(x^{\text{opt}}) + v_l(x).
\]

Furthermore, by construction, \( \hat{b}_l(x) \geq b_l(x) \) and \( \hat{b}_l(x^{\text{opt}}) = b_l(x^{\text{opt}}) \). Consequently,

\[
\hat{b}_l(x) \geq \hat{b}_l(x^{\text{opt}}) - v_l(x^{\text{opt}}) + v_l(x).
\]

Thus,

\[
v_l(x^{\text{opt}}) - v_l(x) \geq \hat{b}_l(x^{\text{opt}}) - \hat{b}_l(x)
= \hat{b}_l(x^{\text{opt}}) - \hat{b}_l(x) - \left[ v_0(x^{\text{opt}}) + \sum_{n=1}^{N} \hat{b}_n(x^{\text{opt}}) \right] + \left[ v_0(x) + \sum_{n=1}^{N} \hat{b}_n(x) \right]
= \left[ v_0(x) + \sum_{n \neq l} \hat{b}_n(x) \right] - \left[ v_0(x^{\text{opt}}) + \sum_{n \neq l} \hat{b}_n(x^{\text{opt}}) \right],
\]

as required.

To establish condition (iv), note that by the construction of \( \hat{b}_k \) and \( \hat{b}_m \), \( v_0(x) + \sum_{n=1}^{N} b_n(x) \) is constant as a function of \( x \). It therefore only remains to establish that for every bidder \( n \), there is some \( x \in X \) for which \( \hat{b}_n(x) = 0 \). Since \( \pi \in Z \), for every bidder \( n \), there exists some \( x \in X \) for which \( \pi_n \geq v_n(x) \). For that \( x \), \( b_n(x) = 0 \). Therefore, provided \( n \notin \{k, m\} \), we also have \( \hat{b}_n(x) = 0 \). For bidder \( k \), \( \hat{b}_k(x') = b_k(x') = \max\{v_k(x') - \pi_k, 0\} = 0 \). Finally, for bidder \( m \), \( b_m(x'') = \hat{b}_m(x'') = \max\{v_m(x'') - \pi_m, 0\} = 0 \).

Finally, to verify that the equilibrium is in fact a test-set equilibrium, by Lemma 8, it suffices to check that for any bidder \( n \) and any element \( x \in X \), that

\[
\hat{b}_n(x) \geq \hat{b}_n(x^{\text{opt}}) - v_n(x^{\text{opt}}) + v_n(x).
\]

However, this was already established in the process of verifying condition (iii). \( \square \)

**Proof of Example 1.** It is easy to verify that these bids result in the allocation \((w, l, l, l, l, l)\). This allocation is inefficient because it yields a total surplus of 29, whereas the allocation \((l, l, w, w, w, w)\) yields a total surplus of 30. It is also easy to check that these bids are a Nash equilibrium.

Bidder 1 has a unique best response to the equilibrium bids of the other bidders: its equilibrium bid of \( b_1(l) = 0 \) and \( b_1(w) = 28 \). The best responses of bidder 2 are those of the form \( b_2(l) = 0 \) and \( b_2(w) \in [0, 19] \). For all bidders \( n \in \{3, 4, 5, 6\} \), their best responses are those of the form \( b_n(l) = 0 \) and \( b_n(w) \in [0, 9] \). Given this, it is easily checked that each bidder’s test-set condition is satisfied. \( \square \)