1. Introduction

This paper concerns whether or not Bayesian decision theory has any revealed-preference implications regarding the asymptotic behavior of a rational agent. Specifically, it concerns ideas that might be thought to follow from two features of Bayesian statistical theory.

(1) A premise of the Bayesian theory—and presumably of all theories of rational decision under uncertainty—is that an agent does not “throw information away” after having made relevant observations. Bayesian statistics models the agent’s prior belief, and the agent’s posterior belief at each stage after having made a new observation, as probability measures.

(2) A theorem of Bayesian statistics is that, with respect to the prior probability measure over possible sequences of observations, the agent’s posterior belief converges in probability, and almost surely, to some probability-measure-valued random variable.\(^1\)

In view of these features of the theory, it is intuitive to suppose two things about the asymptotic behavior of an agent has repeated opportunities to revise a choice of an action after making new observations, and who maximizes posterior expected utility.

(1) Eventually, the chosen action will not depend solely on the most recent observation, since the cumulative information from past observations outweighs the incremental information from the most recent observation. That is, for any possible value that a single observation might take, there are two sequences of observations, both of which terminate with observations having that value but following which the agent makes distinct action choices.

(2) Along (almost) any infinite sequence of observations, the agent’s choice of action after finite initial sequences will eventually “settle down” to a constant action, since the agent is maximizing expected utility with respect to a posterior probability measure that is converging to a limit.

However, despite their intuitive plausibility, it will be shown here that neither of these suggestions is an implication of Bayesian decision theory. In fact, the theory has no implication specifically about the agent’s asymptotic behavior. Its only restriction on that behavior is an elementary one, immediately implied by Savage’s sure-thing principle, concerning how the chosen action can evolve after a single incremental observation has been made.

2. A Fragment of Formal Decision Theory

2.1. The situation to be modeled. Consider an agent who, at each stage \( t \in \mathbb{N}^+ \) makes an observation having value \( x_t \in X \).\(^2\) At some stage \( t \in \mathbb{N} \), and after having observed the values

\(^1\)This result is often called “convergence to truth.” However, if the observations are not completely informative about the truth, then the limiting probability will not necessarily have one-point support. Moreover, the informal statement of the result that we make here has implicit premises, notably that an observation can take only finitely many possible values.

\(^2\)\(\mathbb{N} = \{0, 1, 2 \ldots \} \) and \( \mathbb{N}^+ = \{1, 2, 3 \ldots \} \).
\(x_1, \ldots, x_t\), the agent must take an action \(a\) from a set \(A\). (The action may have to be taken at stage 0, before having made any observation.) We have in mind a situation in which the observations could potentially influence the agent’s preference among available actions, but in which that preference does not depend on the stage at which the action must be taken.

In the formal theory, we will assume that
\[
\text{X} \text{ and } A \text{ are finite sets.} 
\]

2.2. Observation sequences and contingent plan. Define the set of observation sequences, \(X^* = \bigcup_{t \in \mathbb{N}} X^t\), and define \(\lambda: X^* \to \mathbb{N}\) by \(\lambda(\vec{x}) = t \iff \vec{x} \in X^t\). Let \(\emptyset\) denote the null sequence (defining \(\lambda(\emptyset) = 0\)) and let \(\vec{x} \preceq \vec{y}\) denote that \(\vec{y}\) is an extension of \(\vec{x}\) and let \(\vec{x} \prec \vec{y}\) denote that \(\vec{y}\) is a proper extension of \(\vec{x}\). If \(\vec{x} = \langle x_1, \ldots, x_n \rangle\), then for \(1 \leq t \leq n\), \(x_t\) denotes \(x_t\). The fact that \(X^*\) is a countable set will be invoked, without explicit mention, in proofs below.

The immediate extensions of \(\vec{x} \in X^*\) consist of \(\vec{x}\) followed by a single observation. Let \(\vec{x} \ast y\) denote the immediate extension of \(\vec{x}\) by \(y\).³

The usual representation of observations in terms of measurable functions is adopted here. A measurable space \((\Omega, \mathcal{B})\) is constructed from \(X\) by taking \(\Omega = X^\mathbb{N}\) and \(X_t(\omega) = \omega_t\), and by taking \(\mathcal{B}\) to be the smallest \(\sigma\)-algebra with which all of the projection functions \(X_t\) are measurable. \(\mathcal{B}_t \subseteq \mathcal{B}\) is the \(\sigma\)-algebra (a finite Boolean algebra, given the assumption that \(X\) is finite) generated by \(X_1, \ldots, X_t\). In particular, every set
\[
B^*(\vec{x}) = \{\omega | \forall t \leq \lambda(\vec{x}) \ X_t(\omega) = \vec{x}_t\} \tag{2}
\]
is measurable with respect to \(\mathcal{B}\) and to \(\mathcal{B}_t\) for \(\lambda(\vec{x}) \leq t\). If \(\forall t \leq \lambda(\vec{x}) \ X_t(\omega) = \vec{x}_t\), then \(B^*(\vec{x}) = B(\omega, \lambda(\vec{x}))\). Define \(\phi: \Omega \times \mathbb{N} \to X^*\) and \(\alpha^* : \Omega \times \mathbb{N} \to A\) by
\[
\xi(\omega, 0) = \emptyset \text{ and, for } t > 0, \xi(\omega, t) = \langle X_1(\omega), \ldots, X_t(\omega) \rangle. \tag{3}
\]
It will be convenient to enlarge the domain of \(\xi, t\circ \Omega \cup X^*\) by defining
\[
\xi(\vec{x}, 0) = \emptyset \text{ and, for } t > 0, \xi(\vec{x}, t) = \langle X_1(\vec{x}), \ldots, X_{\min\{t, \lambda(\vec{x})\}}(\vec{x}) \rangle. \tag{4}
\]
Define
\[
\vec{x} \prec \omega \iff \xi(\omega, \lambda(\vec{x})) = \vec{x}. \tag{5}
\]

A (single-valued) contingent plan is a correspondence \(\alpha: X^* \to A\). Define
\[
\alpha^*(\omega, t) = \alpha(\xi(\omega, t)). \tag{6}
\]

2.3. Bayesian rationality. The Bayesian concept of rationality interprets the agent as having beliefs represented by a probability space \((\Omega, \mathcal{B}, P)\) and a bounded, state-contingent utility function \(u: A \times \Omega \to \mathbb{R}\), and interprets the observations \(x_t\) as the values of random variables \(X_t: \Omega \to X\). \(P\) is assumed to be countably additive.

A Bayesian-rational agent almost surely (that is, after any sequence of observations that has positive probability according to the agent’s prior beliefs) chooses an action that maximizes the posterior expected utility
\[
U(a, \omega, t) = \frac{\int_{B(\omega, t)} u(a, \psi) \, dP(\psi)}{P(B(\omega, t))} = \frac{\int_{B^*(\xi(\omega, t))} u(a, \psi) \, dP(\psi)}{P(B^*(\xi(\omega, t)))}, \tag{7}
\]
where
\[
B(\omega, t) = \{\psi \in \Omega | \forall s \leq t \ X_s(\psi) = X_s(\omega)\} = B^*(\xi(\omega, t)). \tag{8}
\]
Define
\[
U^*(a, \vec{x}) = \frac{\int_{B^*(\vec{x})} u(a, \psi) \, dP(\psi)}{P(B^*(\vec{x}))}. \tag{9}
\]
³In general, let \(\vec{x} \ast \vec{y}\) be the concatenation of sequences \(\vec{x}\) and \(\vec{y}\). Then \(\vec{x} \ast y\) is shorthand for \(\vec{x} \ast \langle y \rangle\).
so that, if \( P(\xi(\omega, t)) > 0 \), then
\[
U(a, \omega, t) = U^*(a, \xi(\omega, t)).
\] (10)

Intuitively, a Bayes contingent plan is one that might be followed by an agent who maximizes posterior objective utility, conditional on any sequence of observations. Formally, a contingent plan \( \alpha \) is Bayes for \( \langle \Omega, \mathcal{B}, P, \{X_t\}_{t \in \mathbb{N}_+} \rangle \) (or simply Bayes for \( P \)) if \( \alpha \) if it satisfies, for all \( \vec{x} \in X^* \),
\[
P(B^*(\vec{x})) > 0 \implies \forall a \in A \left[ a = \alpha(\vec{x}) \iff U^*(a, \vec{x}) = \max_{a' \in A} U^*(a', \vec{x}) \right]
\] (11)
with respect to some von Neumann-Morgenstern utility function and to that stochastic process. Call \( \alpha \) Bayes if it is Bayes for some \( P \), and Bayes for \( u \) and \( P \) if \( u \) induces \( U \), which satisfies (11) for \( \alpha \) and \( P \).

Stochastic processes for which every observation sequence occurs with positive probability will be of special interest. That is, statements of some results will involve the condition that
\[
\forall \vec{x} \in X^* \ P(B^*(\vec{x})) > 0.
\] (12)
Conceptually, condition (12) ensures that Bayes rule is always applicable.\(^4\)

3. Nearly sound conjectures

3.1. The conjectures are not vacuous. At the end of section 1, it was conjectured that Bayes contingent plans might have two properties.

**Conjecture 1.** A Bayes contingent plan depends non-trivially on past observations, as well as on the current observation, after some observation sequences.

**Conjecture 2.** Almost surely, a Bayes contingent plan is asymptotically constant (if, as is assumed in this paper, there are only finitely many feasible actions).

It is obvious that there are contingent plans that violate each of those conjectures. For example, let \( X = \{1, 2\} \) and \( A = \{0, 1, 2\} \), and define
\[
\alpha_1(\emptyset) = \alpha_2(\emptyset) = 0 \text{ and } \\
\alpha_1(\vec{x}) = \vec{x}_{\lambda(\vec{x})} \text{ and } \alpha_2(\vec{x}) = \sum_{t=1}^{\lambda(\vec{x})} x_t \pmod{3}.
\] (13)
Contingent plan \( \alpha_1 \) clearly violates conjecture 1. Also \( \alpha_2 \) violates conjecture 2. If \( \vec{x} \in \Omega \) and \( y \in X \), then \( \alpha_2(\vec{x} * y) \equiv \alpha_2(\vec{x}) + y \pmod{3} \) so \( \alpha_2(\vec{x} * y) \neq \alpha_2(\vec{x}) \).

3.2. Qualified versions of the conjectures are sound. When qualified by some hypotheses, the two conjectures become sound propositions.

**Proposition 1.** Suppose that \( f: X \to A \) and contingent plan \( \alpha \) satisfy
\[
\forall \vec{x} \forall y \ \alpha(\vec{x} * y) = f(y)
\] (14)
and that, for some observation value \( y \) such that \( f(y) \neq \alpha(\emptyset) \),
\[
P(\exists t X_t = y) = 1.
\] (15)
Then \( \alpha \) is not Bayes for \( P \)

\(^4\)Mathematically, this condition states that \( P \) has full support with respect to the product topology of \( X^{\mathbb{N}_+} \), since the sets \( B^*(\vec{x}) \) are a sub-base of that topology.
Proof. Define $Y = \{ \bar{x} \mid P(B^*(\bar{x})) > 0 \text{ and } \bar{x}_{\lambda(\bar{x})} = y \text{ and } \forall t < \lambda(\bar{x}) \bar{x}_t \neq y \}$. Define $Y^* = \bigcup_{\bar{x} \in Y} B^*(\bar{x})$. Then, by (15), $P(Y^*) = 1$. By condition (11), $\int_{B^*(\bar{x})} u(f(y), \omega) dP(\omega) > \int_{B^*(\bar{x})} u(\alpha(\emptyset), \omega) dP(\omega)$ for every $\bar{x} \in Y$. Note that, if $\bar{w} \neq \bar{x}$ and $\bar{w} \in Y$ and $\bar{x} \in Y$, then $B^*(\bar{w}) \cap B^*(\bar{x}) = \emptyset$. Therefore, since integrals of the $L_1$ functions $u(f(y), \omega)$ and $u(\alpha(\emptyset), \omega)$ with respect to $P$ define countably additive measures, $\int_{Y^*} u(f(y), \omega) dP(\omega) > \int_{Y^*} u(\alpha(\emptyset), \omega) dP(\omega)$. Since $P(Y^*) = 1$, $\int_{Y^*} u(f(y), \omega) dP(\omega) = U^*(f(y), \emptyset)$ and $\int_{Y^*} \alpha(\emptyset), \omega) dP(\omega) = U^*(\alpha(\emptyset), \emptyset)$. Thus $U^*(f(y), \emptyset) > U^*(\alpha(\emptyset), \emptyset)$, contradicting condition (11).

\begin{proposition}
Let $B \in \mathcal{B}$ and $C \in \mathcal{B}$ be defined, for some $a^* \in A$, by

$$B = \{ \omega \mid \forall a \neq a^* [u(a^*, \omega) > u(a, \omega)] \} \text{ and } C = \{ \omega \mid \forall s \in \mathbb{N} \exists t > s [\alpha^*(\omega, s) \neq a^*] \}. \quad (16)$$

Suppose that

$$P(B \cap C) > 0. \quad (17)$$

Then $\alpha$ is not Bayes for $P$.
\end{proposition}

This proposition will be proved in section 8

3.3. Discussion. Proposition 1 and proposition 2 qualify conjecture 1 and conjecture 2, respectively, in ways that make them sound. Condition (15) of proposition 1 shows that if observations are dichotomous, for example, then the agent would have to attribute positive prior probability to the event of making identical observations forever, in order for contingent plan $\alpha_1$ (which satisfies condition (14)) to be Bayes with respect to prior beliefs. Condition (17) of proposition 2 shows that the agent would have to believe that almost surely several actions will be tied for being the best, in order for contingent plan $\alpha_2$ to be Bayes with respect to prior beliefs.

The hypotheses of both of these results apply straightforwardly to canonical examples of decision making according to posterior-expected-utility maximization, such as the exchangeable-distribution model of a risk-averse agent’s decision whether to bet on Heads, on Tails, or not at all, on the toss of a coin of unknown bias, after a finite sample of tosses has been observed.

Nevertheless, the hypotheses of these results that rule out divergent sequences of posterior-EU-maximizing actions are considerably more restrictive than those of the corresponding results that rule out divergent sequences of posterior beliefs for a Bayesian agent. At best, the results presented in section 3.1 only suggest asymptotic properties that Bayes contingent plans will possess under conditions that are ad hoc, albeit typically satisfied. These results do not identify intrinsic properties that are dictated by the Bayesian decision criterion itself. In fact, the remainder of this paper is devoted to showing that there are no such intrinsic asymptotic properties.

4. Consistent contingent plans

Define $\alpha$ to be consistent at $\bar{x}$ iff

$$\forall a \in A \setminus \{ \alpha(\bar{x}) \} \exists y \in X \alpha(\bar{x} * y) \neq a. \quad (18)$$

Consistency is an obvious revealed-preference implication of the sure-thing principle formulated by Savage-1954.\(^5\) Therefore, a Bayesian agent must consistent.

If there is a finite bound on the length of observation sequences, then the converse is true: for every $P$, a finite contingent plan that is consistent at every node must be Bayes for $P$ and some $u$. GreenPark-1996 prove this fact, in the course of proving theorem 1 in that paper. (Antecedent investigations of the relationship between being consistent and being Bayes include Weller-1976,

\(^5\)In the formal theory introduced by Savage, which has been adopted by most subsequent researchers, acts are defined entities. Despite that difference from the present theory (in which $A$ is a primitive of the theory), clear parallels can be drawn.
Hammond-1988, and EpsteinLeBreton-1993.) However, they asserted a result that encompasses both finite- and infinite-horizon plans, and the assertion is false as it applies to infinite-horizon plans. (Green and Park’s proof makes the unsound assumption that an a.s. convergent martingale must always converge also in $L_1$.)

Proposition 6 in the present paper, to be stated and proved below, is a sound weakening of Green and Park’s theorem 1. The lemmas following the next definition will be used in the proof of proposition 6.

Define
\[
C(\omega, t) = \{ \alpha^*(\omega, \lambda(\bar{x}) + n) \mid n \in \mathbb{N} \}
\]
\[
D(a, \bar{x}) = \{ \omega \mid \bar{x} < \omega \text{ and } \forall n \in \mathbb{N} \alpha^*(\omega, \lambda(\bar{x}) + n) \neq a \}
\]

(19)

Lemma 3. If $\alpha$ is consistent and $a \neq \alpha(\bar{x})$, then $D(a, \bar{x}) \neq \emptyset$.

Proof. A recursive construction, using (5) at each stage $n > 0$, establishes that there is a sequence $\bar{x}^0, \bar{x}^1, \ldots$ such that $\bar{x}^0 = \bar{x}$ and, for every $n \in \mathbb{N}$, $\lambda(\bar{x}^n) = \lambda(\bar{x}) + n$ and $\bar{x}^n < \bar{x}^{n+1}$ and $a \neq \alpha(\bar{x}^n)$. Then, defining $\omega_t = x \iff \exists n \bar{x}^n_t = x$ produces an element of $\Omega$ that satisfies the lemma. \[\square\]

The next lemma follows immediately from lemma 3.6

Lemma 4. If $\alpha$ is consistent and there exists a function $d: A \times X^* \rightarrow \Omega \cup \{\emptyset\}$ such that, for every $\bar{x}$,
\[
d(\bar{x}, \alpha(\bar{x})) = \emptyset \text{ and } \forall a \neq \alpha(\bar{x}) \ d(a, \bar{x}) \in D(a, \bar{x}).
\]

(20)

Define $\Psi \subseteq \Omega$ by
\[
\Psi = \{ d(a, \bar{x}) \mid a \neq \alpha(\bar{x}) \}.
\]

(21)

$\Psi$ is countable, since it is a subset of the range of a function with a countable domain. Therefore

Lemma 5. There is a countably additive probability measure $P$ on $\Omega$ that satisfies
\[
\forall \psi \in \Psi \ P(\{\psi\}) > 0.
\]

(22)

Condition (44) entails the full-support condition (12).

5. The main result

From lemmas 3–5, we will prove the following result.

Proposition 6. If $\alpha$ is consistent, then there are a bounded utility function $u$ and a countably additive probability measure $P$ such that $\alpha$ is Bayes for $u$ and $P$, and every finite sequence of observations receives positive probability according to $P$.

6It can be proved that, for each $\bar{x}$, at most one of the sets $D(a, \bar{x})$ (for $a \neq \alpha(\bar{x})$) can be finite. It would be straightforward to define $d$ in lemma 4 so that
\[
\text{If } \bar{y} < \bar{x} < d(a, \bar{y}) \text{ and } a \neq \alpha(\bar{x}), \text{ then } d(a, \bar{x}) = d(a, \bar{y})
\]

would also be satisfied.
Set of signals: \( X = \{1, 2\} \)
Action set: \( A = \{0, 1, 2\} \)
A typical element in \( \Omega = \{1, 2\}^N \) is shown by \( \omega = (\omega_1, \omega_2, ...) \)
\( \omega^t = (\omega_1, \omega_2, ..., \omega_t) \) is the initial segment of \( \omega \).
At each \( t \), the agent has observed (and recalls) \( \omega^t \).
\( \omega^t * i = (\omega_1, \omega_2, ..., \omega_t, i) \)
A contingent plan assigns an action to each finite sequence \( \omega^t \)
Contingent plan: \( \alpha(\omega^t) = \sum_{t' \leq t} \omega_{t'} \mod 3 \) \(^7\)

---

\(^7\)This is the same contingent plan that was denoted by \( \alpha_2 \) in equation (13).
\(\alpha(\omega^t), \alpha(\omega^t \ast 1)\) and \(\alpha(\omega^t \ast 2)\) are distinct values from one another. Therefore for any \(\omega\) chosen actions never settle down. But the beliefs of a Bayesian agent typically converge. So, isn’t behavior according to \(\alpha\) evidence against the agent being Bayesian rational?

Contingent plan \(\alpha\) is consistent at \(\vec{x}\) iff

\[\forall a \in A \setminus \{\alpha(\vec{x})\} \quad \exists y \in X \quad \alpha(\vec{x} \ast y) \neq a\]

\(\alpha\) is consistent.

By proposition 6, there is a probability/utility framework that rationalizes \(\alpha\).

The key to rationalizing \(\alpha\) is to find a set of states \(\Psi\) which satisfies three properties. Subsequently we will assign positive probabilities, summing to one, to the elements of \(\Psi\).

- \(\Psi\) is countable.
- For any finite initial sequence of observation, there exists some \(\psi \in \Psi\) which is not in contradiction with observations.
- For any \(\omega \in \Psi\) at least one action is chosen by \(\alpha\) only finitely many times along \(\omega\).
  - We construct \(\omega\) by recursively applying consistency.
  - \(\tau(\omega)\) indicates the first time that some action is never chosen at it or later along \(\omega\).

Define \(\Psi\) to be the set of states that, after some finite time, oscillate with period 2 (zigzag forever between 1 and 2).

\[\Psi = \{\omega \in \Omega \mid \exists t \in \mathbb{N} \quad \forall t' \geq t \quad \omega_{t'} \neq \omega_{t'+1}\}\]  

Let \(\tau: \Omega \rightarrow \mathbb{N}\) specify the date at which each state in \(\Psi\) becomes periodic.

\[\tau(\omega) = \min\{t \mid \forall t' > t \quad \omega_{t'} \neq \omega_{t'+1}\}\].

Once a state begins to oscillate, \(\alpha\) also begins to oscillate between only 2 of the 3 actions in \(A\). The third action will never be picked by \(\alpha\) and this makes this definition of \(\tau\) the same as previous slide.

Let set \(C(\psi)\) be such that for \(\psi \in \Psi\), it consists of the two actions that are picked by \(\alpha\) at time \(\tau(\psi)\) and later:

\[C(\psi) = \left\{ \sum_{t \leq \tau(\omega)} \omega_t \pmod{3}, \sum_{t \leq \tau(\omega)+1} \omega_t \pmod{3} \right\}\].

For every finite signal sequence, to define the set of states in \(\Psi\) which begin with that sequence, and oscillate immediately thereafter.

We are going to assign a high probability to every state in this set, conditional on the sequence being observed.

Let \(B^*(\omega^t)\) be the cylinder set defined as

\[B^*(\omega^t) = \{\zeta \in \Omega \mid \zeta^t = \omega^t\}\].

Define \(\hat{\Psi}(\omega^t)\) to be the set of states in \(\Psi\) such that their initial part be equal to \(\omega^t\) and begin to zigzag at \(t\) or before it.

\[\hat{\Psi}(\omega^t) = \{\psi \in \Psi \cap B^*(\omega^t) \mid \tau(\psi) \leq t\}\].

For any \(\omega^t\) the set \(\hat{\Psi}(\omega^t)\) contains only two states and

\[\bigcap_{\zeta \in \hat{\Psi}(\omega^t)} C(\zeta) = \{\alpha(\omega^t)\}\].

Pick \(\delta \in (0, \frac{1}{2})\) and consider additive probability function and state dependent utility function which satisfy
Figure 3. $\tau(\omega) = 7, C(\omega) = \{0, 1\}, \tau(\omega') = 8, C(\omega') = \{0, 2\}$

$$P(\{\omega\}) = \kappa \delta^{\tau(\omega)} 1_{\Psi}(\omega)$$  \hspace{1cm} (29)$$

$$u(a, \psi) = 1_{C(\psi)}(a) \hspace{1cm} (30)$$

Every state in $\Psi$ has positive probability. Therefore every finite initial sequence of observations is observed with positive probability.

For every $\omega^t$, $p(\Psi(\omega^t)) > p(B^*(\omega^t))/2$.

After having observed $\omega^t$, agent assigns a very high probability to each of the two states in $\hat{\psi}(\omega^t)$. $\alpha(\omega^t)$ is the only action with highest possible utility in both states in $\hat{\psi}(\omega^t)$.

Probability assigned to any state in $\hat{\psi}(\omega^t)$, is higher than the summation of probabilities assigned to all the still possible states outside this set.

The maximum utility attainable in any state is one.

Therefore $\alpha$ is the unique expected utility maximizer.

So $(\Omega, B^*, P)$ and $u$ rationalize $\alpha$ as a Bayes contingent plan.

7. Sketch of Proof of Proposition 6

The proof generalizes the preceding example. Set $\Omega = X^{\mathbb{N}^+}$.

First we define $\Psi$ a subset of $\Omega$ which has three properties:

- It is countable.
- For any finite initial sequence of observation, there exists some $\psi \in \Psi$ which is not in contradiction with observations.
- For any $\omega \in \Psi$ at least one action is chosen only finitely many times along $\omega$.
  - $\tau(\omega)$ indicates the first time that an action is never chosen at it or later along $\omega$.

(These are the features of $\Psi$ that we used to construct a Bayes representation for the example.)

Lemma 3 to lemma 10 in the paper are used in defining $\Psi$ and proving these properties.

Given the first two properties of $\Psi$, we define

- a probability measure such that every finite initial sequence has positive probability of being observed
- a bounded state dependent utility function
Probability measure
- \( P(\Psi) = 1 \).
- Any state in \( \Psi \) has positive probability.
- Every finite initial sequence of observations is observed with positive probability.
- \( P \) assigns probability to states according to their index \( \tau(\omega) \).
- States with smaller index are assigned a much higher probability.

State dependent utility function
\[
u(a, \omega) = \begin{cases} 
1 & \text{if } a \in C(\omega) \text{ and } \omega \in \Psi \\
0 & \text{otherwise.} 
\end{cases}
\] (31)

Set \( C(\omega) \) consists of all actions chosen along \( \omega \) but the action(s) which is abandoned first.

We will show that \( \alpha \) is Bayes for \( P \) and \( u \).

7.1. **Step 1: defining \( \Psi \subseteq \Omega \).** Define
\[
C(\omega, t) = \{ \alpha(\omega^{t+n}) | n \in \mathbb{N}_+ \}
\] (32)
\[
D(a, \bar{x}) = \{ \omega | \bar{x} \prec \omega, a \neq C(\omega, t) \}
\] (33)

**Lemma 7.** If \( \alpha \) is consistent and \( a \neq \alpha(\omega, t) \), then \( D(a, \bar{x}) \neq \emptyset \).

**Proof.** A recursive construction, using consistency at each stage \( n > 0 \), establishes that there is a sequence \( \bar{x}^0, \bar{x}^1, \ldots \) such that \( \bar{x}^0 = \bar{x} \) and, for every \( n \in \mathbb{N} \), \( \lambda(\bar{x}^n) = \lambda(\bar{x}) + n \) and \( \bar{x}^n \prec \bar{x}^{n+1} \) and \( a \neq \alpha(\bar{x}^n) \). Then, defining \( \omega_t = x \iff \exists n \bar{x}^n_t = x \) produces an element of \( \Omega \) that satisfies the lemma. \( \square \)

**Lemma 8.** If \( \alpha \) is consistent and there exists a function \( d : A \times X^* \to \Omega \cup \{\emptyset\} \) such that, for every \( \bar{x} \),
\[
d(\alpha(\bar{x}), \bar{x}) = \emptyset \text{ and } \forall a \neq \alpha(\bar{x}) \ d(a, \bar{x}) \in D(a, \bar{x}).
\] (34)

Define \( \Psi \subseteq \Omega \) by
\[
\Psi = \{ d(\alpha(\bar{x}), \bar{x}) | a \neq \alpha(\bar{x}) \}
\] (35)

Define \( \tau : \Psi \to \mathbb{N} \) that satisfies
\[
\tau(\psi) = \min \{ t \in \mathbb{N} | \text{for some } a \neq \alpha(\psi^t), a \notin C(\psi^t). \}
\] (36)

Define \( \hat{\Psi}(\omega^t) \subseteq B^*(\omega^t) \cap \Psi \) by
\[
\hat{\Psi}(\omega^t) = \{ \psi \in \Psi \cap B^*(\omega^t) | \tau(\psi) \leq t \}.
\] (37)

Note that equivalently we can restate \( \hat{\Psi}(\omega^t) \) as
\[
\hat{\Psi}(\omega^t) = \bigcup_{t' \leq t} \bigcup_{a \neq \alpha(\omega^{t'})} \{ d(a, \omega^t') \}.
\] (38)

Therefore \( \hat{\Psi}(\omega^t) \) is finite and
\[
(|A| - 1) \leq |\hat{\Psi}(\omega^t)| \leq (1 + t) \times (|A| - 1).
\] (39)

**Lemma 9.** \( \Psi \) is countable.

**Proof.** \( \Psi \) is a subset of the range of a function with a countable domain therefore it is countable. \( \square \)
Lemma 10.

\[ \bigcap_{\psi \in \hat{\Psi}(\omega^t)} C(\psi^\tau(\psi)) = \alpha(\omega^t) \]  

(40)

Proof. Please note that

\[ \forall \psi \in \hat{\Psi}(\omega^t) \quad \alpha(\omega^t) \in C(\psi^\tau(\psi)), \]

(41)

\[ \bigcap_{\{\psi \in \hat{\Psi}(\omega^t)\mid \tau(\psi) = t\}} C(\psi^\tau(\psi)) = \alpha(\omega^t). \]  

(42)

Result immediately follows from the above two equations. \(\square\)

7.2. Step 2: probability measure and utility function. Since \(\Psi\) is countable and every \(\hat{\Psi}(\omega^t)\) is nonempty and finite, and all information partitions are countable, we can specify a countably additive probability measure \(P\) such that

\[ P(\Psi) = 1, \]

(43)

\[ \forall \psi \in \Psi \quad P(\{\psi\}) > 0 \]  

(44)

and

\[ \sum_{\zeta \in \Psi \cap B^*(\omega^t) \setminus \hat{\Psi}(\omega^t)} P(\{\zeta\}) = \delta \times \min_{\hat{\Psi}(\omega^t)} \{P(\{\omega\})\} > 0 \]  

(45)

for \(\delta \in (0, 1)\).

Note that condition (44) entails that every finite sequence of initial observations has positive probability.

Define bounded utility function

\[ u(a, \omega) = \begin{cases} 1 & \text{if } a \in C(\psi^\tau(\psi)) \text{ and } \omega \in \Psi \\ 0 & \text{otherwise.} \end{cases} \]  

(46)

Next we show that the consistent contingent plan, \(\alpha\), is Bayes with respect to \(P\) and \(u\) so defined.

7.3. Step 3: rationalization.

Lemma 11. If \(P\) and \(u\) satisfy conditions (44) and (46), and \(a \neq \alpha(\omega, t)\), then

\[ U(a, \omega, t) < U(\alpha(\omega^t), \omega, t). \]  

(47)

where \(U\) is derived from \(u_\delta\) according to

\[ U(a, \omega, t) = \frac{\int_{B^*(\omega^t)} u(a, \psi) dP(\psi)}{B^*(\omega^t)} \]  

(48)

Proof. Maximum gain of deviating from \(\alpha(\omega^t)\):

\[ \sum_{\zeta \in \Psi \cap B^*(\omega^t) \setminus \hat{\Psi}(\omega^t)} P(\{\zeta\}) \times 1 = \delta \times \min_{\hat{\Psi}(\omega^t)} \{P(\{\omega\})\} \]  

(49)

Minimum loss of deviating from \(\alpha(\omega^t)\):

\[ \min_{\zeta \in \hat{\Psi}(\omega^t)} \{P(\{\zeta\})\} \times 1 \]  

(50)

By comparing the above two equations we can see that the maximum gain is strictly smaller than the minimum loss. \(\square\)

Proposition 6 follows immediately from lemma 11.
8. Proof of Proposition 2

Here we prove proposition 2:
Let $B \in \mathcal{B}$ and $C \in \mathcal{B}$ be defined, for some $a^* \in A$, by
\[ B = \{ \omega \mid \forall a \neq a^* \ [u(a^*, \omega) > u(a, \omega)] \} \quad \text{and} \quad C = \{ \omega \mid \forall s \in \mathbb{N} \ \exists t > s \ [\alpha^*(\omega, s) \neq a^*] \}. \]

Proof. Suppose that $\alpha$ is Bayes for $P$. A contradiction will be derived.

$R$: the smallest ring containing $\{B^*(\bar{x}) \mid \bar{x} \in X^*\}$

$S(R)$: the smallest $\sigma$-ring containing $R$.

$E_n = \{ \omega \in B(a^*) \cap C(a^*) \mid u(a^*, \omega) > \frac{1}{n} + \max_{\alpha \neq a^*} u(a, \omega) \} \in S(R)$

By countable additivity, there is some $n$ large enough so that $P(E_n) > 0$.

By a theorem in Halmos-1970\(^8\) for any $\epsilon > 0$ there exists a set $E_\epsilon \in R$ such that $P(E_n \Delta E_\epsilon) < \epsilon$.

Since $E_\epsilon \in R$ there exists a finite set $\hat{O}_\epsilon \subset X^*$ such that $E_\epsilon = \bigcup_{\bar{x} \in \hat{O}_\epsilon} B^*(\bar{x})$. Let $\hat{O}_\epsilon$ be such set with minimum size. Then for any $\bar{x}, \bar{y} \in \hat{O}_\epsilon$ if $\bar{x} \neq \bar{y}$

\[ B^*(\bar{x}) \cap B^*(\bar{y}) = \emptyset, \]

set $Q_{\bar{x}}$ of extensions of $\bar{x}$ such that $P(B^*(\bar{x}) \setminus \bigcup_{\bar{y} \in Q_{\bar{x}}} B^*(\bar{y})) = 0$ and for any $\bar{y}, \bar{y'} \in Q_{\bar{x}}$, $\alpha(\bar{y}) \neq a^*$ and $B^*(\bar{y}) \cap B^*(\bar{y'}) = \emptyset$ if $\bar{y} \neq \bar{y'}$.

Claim For any $\bar{x} \in \hat{O}_\epsilon$ there exists a countable set $Q_{\bar{x}}$ of extensions of $\bar{x}$ such that
\[ \alpha(\bar{y}) \neq a^* \quad \text{for all} \quad \bar{y} \in Q_{\bar{x}} \]

and
\[ B^*(\bar{y}) \cap B^*(\bar{z}) = \emptyset \quad \text{if} \quad \bar{y} \neq \bar{z} \quad \text{for all} \quad \bar{z}, \bar{y} \in Q_{\bar{x}} \]

and
\[ P(E_\epsilon \setminus \bigcup_{\bar{x} \in \hat{O}_\epsilon} \bigcup_{\bar{y} \in Q_{\bar{x}}} B^*(\bar{y})) < \epsilon. \]

Proof. If we can find \( \{ Q_{\bar{x}} \}_{\bar{x} \in \hat{O}_\epsilon} \) that satisfies (52) and (54) we can refine it such that (53) also be satisfied. We do this by deleting $\bar{y}$ from $Q_{\bar{x}}$ if it is an extension of some other member of $Q_{\bar{x}}$. Thus we just need to show for some $\{ Q_{\bar{x}} \}_{\bar{x} \in \hat{O}_\epsilon}$ both (52) and (54) are satisfied.

Suppose for every choice of $\{ Q_{\bar{x}} \}_{\bar{x} \in \hat{O}_\epsilon}$ that satisfies (52), (54) is not satisfied. A contradiction will be derived.

Consider $Q_{\bar{x}} = \{ \bar{y} \geq \bar{x} | \alpha(\bar{y}) \neq a^* \}$.

Define $O_\epsilon = \bigcup_{\bar{x} \in \hat{O}_\epsilon} Q_{\bar{x}}$ and $F_\epsilon = \bigcup_{\bar{x} \in \hat{O}_\epsilon} \bigcup_{\bar{y} \in Q_{\bar{x}}} B^*(\bar{y})$. Then $F_\epsilon = \bigcup_{\bar{y} \in O_\epsilon} B^*(\bar{y})$.

For any $\bar{x}, \bar{y} \in X^*$ the following is satisfied:
\[ B^*(\bar{x}) \setminus B^*(\bar{y}) = \begin{cases} \emptyset & \text{if} \ \bar{x} \geq \bar{y}, \\ \bigcup_{\bar{z} \mid \bar{x} \geq \bar{y}, \lambda(\bar{z}) = \lambda(\bar{y})} B^*(\bar{z}) & \text{if} \ \bar{y} \nless \bar{x}, \\ B^*(\bar{x}) & \text{otherwise}. \end{cases} \]

Therefore
\[ E_\epsilon \setminus F_\epsilon = \bigcup_{\bar{x} \in \hat{O}_\epsilon} B^*(\bar{x}) \setminus \bigcup_{\bar{y} \in O_\epsilon} B^*(\bar{y}) = \bigcup_{\bar{x} \in \hat{O}_\epsilon} \bigcup_{\bar{y} \in Q_{\bar{x}}} B^*(\bar{y}). \]

Second equality in the above equation comes from (51).

Considering that $Q_{\bar{x}}$ is a subset of extensions of $\bar{x}$ we have
\[ B^*(\bar{x}) \setminus \bigcup_{\bar{y} \in Q_{\bar{x}}} B^*(\bar{y}) = \bigcup_{\bar{x} \in \{ \bar{y} \mid \bar{y} \nless \bar{x}, \bar{y} \neq \bar{y'} \text{ for any } \bar{y'} \in Q_{\bar{x}} \}} B^*(\bar{z}). \]

\(^8\)Page 56, Theorem D.
Thus
\[ E_{\epsilon} \setminus F_{\epsilon} = \bigcup_{\tilde{x} \in \tilde{O}_n} \bigcup_{\tilde{z} \in \tilde{W}_n} B^*(\tilde{z}). \] (58)

Define \( W_{\epsilon} = \bigcup_{\tilde{x} \in \tilde{O}_n} \{ \tilde{y} \succ \tilde{z} | \tilde{y} \notin \tilde{z} \text{ for any } \tilde{y} \in \tilde{Q}_{\tilde{\epsilon}} \} \) then \( E_{\epsilon} \setminus F_{\epsilon} = \bigcup_{\tilde{x} \in \tilde{W}_n} B^*(\tilde{z}). \) For any \( \tilde{z} \in W_{\epsilon}, \)
for any \( \tilde{y} \succeq \tilde{z}, \alpha(\tilde{y}) = a^*. \) Thus \( B^*(\tilde{z}) \cap E_n = \emptyset \) and as a result \( E_{\epsilon} \setminus F_{\epsilon} \) has an empty intersection with \( E_n \) and therefore \( [E_{\epsilon} \setminus F_{\epsilon}] \setminus E_n = E_{\epsilon} \setminus F_{\epsilon}. \) Therefore \( P(E_{\epsilon} \setminus E_n) \geq P([E_{\epsilon} \setminus F_{\epsilon}] \setminus E_n) = P(E_{\epsilon} \setminus F_{\epsilon}) \geq \epsilon. \)
This contradicts \( P(E_n \Delta E_{\epsilon}) < \epsilon \) \( \square \)

Please note that since \( P(E_{\epsilon} \setminus F_{\epsilon}) < \epsilon \) and \( P(E_n \Delta E_{\epsilon}) < \epsilon \) we have \( P(F_n \Delta E_{\epsilon}) < 2\epsilon. \)
For any \( \omega \in E_{\epsilon} \) let \( \tilde{x}_\omega \in O_{\epsilon} \) be defined such that \( \omega \in B^*(\tilde{x}). \) \( \tilde{x}_\omega \) is unique.
Consider an alternative contingent plan \( \hat{\alpha} \) defined below.

\[ \hat{\alpha}(\tilde{x}) = \begin{cases} a^* & \text{if } \tilde{x} \in O_{\epsilon}, \\ \alpha(\tilde{x}) & \text{otherwise.} \end{cases} \] (59)

By assumption \( P(B^*(\tilde{x})) > 0 \) for any \( \tilde{x} \) therefore if \( \alpha \) is Bayes then
\[ \sum_{\tilde{x} \in O_{\epsilon}} [U(\alpha(\tilde{x}, \omega), \omega) - U(\hat{\alpha}(\tilde{x}), \omega)] P(B^*(\tilde{x})) > 0. \] (60)

Left hand side in turn is equal to
\[ \int_{\omega \in F_{\epsilon} \setminus E_n} [u(\alpha(\tilde{x}_\omega), \omega) - u(a^*, \omega)] dp(\omega) + \int_{\omega \in F_{\epsilon} \cap E_n} [u(\alpha(\tilde{x}_\omega), \omega) - u(a^*, \omega)] dp(\omega). \] (61)
Which is less than \( 2M.2\epsilon - \frac{1}{\alpha}(P(E_n) - 2\epsilon). \) But for small enough \( \epsilon \) this expression is negative and this is a contradiction to (60). \( \square \)

\[ E-mail \ address: \ fzb5027@psu.edu \]
\[ E-mail \ address: \ eug2@psu.edu \]