When markets never fail: Reciprocal aggregation and the duality between persons and groups

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Abstract

When markets fail, at equilibrium shareholders typically disagree on how to run the firms, and genuine problems of social choice appear. Hence the necessity for an aggregation mechanism. This paper assumes aggregation simultaneously at the collective and individual levels, proposing a general equilibrium notion of reciprocal aggregation. The central principle the latter builds on is the unanimity principle, an admittedly weak requirement for aggregation mechanisms. Applied to the two leading cases of externalities in production and incomplete financial markets, we show how the strong unanimity principle restores Pareto optimality, even in case of severe market failures.

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1 Introduction

1.1 The problem

Do market failures matter? Sometimes probably they do. That is the reason why, to address the threat of acid rains for example, or the even more dreadful threat of global warming — two major instances of the tragedy of the commons — States and international authorities have established markets to trade emission permits of sulfur and carbon dioxide. That is also why, to hedge against risk for another example, financial innovation is galloping to complete the financial market structure. Nevertheless markets to trade externalities are the exception, not the rule. As for the financial structure, lots of neoclassical economists find it hard to believe that it could be complete, given the extent of uncertainty in the economy, and given moreover that financial innovation creates its own, endogenous uncertainty. Hence failures seem not all taken care of.

But to which extent in this really a problem? If agents nevertheless optimize and markets do come to a well-defined equilibrium, maybe first-best allocations are not reached at equilibrium, but are we so far away from them? There is an influential view according to which, if residual market failures were mattering so much, then individual economic genius motivated by arbitrage opportunities, or collective political wisdom driven by the sense of common good, would tackle them.1 If they don’t, it is probably because the potential gains are not worth the (transaction) costs (Coase, 1960). Hence not worth much.

There is a matter in which market failures are a problem though, both in reality and from an intellectual point of view: it is when collective decisions in the private arena have to be taken. Typically, in corporations or partnerships. It is not clear what a meaningful notion of equilibrium is at the first place. Indeed, discard the question of collective decision making for the moment, and suppose that the production plans of the firms are fixed exogenously. In presence of externalities, or imperfect competition, or market incompleteness, at the exchange equilibrium shareholders typically disagree on how to operate the firms: they want to modify the production plans, but disagree on the way it should be modified. Hence a genuine problem of social choice appears in each and every firm. How are production plans endogenized then? The social choice literature is spotted with impossibility results, and thus quite powerless at providing a compelling notion of (political) equilibrium which would generate the hope that, even if efficiency is a dream, at least some stability is reachable.

This difficulty has been discarded in the classical literature, either by assuming that the firm is a monolithic decision maker endowed with an ad hoc objective function, or by assuming that the space of political heterogeneity is one-dimensional, and then resorting to the median voter theorem.2 The latter trick is quite desperate given the high number of decision variables in a firm. On top of that, it offers little hope to reconcile equilibrium and efficiency: indeed the median voter has no incentive to promote efficiency, unless it happens to be, by coincidence, also the mean voter — e.g., for symmetric distribution of characteristics.3

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1This view is supported by, e.g., the so-called Austrian school. For the role of entrepreneurs, see Kirzner (1963).
3See Bowen (1943), Bergstrom (1970).
The attempts to more ambitiously address that question in a general equilibrium perspective, and open the black box of the firm (Drèze, 1974, 1989), have been severely exposed to the lack of structure and order at the aggregate level. There is no aggregation mechanism, as simple, parsimonious and robust as the market mechanism, that yields the same remarkable (coincidental) combination of stability and efficiency as the latter offers. This seems inescapable, at least when one sticks to the strict view of a monolithic and immutable individual decision maker.

1.2 The argument

Our analysis starts from here, and opens the question of aggregation simultaneously at the collective and individual levels. An individual agent has to take actions, make decisions, or express opinions in many different capacities, and various arenas. As an individual consumer, he trades on the market to buy goods and services; as an individual investor, he rebalances his portfolio according to his insurance needs; and as an individual shareholder, he weighs in the decision process in firms. In the latter capacity, the individual is exposed to the decision dynamics of many groups – in our setup: of all boards, or assemblies of shareholders, of which he is a member.

The appropriate tool to represent this problem is a graph with individual nodes linked to collective nodes according to whether the concerned individual is a member of the concerned board. It is naturally accepted that an aggregation mechanism takes place at every collective node, either through deliberation, bargaining, log-rolling, side-paying, voting... or all at once, at the end of which some decision is taken. It is less frequently assumed that aggregation at the individual level yakes place.4 Nevertheless, it is hard to believe that the individual nodes are not impacted, one way or another, by these dynamics, and their outcomes. We argue that what happens at the collective level backward loops to the individual level, providing some new parameters for individual decision making. These parameters can convey positive, informational content, and feedback the cognitive process of the individual; they can also convey some normative content and feedback the affective process of the individual. We explore how these parameters could be aggregated, or synthesized, at the individual level, and study the resulting general equilibrium.

Before we develop this argument, let us pursue the basic intuition of the paper, and accept for a moment that some aggregation mechanisms take place simultaneously at collective and individual nodes, without prejudice about their nature. Then we face a simple, flat graph, denoted $G$. Flat in the sense that there isn’t an individual level and, ‘above’ it, an aggregate level, but only one level with aggregation at all nodes (Latour, 2005). Another way to envisage it is the dual perspective: The individual level is the dual of the collective level, and individuals

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4That individual preferences are formed by aggregation is well accepted when taking households as the individual decision making unit (see, e.g. Chiappori & Ekeland, 2006, and the survey therein). It is also a major avenue in the theory of choice rationalization (see the survey in Ambrus & Rozen, 2008). The literature on aggregation of experts judgements also assumes that the preferences of the individual decision maker are shaped by aggregation (e.g., Crès et al., 2012, and the survey therein). Finally, preference incompleteness is often axiomatized, if not by sheer aggregation, at least by multi-utility representation (see the survey in Evren & Ok, 2011).
are merely collectives of collectives. In this paper, we try to stick as close as possible to this duality between nodes, in order to grasp the essence of what we call *reciprocal aggregation*.

The intuition that there is a duality between persons and groups has a long tradition in sociology. It dates back to Simmel. The question of the interpenetration of group-affiliation and individual personality is central in his theory. A first quote of Simmel (1955) will help catch the essence of our argument. One could naively translate it in the following terms: homo economicus and homo sociologicus are the two parents of sapiens — and, we add, very interfertile.

"[...] as individuals, we form the personality out of particular elements of life, each of which has arisen from, or is interwoven with, society. This personality is subjectivity par excellence in the sense that it combines the elements of culture in an individual manner. There is here a reciprocal relation between the subjective and the objective. As the person becomes affiliated with a social group, he surrenders himself to it. A synthesis of such subjective affiliations creates a group in an objective sense. But the person also regains his individuality because his pattern of participation is unique: hence the fact of multiple group-participation creates in turn a new subjective element. Causal determination of, and purposive action by, the individual appear as two sides of the same coin." (p. 141)

In short, individuals shape the collectives they join, *and thereafter they are shaped by them*. This somewhat challenges a rigorist obedience to methodological individualism,\(^5\) but we argue that it can do so in ways that are already widely accepted in the realm of mainstream economic analysis.

A diagram can help fix ideas. Nodes, whether individual or collective, are equipped with a device to take decisions — represented in our setup by valuation vectors, or shadow prices. Let us denote ∇ a valuation vector for a person \(i\), or a group \(j\). The economic analysis starts from an individual, subjective valuation vector \(\nabla_i\), which is then ‘socialized’ into a vector \(\nabla_{ij}\) translating how individual \(i\)’s self-interest is expressed in group \(j\). Then it proposes an aggregation mechanism \(E\) that generates a collective, objective^6 valuation vector \(\nabla_j\) from the collection \((\nabla_{ij})_i\).

Reciprocally the sociological analysis starts from a group, objective valuation vector \(\nabla_j\); it then ‘individualizes’ \(\nabla_j\) into a new subjective valuation vector \(\nabla_{ji}\) translating how group \(j\)’s actions resonate in individual \(i\)’s decision environment;\(^7\) and finally it proposes an aggregation mechanism \(S\) that generates a new individual characteristic vector \(\nabla_i\) from the collection \((\nabla_{ji})_j\).

\(^5\)List & Spiekermann (2013) argue that ‘supervenience individualism’ (i.e., the view according to which the individual-level facts fully determine the social facts) is compatible with ‘causal-explanatory holism’ (i.e., the view according to which some causal relations are distinct from any individual-level causal relations). They illustrates this compatibility with, among others, the example of social-network theory.

\(^6\)We come back latter to what we mean by ‘objective’.

\(^7\)The is reminiscent of Arrow-Lindahl ‘individualization’ of collective consumption \(x_j\) for individual \(i\). The aggregation mechanism in that case goes through the market: there is a price \(p_{ji}\), according to which individual \(i\) expresses a demand for collective consumption \(j\), \(x_{ji}\), and the price level is fixed through the market clearing equation: \(x_{ji} = x_j\).
Hence the diagram:

\[
\begin{align*}
\nabla_i & \quad \text{socialization} \quad \rightarrow \quad \nabla_{ij} \\
S & \uparrow \quad \downarrow \quad E \\
(\nabla_{ji})_i & \quad \text{individualization} \quad \leftarrow \quad (\nabla_j)_j
\end{align*}
\]

(1)

Our notion of reciprocal aggregation equilibrium identifies the fixed points along this loop in a general equilibrium perspective.

1.3 The result

We try to remain as little specific as possible on the nature of the aggregation mechanism taking place at nodes. The central principle that we build on is the strong unanimity principle. This principle establishes that if all neighbors of a decision node unanimously agree that action \(a\) is at least as good as action \(b\), and one neighbor thinks \(a\) is better than \(b\), then so should the considered node. This principle is probably one of the mildest, and least demanding when thinking of an aggregation mechanism. The main aim of the paper is to follow this route and explore the consequences of assuming only that. Our, quite intuitive, finding is that with sufficient interconnectedness in the graph, a great convergence occurs: At equilibrium all valuations are identical. In a general equilibrium set-up, this finding translates in the following terms: Assuming reciprocal aggregation, then with sufficient portfolio diversification, the strong unanimity principle leads to Pareto optimality, even in case of severe market failures.

The unanimity principle applied to aggregation at the collective nodes has not been much questioned. It is a founding assumption of the literature on aggregation of individual preferences, starting with Arrow (1951). It is compatible with all types of aggregation processes that naturally come to mind, including voting mechanisms of all sorts. The unanimity condition is also a founding assumption of the logical aggregation theory (Kornhauser & Sager, 1986). It is also a founding assumption of the literature on aggregation of judgements and beliefs, starting with Harsanyi (1955).

We build on the latter literature to apply the unanimity principle also at the individual level. We argue that it does not necessarily requires a revision of deep, primitive characteristics of the individual. We show that in the case of production externalities, this principle is vehicled by a mere rebalancing of portfolios, in a way which is perfectly compatible with the standard competitive analysis, with the exception that the rationale for rebalancing is not to meet the usual hedging needs, but to restore internal consistency of choice. In the case of incomplete markets, when preferences are of the expected utility form, the unanimity principle is mostly vehicled by a mere updating of beliefs, i.e. subjective probabilities over states of nature. It builds on the rationale that individual agents recognize some wisdom, or authority, to the boards: they consider as compelling judgements the outcome of the aggregation mechanism occurring at the collective level.

The argument to advocate for the latter assumption is threefold, summed up in: objectivity, affectio societatis, and amor fati.
Firstly, what comes out of aggregation at collective nodes is not as subjective as individual beliefs, opinions or tastes. It loses its subjectivity to the extent that its link with individual experiences, desires, perceptions or emotions becomes more distant. Another way to put it is: In the aggregate, idiosyncracies are purged, and subjectivity is averaged away (see Proposition 1). Losing subjectivity, it can hence be argued that the aggregate outcome somehow acquires objectivity. Not in the sense that it is related to actual, external facts, but because it is less distorted by feelings and personal bias.8

Secondly, individuals are free to choose the nodes they link to, at least when portfolios are traded, or in case of partnerships. There is a will to invest in common, an affectio societatis. And this carries two forces. The first one is that by joining a group/board individuals make a sort of meta-choice. They remain free to disagree with the group’s view, but freedom of group affiliation makes it less likely to observe blindly opposing patterns of behavior as, e.g., a teenager opposing a view for the very reason that it is his parents’ (a reverse unanimity condition). The second force is the character of local public good of the firm. If an investor takes shares in some firm, it is because its production plan provides an insurance service that fits the investor’s needs. Hence all investors in a given firm share features in common.9 More generally, in affiliations which are freely chosen, an individual can make his beliefs and tastes felt by the collective. We argue that the joint operation of the force of the meta-choice, together with the sense of commonality of needs, fosters a tendency to subscribe to the views of a group one freely joins,10 especially when these views have unanimous support from all groups to which one adheres11.

Thirdly, and this argument will be developed extensively in Section 8, there are good reasons (if not a full-fledged ‘rationality’) to ‘adapt’ to (if not adopt) the unanimous views expressed by groups to which one chooses to adhere. Given unanimity, most probably these views will be imperative. They will generate outcomes that everyone will have to put up with. Adapting to these ineluctable circumstances is reasonable - a bit like a trader has to conform with the mainstream expectations, otherwise she loses money. If all firms in which one invests unanimously value a type of investment policy, then most probably this type will be widely adopted, and therefore the concerned shareholder would be better off resolving to subscribe to such policies, or vote with his feet, and short-sell the firm’s stock. Updating one’s beliefs for the sake of welfare is rational. Adapting one’s tastes for the same sake sounds reasonable, in which case it shows the strength, and freedom, to embrace the ineluctable, amor fætæ, the ‘love of fate’ or ‘faire de nécessité vertu’ (Elster, 1983).

The paper is organized as follow. Section 2 formalizes the main modelling tools: production

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8Following Gilboa & ali (2010), a more precise, although more cumbersome term would be ‘inter-subjectivity’ instead of sheer objectivity, that would refer to some external, validated ‘truth’.

9Otherwise, in general equilibrium, investor rebalance their portfolio and exit from the capital of the firm, instead of stay and voice (see the ‘political sunspots’ described in Crès & Tvede, 2009, based on Hirschman, 1970).

10Simmel, in the quote above, uses the word ‘surrender’, although here there is no sense of constraint, all the contrary, as one surrenders to the person one loves.

11Section 8 discusses why this is far from assuming conformism, or imitation. In particular, recall that these last words apply generally to direct inter-individual influences; here these influences are mediated by collectives through institutional designs.
sets, valuation vectors and the link between optimization and the unanimity principle. Section 3 introduces the concept of reciprocal aggregation through a partial equilibrium analysis of a network of boards and directors. It formalizes the notion of duality between persons and groups and provides the main insight of the paper, i.e. the global convergence of valuations when there is enough interlock in the board membership (Proposition ?? and Theorem 1). Section 4 introduces a standard general equilibrium model of an exchange economy and provides the micro-foundations of valuation vectors. Through two leading cases of market failures, namely production externalities and incomplete financial markets, it highlights how market failures give rise at equilibrium to conflicting valuations between shareholders. Section 5 extends the general equilibrium analysis of Section 4 by endogenizing production and individual characteristics to provide a full-fledged notion of general equilibrium based on reciprocal aggregation. Section 6 and Section 7 apply the analysis to the two leading cases of market failures, respectively, and show how the first welfare theorem is restored in both cases (Theorem 2 and Theorem 3). Finally Section 8 offers a discussion of stable preferences, as opposed to endogenous, or adaptive preferences, and Section 9 concludes with various comments.

2 Valuation and the unanimity principle

Consider an economy with \( \ell \) commodities, \( I \) consumers with \( i \in \mathcal{I} = \{1, \cdots , I\} \), and \( J \) firms with \( j \in \mathcal{J} = \{1, \cdots , J\} \).

Firms are characterized by their sets of production plans, or actions, denoted \( A_j \subset \mathbb{R}^\ell \) for firm \( j \). These sets are supposed to be convex and have a smooth frontier.

Assumption (A): The set of production plans \( A_j \) is compact and there is a smooth and concave mapping \( \gamma_j : \mathbb{R}^\ell \to \mathbb{R} \) such that \( A_j = \{a_j \in \mathbb{R}^\ell | \gamma_j(a_j) \leq 0\} \).

Observation 1 Suppose \( A_j \) satisfies assumption (A), then \( g_j(a_j) = 0 \) implies \( Dg_j(a_j) \neq 0 \).

Proof: Since \( g_j \) is concave, \( g_j((1-t)a_j + ta'_j) \geq (1-t)g_j(a_j) + tg_j(a'_j) \) for all \( t \in [0,1] \). Therefore \( Dg_j(a_j) \cdot (a'_j - a_j) \geq g_j(a'_j) - g_j(a_j) \). There is \( a'_j \) such that \( g_j(a'_j) > 0 \) because \( A_j \) is compact. Suppose \( g_j(a_j) = 0 \) and \( g_j(a'_j) > 0 \), then \( Dg_j(a_j) \cdot (a'_j - a_j) \geq g_j(a'_j) - g_j(a_j) > 0 \). Hence \( Dg_j(a_j) \neq 0 \). \( \square \)

Denote \( a = (a_1, \cdots , a_J) \in \mathcal{A} = \Pi_j A_j \) a global production plan.

Denote \( k^\ell \) the simplex of vectors of \( \mathbb{R}^\ell_+ \) whose coordinates sum up to 1:

\[
    k^\ell = \left\{ \nabla \in \mathbb{R}^\ell_+ \mid \sum_{l=1}^{\ell} \nabla^l = 1 \right\}.
\]

Let \( k^\ell_+ \) be its interior. In the sequel valuation vectors will be normalized and taken in \( k^\ell_+ \).


2.1 Valuation vectors

We consider cases where all efficient production plans are on the boundary of the production sets, hence, without loss of generality, attention is restricted to production plans $a_j \in \partial A_j$.

**Definition 1** A production plan $a_j \in A_j$ maximizes value with respect to the valuation vector $\nabla \in \mathbb{R}_{++}^l$ provided there exists $\beta > 0$ such that $\nabla = \beta D g_j(a_j)$. If $a_j$ maximizes value with respect to $\nabla$, then $a_j$ is optimal for $\nabla$.

The following observation is immediate, but important. It allows to represent a firm, hence a collective entity, in a similar fashion as an individual. All individuals are equipped with a valuation vector, and Section 4 makes clear how these valuations stem from individual preferences.

**Observation 2** Suppose $A_j$ satisfies assumption (A). Then for all $a_j \in \partial A_j$ there exists a valuation vector $\nabla \in \mathbb{R}^l \setminus \{0\}$ for which $a_j$ is optimal.

Suppose that firms are governed by a board of directors. Let $K = \{1, \cdots, K\}$ be the set of potential directors. Let $K_j \subset K$ be the set of directors of firm $j$. A director $k$ is endowed with a valuation vector $\nabla_k \in \mathbb{R}_{++}^l$, with respect to which she maximizes value.

**Definition 2** Director $k$ values production plan $a'_j$ at least as much as (resp. better than) $a_j$ in firm $j$ if and only if $\nabla_k \cdot a'_j \geq \nabla_k \cdot a_j$ (resp. $\nabla_k \cdot a'_j > \nabla_k \cdot a_j$).

Directors can disagree on how to value production plans in firms. In the standard neoclassical model with perfectly competitive and complete markets, all individual agents use the equilibrium price vector to value production plans in firms. Hence all directors do the same and therefore there is no conflict. But in case of market failures, no general equilibrium condition guarantees equality of individual valuation vectors, as will be illustrated in Section 4.

In this early section of the paper, we do not micro-found valuations of directors into preferences through individual optimization. In this sense it is a partial equilibrium model. Directors are simply considered as experts, and directors’ valuation vectors are taken as expert judgments. Although we do not model this process at this stage, we assume that experts are chosen according to their capacity to synthesize voluminous and diffuse information relevant to the concerned decision, in a balanced way.

Boards are thus collective of experts, and a firm is modelled as aggregating these experts’ valuations according to some mechanism. We assume that the latter satisfies the strong unanimity principle.

2.2 The strong unanimity principle and value maximization

An aggregation mechanism in firm $j$ is said to satisfy the strong unanimity principle if the following holds: suppose directors unanimously value production plan $a'_j$ at least as much as $a_j$, and one values $a'_j$ better than $a_j$, then so should firm $j$.

A concept of stability naturally follows.
Definition 3 A global production plan $a \in A$ is stable with respect to the strong unanimity (of directors) principle if and only if within each firm $j$, there is no change of production plan that is at least as much valued by all directors, and better valued by at least one director:

$\not\exists a'_j \in A_j$ such that $\nabla_k \cdot a'_j \geq \nabla_k \cdot a_j$ (with at least one $>$) for all $k \in K_j$.

The unanimity principle is considered as a weak, low-demanding requirement. It is a founding assumption of the literature on aggregation of individual preferences, starting with Arrow (1951), followed by Sen (1970). It is also a building block of welfare economics, especially of the literature on the joint aggregation of beliefs and tastes, starting with Harsanyi (1955)’s social aggregation theorem, extended to Savage’s framework$^{12}$ by Hylland & Zeckhauser (1979) and Mongin (1995), among others, and more recently to the case of multiple priors by Crès et al. (2012).

The unanimity condition is finally an influential principle of normative economics and moral philosophy. In particular, it appears as a founding assumption of the literature on aggregation of judgements and logical aggregation theory, rendered popular by the doctrinal paradox of Kornhauser & Sager (1986), and the discursive dilemma of List & Pettit (2002) — see, e.g., Mongin (2012) for a panoramic and authoritative survey of these questions.

Unanimity responsiveness, and therefore stability, can be characterized in terms of value maximization. The following result is well-known and widely used throughout this paper. It shows that if there is no alternative production plan unanimously supported by the board against the status quo $a_j$, then $a_j$ is value-maximizing for some valuation vector in the positive cone of the valuation vectors of directors.

Proposition 1 Let $a_j$ be a production plan. The two following assertions are equivalent:

(i) $\not\exists a'_j \in A_j$ such that $\nabla_k \cdot a'_j \geq \nabla_k \cdot a_j$ for all $k \in K_j$ (with at least one $>$);

(ii) $\exists \beta_j > 0$ and $\lambda_j \in \mathbb{R}_{++}^{|K_j|}$ such that $\beta_j Dg_j(a_j) = \sum_{k \in K_j} \lambda_{jk} \nabla_k$.

Proof: This is a direct consequence of Stiemke’s theorem of the alternative (see Border, 2013, Theorem 17).

Denote $B$ the $(|K_j| + 1) \times \ell$ matrix with first $|K_j|$ rows: $\nabla_k^T$ (the transpose of $\nabla_k$), $k \in K_j$, and last row: $-Dg_j(a_j)$. It is immediate to prove that (i) is equivalent to: $\not\exists x_j \in \mathbb{R}^\ell$ such that $Bx > 0$ (where $x > y$ stands for $x_j \geq y_j$ for all $j$, and $x_j \neq y_j$).

By Stiemke’s theorem, this is equivalent to: $\exists (\lambda_j, \beta_j) \in \mathbb{R}^{|K_j|} \times \mathbb{R}$ such that:

$$
\begin{align*}
(\lambda_j \beta_j)B &= 0 \\
(\lambda_j, \beta_j) &\gg 0
\end{align*}
$$

(where $x \gg y$ stands for $x_j > y_j$ for all $j$), which is (ii). □

Hence aggregation of experts’ valuations in firms gives rise, under the strong unanimity principle, to boards’ valuation vectors $(\nabla_j)_{j \in J}$ — where $\nabla_j \in \mathbb{R}^\ell$ is the normalized $Dg_j(a_j)$.

$^{12}$Note that it has been fingered that the unanimity of individual preferences can be ‘spurious’, i.e., falsely driven by the fact that disagreement over tastes and disagreement over beliefs neutralize each other — see, e.g., Mongin (1997) and Gilboa et al. (2004).
Equations (2) can be rewritten, up to rescaling of $\lambda_j$: for all $\nu$, $\nabla \nu = \sum_{k \in K_j} \lambda_{jk} \nabla_k$; then the normalization of valuation vectors yields $\lambda_j \in k^{K_j}$ for all $j$. Define $\Lambda$ as the $J \times K$ matrix with entries $(\lambda_{jk})$, with $\lambda_{jk} = 0$ if $k \not \in K_j$. Since entries across each row are normalized to sum up to 1, $\Lambda$ is stochastic. Now denote $\nabla_J$ the $J \times \ell$ matrix whose $j$-th row is the transposed of $\nabla_j$, denoted $\nabla_j^T$. Then Equations (2) can be written in matricial form:

$$\nabla_J = \Lambda \nabla_K. \tag{3}$$

As argued in the introduction, an interpretation is that in the aggregate idiosyncracies are purged, and subjectivity is averaged away.\textsuperscript{13} Therefore each $\nabla_J$ proposes an ‘objective’ collective judgement. How these various collective judgements might interact through the network of board membership is the object of the next section.

### 3 Reciprocal aggregation and the great convergence

There is one aspect of the question that we want to underline: as argued in the introduction, what comes out of aggregating individual beliefs, opinions or tastes acquires objectivity. This more objective nature of the aggregate valuation conveys a compelling judgement, and carries authority. From a firm’s perspective, the judgement coming out of the aggregation of the expertise of board members in other firms has necessarily some value.\textsuperscript{14} This is certainly one of the determinants of the interlocked nature of board membership in large corporations: if a multi-affiliated director is invited to sit in the board of a given firm, it is certainly, at least partially thanks to the expertise he draws from sitting in the other boards.\textsuperscript{15} That is why we assume that, in forming their valuation, directors aggregate the collective valuations of the boards of which they are members.

The present section aims at catching a simple idea: if boards aggregate directors’ valuations according to a unanimity responsive mechanism, and if, reciprocally, directors aggregate boards’ valuations according to a unanimity responsive mechanism, then with enough interlock in board membership all directors, and all boards, have common valuations at equilibrium.

#### 3.1 Interlocking directorates

For any director $k$, let $J_k \subset J$ be the set of firms of which $k$ is a director. Along the preceding line, we assume that directors consider boards as experts. More precisely, the judgement arising

\textsuperscript{13}An opposite view is what Janis (1972) calls ‘group-think’, i.e. mutually reinforcing bias. But Proposition 1 formally supports Elster (1983): “The random errors of private and selfish preferences may to some extent cancel each other, and thus be less to be feared than the massive and coordinated errors that arise through group-think” (p. 40).

\textsuperscript{14}Mace (1971) p.197: “The opportunity to learn through exposure to other companies operations is something of value that might be useful in their own situation [...]”

\textsuperscript{15}See Davis & al. (2002), p. 305: “[...] board interlocks may be a fortuitous by-product of board preferences for recruiting experienced directors [...] The prior experience of directors is part of the raw material of board decision making, and it is unsurprising that a director who has been involved in acquisitions, alliances [...] or any other board-level decision (including recruiting other directors) would bring that expertise to bear.”
from the aggregation of directors’ expertise in boards is objective enough to convey, in return, some compelling expertise for directors and carry authority. We assume that when expressing her valuations in a board, or when re-assessing her valuations if necessary, a director neither follows a personal agenda (contrary to a shareholder who is endowed with exogenously given preferences), nor faces conflicts of interests.

In assessing the value of a plan \( b \in \mathbb{R}^\ell \), director \( k \) faces \( |\mathcal{J}_k| \) potentially different judgements in the firms of which he is a director, represented by the valuation vectors \((\nabla_j)_{j \in \mathcal{J}_k}\). We assume that these judgements are compelling to the extent described by the strong unanimity principle: If boards in \( \mathcal{J}_k \) unanimously value plan \( b_0 \) at least as much as \( b \), and one values \( b_0 \) better than \( b \), then so should director \( k \).

**Assumption (SUP).** The strong unanimity (of board) principle:

\[
\nabla_j \cdot b' \geq \nabla_j \cdot b \quad \forall j \in \mathcal{J}_k \text{ (with at least one \( > \)) \Rightarrow } \nabla_k \cdot b' > \nabla_k \cdot b
\] (4)

The dual to Proposition 1 holds.

**Proposition 2** Assumption (SUP) holds if and only if:

\[
\forall k, \exists \xi_k \in [0,1]^{\mathcal{J}_k} \text{ such that } \nabla_k = \sum_{j \in \mathcal{J}_k} \xi_{kj} \nabla_j
\] (5)

**Proof:** The proof, as for Proposition 1, is a direct consequence of Stiemke’s theorem of the alternative. Denote \( B \) the \((|\mathcal{J}_k| + 1) \times \ell \) matrix with first \(|\mathcal{J}_k| \) rows: \( \nabla^T_j, \quad j \in \mathcal{J}_k \), and last row: \(-\nabla^T_k\). It is immediate to prove that (SUP) is equivalent to: \( \exists x_j \in \mathbb{R}^\ell \text{ such that } Bx > 0 \). By Stiemke’s theorem, this is equivalent to (5).□

A necessary and sufficient condition for the Assumption (SUP) is therefore that, when assessing her valuation vector, a director gives strictly positive weight to all valuations of boards to which she affiliates.

Consider the graph \( \mathcal{G} \) defined in the introduction, with director nodes linked to board nodes according to whether the concerned director is a member of the concerned board. It is strongly connected if there is a path from any node to any other node.\(^{16}\) Under this condition, at a stable global production plan all firms agree on how to value production plans; and so do all directors.

**Theorem 1** Suppose \( \mathcal{G} \) is strongly connected. Then a global production plan which is stable for the strong unanimity (of directors) principle, and at which the strong unanimity (of boards) principle holds is such that:

\[
\exists \nabla \text{ such that } \forall j \in \mathcal{J}, \forall k \in \mathcal{K}, \nabla_j = \nabla_k = \nabla.
\]

\(^{16}\)Most studies of actual directorate networks tend to support strong connectedness (see, e.g., Burt, 2006). Davis (1996) states that "the median Fortune 500 firm during the 1980s collectively sat on the boards of seven other Fortune 500 firms, and some firms [..] shared directors (‘interlocked’) with 40 or more large firms. The aggregate result is the creation of an interlocking directorate linking virtually all large American firms into a single network based on shared board members."
Proof: With the obvious extension $\xi_{kj} = 0$ if $j \notin J_k$, and denoting $\Xi$ the $K \times J$ stochastic matrix with entries $(\xi_{kj})$, with $\nabla_J$ defined as above, and $\nabla_K$ the $K \times \ell$ matrix whose $k$-th row is $\nabla_k^T$, then Equations (5) can be written in matricial form: $\nabla_K = \Xi \nabla_J$.

Hence, given Equations (3), at a stable global production plan: $\nabla_J = \Lambda \Xi \nabla_J$. Reciprocally, $\nabla_K = \Xi \Lambda \nabla_K$.

The product of two stochastic matrices being stochastic, both $\Lambda \Xi$ and $\Xi \Lambda$ are square and stochastic. And $\mathcal{G}$ being strongly connected, both $\Lambda \Xi$ and $\Xi \Lambda$ are strongly connected: there is a path from any board (resp. director) node to any other board (resp. director) node.

Since $\Lambda \Xi$ is strongly connected and aperiodic (it has positive diagonal entries), its powers $(\Lambda \Xi)^n$ converge toward a matrix with identical rows $\lambda \xi^T$ when $n \to \infty$ (see, e.g., Jackson, 2008). Since $\nabla_J = (\Lambda \Xi)^n \nabla_J$ for all $n$, at the limit $\nabla_J^T = \lambda \xi^T \nabla_J$ for all $j$. The same argument holds for $\Xi \Lambda$. □

Although not a central message of the paper (and presented only to introduce the basic insight of the restored first welfare Theorem 2 and Theorem 3 below) this result seems to theoretically support empirical findings that interlocking directorates create "an informational and normative context – an ‘embeddedness’– for board decision (Granovetter, 1985). Decisions at one board in turn become part of the raw material for decisions at other boards. In the aggregate, the structure of the network [...] will influence how the field as a whole evolves [...]" (Davis 1996, p.154).

3.2 Reflexivity in question

The proof of Theorem 1 shows that in shaping valuation vectors, reciprocal aggregation entails some reflexivity as soon as the matrix $\Xi \Lambda$ has positive diagonal entries. The latter always holds under Assumption (SUP). Indeed the $k$-th diagonal entry is $\sum_j \xi_{kj} \lambda_{jk}$, a sum of non-negative numbers, positive when $j \in J_k$.

But suppose that directors remain more aloof, and still consider some idiosyncratic, primitive valuation vector $\tilde{\nabla}_k$ when they aggregate at the individual level. Assumption (SUP) could be reformulated as strong unanimity principle with reflexivity.

Assumption (SUPR).

$$\tilde{\nabla}_k \cdot b' \geq \tilde{\nabla}_k \cdot b \text{ and } \nabla_j \cdot b' \geq \nabla_j \cdot b \quad \forall \ j \in J_k \ (\text{with at least one } >) \Rightarrow \nabla_k \cdot b' > \nabla_k \cdot b \quad (6)$$

Then Proposition 2 would still apply, yielding for all $k$

$$\nabla_k = \xi_k \tilde{\nabla}_k + \sum_{j \in J_k} \xi_{kj} \nabla_j,$$

with $(\xi_k, (\xi_{kj}))_j \in \mathbb{R}^{1+|J_k|}$. So when aggregating, the director takes into account another self, endowed with primitive valuations. We assume that aggregation leaves the individual at personal equilibrium, i.e. with agreeing selves.

Assumption (PE). Individual aggregation yields personal equilibrium, i.e. $\forall \ k$, $\nabla_k = \tilde{\nabla}_k$. 12
Since at least one $\nabla_j$ is independent, then $0 \leq \xi_k < 1$, therefore at a personal equilibrium $\nabla_k = \sum_{j \in J_k} \frac{\xi_{kj}}{\xi_k} \nabla_j$, hence we are back to Proposition 2.

**Observation 3** Assuming (SUP) is formally equivalent to Assuming (SUPR) and (PE).

Assumption (SUPR) is more innocuous than Assumption (SUP) to the extent that aggregation can only change an individual ranking from an indifference to a (strict) preference. Assumption (PE) means some internal consistency of choice, and can therefore be considered quite demanding.

For the sake of simplicity, we will only consider Assumption (SUP) in the sequel.

### 3.3 The duality of boards and directors

Following Breiger (1974), we can construct the $K$–square (symmetric) matrix $D$ of interpersonal ties, and a $J$–square (symmetric) matrix $B$ of intergroup ties. The entry $d_{kk'}$ of $D$ indicates the number of boards of which both directors $k$ and $k'$ are members. The entry $b_{jj'}$ of $B$ indicates the number of directors who are members of both boards $j$ and $j'$. One can think of the interboard ties as the set of directors in the intersection of the board membership; and ‘dually’ think of the interdirector ties as the set of boards in the intersection of their individual affiliations.

This duality has a mathematical representation. Let $M$ be the $K \times J$ membership (adjacency) matrix, with entry $m_{kj} = 1$ if director $k$ is a member of board $j$; $m_{kj} = 0$ otherwise. The duality reads: $D = MM^T$ and $B = M^T M$; they represent two graphs where the roles of nodes and links are swapped: $D$ is the matrix of a graph where nodes are directors and links are boards; dually, $B$ is the matrix of a graph where nodes are boards and links are directors. Obviously $D$ is strongly connected if and only if the underlying graph $G$ (defined by the adjacency matrix $M$) is strongly connected.

Clearly, the matrix $\Xi$ has positive entries where the matrix $M$ has positive entries; and $\Lambda$ has positive entries where $M^T$ has positive entries. Therefore the product matrix $\Lambda \Xi$ represents the interboards ties, whereas the matrix $\Xi \Lambda$ represents the interdirectors ties. The degree to which director $k$ takes into account director $k'$ valuations is given by the entry $(k, k')$ and linked to the number of boards of which $k$ and $k'$ are both members. In particular, the degree of reflexivity is given by the diagonal of the matrix and (relatively) bigger the more boards of which the concerned director is a member.

To intuitively understand why, suppose that aggregation is ‘impartial’, in the sense that $\xi_{kj} = \xi_{kj'}$ for all $j, j' \in J_k$ and $\lambda_{jk} = \lambda_{jk'}$ for all $k, k' \in K_j$. Assume furthermore that board size is constant across firm: $\forall j, |K_j| = n$. Then the entry $(k, k')$ of matrix $\Xi \Lambda$ at a reciprocal equilibrium is: $\frac{|J_k \cap J_{k'}|}{|J_k|} \frac{1}{n}$. It is always maximum on the diagonal, and equal to $\frac{1}{n}$.

The dual analysis would hold for boards: for a given board, self-reference is (relatively) higher, the larger the board.
4 General equilibrium: Micro-foundations of valuation vectors

We aim in this section at modelling where from valuation vectors come, in a general equilibrium perspective. Consumers are primitively characterized by their identical consumption sets $\mathbb{R}^\ell$, endowment vectors $\omega_i \in \mathbb{R}^\ell$, utility functions $u_i : \mathbb{R}^\ell \to \mathbb{R}$, and portfolios of shares in firms $\delta_i = (\delta_{i1}, \ldots, \delta_{ij})$, where $\delta_{ij} \in \mathbb{R}$ and $\sum_j \delta_{ij} = 1$ for all $j$. Hence the primitive characteristics of an economy are given by the array

$$\{(u_i, \omega_i, \delta_i)_{i \in \mathbb{I}}, (g_j)_{j \in \mathbb{J}}\}.$$ 

Let $x = (x_1, \ldots, x_I) \in \mathbb{R}^{\ell I}$ be an allocation and $p = (p^1, \ldots, p^\ell)$ be a price vector for commodities. We normalize price vectors to be in $k^{\ell+}$. A state $(x, a)$ is a vector of allocation and global production plan such that $\sum_i x_i = \sum_i \omega_i + \sum_j a_j$.

4.1 Consumer choice and individual valuation vector

Given $(p, a)$ a price vector and actions in firms, consumer $i$ chooses the optimal consumption bundle in his budget set $B_i(p, a, \omega_i, \delta_i) \subset \mathbb{R}^\ell$ :

$$\max_{x_i} u_i(x_i) \qquad \text{s.t. } x_i \in B_i(p, a, \omega_i, \delta_i) \quad (7)$$

In some circumstances, consumers are allowed to rebalance their portfolios in order to finance the optimal consumption bundle, in which case a new portfolio $\theta_i$ is endogenously defined by the budget constraint — see Section 4.5 below. When portfolio are not rebalanced, $\theta_i = \delta_i \forall i$.

The modelling strategy mainly rests upon the concept of individual valuation vectors $\nabla_i(p, x, a) \in \mathbb{R}^\ell$ for consumers, as for directors in Definition 2. In the sequel, the derivation of $\nabla_i(p, x, a)$ is detailed in various circumstances, and Appendix A makes precise how valuations are linked with actual preferences.\textsuperscript{17}

For lightness of notation, $\nabla_{ij}$ will stand for $\nabla_{ij}(p, x, a)$ when no confusion is possible. Let $\nabla_i = (\nabla_{i1}, \ldots, \nabla_{ij}) \in \mathbb{R}^{\ell J}$ be individual $i$'s global valuation vector across all firms.

Our modelling strategy, that goes through individual valuation vectors, allows to investigate most cases of market failures through a common apparatus. It is moreover highly compatible with the tools and machinery developed in social choice theory.

4.2 Equilibrium and optimality

Given actions in firms, the standard Walrasian equilibrium concept applies readily.

\textsuperscript{17}It might appear at first sight that the analysis led through individual valuation vectors does not have a global reach. In fact it has, at least in the two leading cases covered in this paper (where market failures are due to production externalities or incomplete financial markets). Appendix A shows that local preferences derived from the $\nabla_{ij}$'s, both for consumers and firms, hold globally.
Definition 4  Fix actions \( a \in A \) in firms. A market equilibrium (for fixed actions) is a vector of price and consumption bundles \((p^*, x^*)\) such that:

(i) individual consumers optimize, so \( \forall i, x^*_i = \arg \max \{ u_i(x_i) | x_i \in B_i(p^*, a, \omega_i, \delta_i) \} \) given \((p^*, a)\);
(ii) markets clear, so \( \sum_i x^*_i = \sum_i \omega_i + \sum_j a_j \), (and \( \sum_i \theta^*_i = 1 \) when portfolio are rebalanced).

And the standard definition of Pareto optimality for states follows.

Definition 5  A state \((x, a) \in \mathbb{R}^I \times A\) is Pareto optimal if there is no other state \((x', a') \in \mathbb{R}^I \times A\) such that \( u_i(x'_i) \geq u_i(x_i) \) for all \( i \), with strict inequality for at least one \( i \).

When no confusion is possible, the allocation \( x \) supporting the state \((x, a)\) will be said Pareto optimal. A crucial point is whether consumers agree on the way to value the actions taken in firms, i.e. whether they agree on how to run the firm. In our modelling strategy, it corresponds to cases where all \( \nabla_{ij} s \) are collinear across \( i's \), and point in the same direction. This is reviewed now.

4.3 Unanimous valuations: The case of perfect markets

Assuming complete and perfectly competitive markets, the budget set of a consumer is:

\[
B_i(p, a, \omega_i, \delta_i) = \left\{ x_i \in \mathbb{R}^I \mid p \cdot x_i = p \cdot \omega_i + \sum_j \delta_{ij} p \cdot a_j \right\}
\]

Hence, assuming competitive behavior, consumers value actions in firm \( j \) through their share of the profit: \( \delta_{ij} p \cdot a_j \).

It is trivial but important to underline that, in this standard Walrasian setup, consumers agree on the way to compute profits. Whether the firm delivers the nominal dividend \( \delta_{ij} p \cdot a_j \in \mathbb{R} \), or the real dividend vector \( \delta_{ij} a_j \in \mathbb{R}^I \), does not make any difference since, at the Walrasian equilibrium, relative values perceived by the consumer — i.e. ratio of marginal utilities represented in short by the utility gradient \( Du_i(x_i) \) — and relative values posted by the market — represented by the market price vector \( p \) — will end up being the same, due to the first order conditions of the consumer’s optimization program (7).

Hence, in the standard Walrasian setup, individual valuation vectors are all equal to the price vector

\[
\nabla_{ij}(p, x, a) = p.
\]

This collinearity between consumers’ utility gradients and market prices does not hold anymore in case of incomplete markets, a case studied in Section 4.5 below.

Not less trivial but not less important it is to underline that, in this standard Walrasian setup, all consumers with positive share agree that the more profit the better. This will not hold anymore if there are production externalities: intuitively, a consumer with shares in only in firm \( j \) will of course want firm \( j \) to maximize profit independently of the impact on other firms. But at the other extreme, suppose that firm \( j \) heavily pollutes firm \( j' \), a consumer with shares only in a firm \( j' \) will probably not seek profit maximization in firm \( j \). This case is studied in detail now.
4.4 Conflicting valuations: The case of production externalities

When firms inflict externalities to each other, and there are no markets to trade externalities, it is in general not the case that all consumers seek to maximize profit. Even though markets succeed in equalizing marginal rates of substitution between goods across consumers, resulting in having consumers agree on how to compute the firms’ profit, in absence of a decentralized mechanism to internalize externalities, consumers will disagree on whether profit should be maximized.

For tractability of the modelling, in addition to sets of actions $A_j \subset \mathbb{R}^d$, firms are characterized by production functions $f_j : A \rightarrow \mathbb{R}^\ell$. A list of firms’ actions $a = (a_1, \ldots, a_J)$ results in a production plan $y_j = f_j(a)$ for firm $j$, so the production plan of each firm may depend on actions taken in all firms (see Crès & Tvede, 2013). And the equilibrium equation of Definition 4 becomes $P_{\xi} = P_{\xi^*} + P_{\phi} f_j(a)$.

Hence the primitive characteristics of an economy are given by the array

$$\{(u_i, \omega_i, \delta_i)_{i \in I}, (g_j, f_j)_{j \in J}\}.$$ 

Given a price vector $p$ and actions $a$ in firms, a consumer, with endowments $\omega_i \in \mathbb{R}^\ell$ and portfolios of shares $\delta_i \in \mathbb{R}^J$, chooses optimal consumption in the budget set:

$$B_i(p, a, \omega_i, \delta_i) = \left\{ x_i \in \mathbb{R}^\ell \mid p \cdot x_i = p \cdot \omega_i + \sum_j \delta_{ij} p \cdot f_j(a) \right\}. \quad (9)$$

Depending upon their portfolio $\delta_i$, consumers will weigh differently the benefits and losses stemming from a change in production plans. This information is summarized into the individual valuation vector.

**Observation 4** The individual valuation vector of consumer $i$ takes the form

$$\nabla_{ij}(p, a) = \sum_{j'} \delta_{ij'} D_{a_j} f_{j'}(a)^T p. \quad (10)$$

**Proof:** Assuming competitive behavior, a change of actions in firm $j$ impacts the consumer through her income, more precisely through her dividends from firms’ profits $p \cdot f_j(a)$, all $j$. As a consequence, consumer $i$ prefers actions $a_{j'}$ to $a_j$ in firm $j$ if and only if its dividend increase, i.e.:

$$\sum_{j} \delta_{ij'} p \cdot f_{j'}(a_{j'}, a_{-j}) > \sum_{j} \delta_{ij} p \cdot f_{j'}(a),$$

with $a_{-j} = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_J)$.

A marginal change of actions $da_j = (da_j^k)_{k=1}^d$ in firm $j$ impacts the profit $p \cdot f_{j'}(a)$ of firm $j'$ through the following matricial expression:

$$\sum_{k=1}^d \left( p \cdot \frac{\partial f_{j'}(a)}{\partial a_j^k} \right) da_j^k = (D_{a_j} f_{j'}(a)^T p) \cdot da_j$$
where \( D_{aj}f_j'(a) \) is the \( \ell \times d \) Jacobian matrix of \( f_j' \) with respect to \( a_j \). Hence the vector \( D_{aj}f_j'(a)^T \mathbf{p} \in \mathbb{R}^d \) measures how a (marginal) change of actions in firm \( j \) changes the profit of firm \( j' \). □

There are obvious sufficient conditions guaranteeing that consumers agree on the way to value actions in firms. One of them follows.

Suppose that all consumers hold a share of the market portfolio:18 \( \delta_i = \alpha_i \mathbf{1}_J \) for some \( \alpha_i \in \mathbb{R} \). Then \( \nabla_{ij}(p, a) = \alpha_i \left[ \sum_{j'} D_{aj}f_{j'}(a)^T \mathbf{p} \right] \), i.e., all consumers have collinear valuation vectors. All consumers agree on how to value production, and all with positive shares agree that value should be maximized.19

But in the generic case, consumers typically disagree on how to value production: \( \nabla_{ij}' \)'s are typically not collinear across \( i' \)'s. As seen form Equations (10), the idiosyncratic element in the valuation vector of agent \( i \) is her portfolio of shares \( \delta_i \), more precisely how \( \delta_i \) distributes the interdependence between profits.

As a consequence, equilibria are in general not Pareto optimal.

4.5 Conflicting valuations: The case of incomplete markets

In the standard Walrasian setup, when consumers disagree with the market on the relative value of goods, they trade with the market. They trade until they agree on these relative values, i.e., until the ratios of marginal utilities equal the ratio of prices: technically until the utility gradient \( Du_i(x_i) \) parallels the market price vector \( p \). Intuitively, if some markets are missing, this process of alignment of the consumers with the market will not hold completely, and as a consequence there will be some residual dimensions of conflict between the shareholders.

This phenomenon is particularly clear in the case where there is uncertainty in the economy, and not enough financial instruments are introduced. To keep the model to the simplest, consider an economy with 2 dates, \( t \in \{0, 1\} \), one state of nature at date \( s = 0 \), and uncertainty modeled by \( S \) possible states of nature at date 1: \( s \in \{1, \ldots, S \} \). Assume furthermore that there is only one commodity at every state, so that the number of commodities is \( \ell = S + 1 \).

In this setup, production plans of firms are real financial assets available to transfer the commodity across dates and states of nature. Obviously the price of the commodity can be normalized at each date and state, and the only market prices are those of the assets: \( q = (q_1, \ldots, q_J) \), where \( q_j \) is the price of asset \( a_j \) on the financial market. At date 0, consumer \( i \) trades assets by rebalancing her portfolio from its initial position \( \delta_i \in \mathbb{R}^J \) to its final one \( \theta_i \in \mathbb{R}^J \).

Given a price vector \( q \) and actions \( a \) in firms, consumer \( i \) with endowments \( \omega_i \in \mathbb{R}^{S+1} \), and initial portfolios of shares in firms \( \delta_i \in \mathbb{R}^J \), will trade shares on the markets and choose a new portfolio \( \theta_i \in \mathbb{R}^J \) to finance the optimal consumption bundle \( x_i \). At period 0, the consumer sells his initial portfolio \( \delta_i \) on the asset markets at prices \( q \), buys the new portfolio \( \theta_i \), and invests

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18 A property which holds in the CAPM; and under wider conditions entailing the two-fund separation theorem, see Cass & Stiglitz (1970).

19 It is worth underlining a virtue of the market portfolio: it secures political agreement between consumers despite the externalities; on top of that is an agreement for efficient actions. Hence with insurance services come along political agreement and economic efficiency, a remarkable alignment of desirable outcome.
in each firm according to her (new) shares \( \theta_i \). At period 1, the consumer receives the dividends from his new portfolio. This gives the constraints:

\[
\begin{align*}
  x_i^0 - \omega_i^0 &= \sum_j \delta_{ij} q_j + \sum_j \theta_{ij} (a_j^0 - q_j) \\
  x_i^s - \omega_i^s &= \sum_j \theta_{ij} a_j^s \quad \text{for all } s \geq 1.
\end{align*}
\]

For commodity of reading, the set of constraints will be written in matrix form. Some definitions and notation are needed. They might appear tedious at first sight but will reveal useful in the sequel because they allow compact equations.

Let \( A = \begin{pmatrix} a_1 & \cdots & a_J \end{pmatrix} \) be the \((S + 1) \times J\) asset matrix whose column vectors are the production plans; for all \( s \), denote \( a^s \in \mathbb{R}^J \) the \( s \)-th row of \( A \). Denote \( A_1 \) be the \( S \times J \) sub-matrix of date 1 returns, called the return matrix. Let \( \hat{A} \) denote the transfer matrix. Hence:

\[
A = \begin{pmatrix} a_1^0 & \cdots & a_J^0 \\
   a_1^1 & \cdots & a_J^1 \\
   \vdots & \vdots & \vdots \\
   a_1^S & \cdots & a_J^S \end{pmatrix} = \begin{pmatrix} a^{0T} \\
   A_1 \end{pmatrix} \quad \text{and} \quad \hat{A} = \begin{pmatrix} a_1^0 - q_1 & \cdots & a_J^0 - q_J \\
   a_1^1 & \cdots & a_J^1 \\
   \vdots & \vdots & \vdots \\
   a_1^S & \cdots & a_J^S \end{pmatrix}.
\]

Define period 0 total resources \( \hat{\omega}_i^0 = \omega_i^0 + q \cdot \delta_i \), i.e. the initial endowment at date 0 augmented with the market value of the initial portfolio. Define \( \hat{\omega}_i = (\hat{\omega}_i^0, \omega_i) \) where \( \omega_i \in \mathbb{R}^S \) denotes the vector of date 1 initial endowments. Then \( x_i = \hat{\omega}_i + \hat{\theta}_i \).

Consumer \( i \) chooses the consumption bundle that maximizes his utility function in the following budget set:

\[
B_i(q, a, \omega_i, \delta_i) = \left\{ x_i \in \mathbb{R}^{S+1} \mid \exists \theta_i \in \mathbb{R}^J \text{ such that } x_i = \hat{\omega}_i + \hat{\theta}_i \right\}.
\]

**Definition 6** Fix actions \( a \). A stock market equilibrium for fixed actions \((q^*, x^*, \theta^*)\) is such that

(i) consumers optimize: for all \( i \), \( x_i^* = \arg \max \{ u_i(x_i) \mid x_i \in B_i(q^*, a, \omega_i, \delta_i) \} \);

(ii) markets clear: \( \sum_i x_i^* = \sum_i \omega_i + \sum_j a_j \) and \( \sum_i \theta_i^* = 1_J \).

The geometry of equilibrium gradients and net trades is transparent. Let \( \langle \hat{A} \rangle \) denote the span of the transfer matrix. Then, at equilibrium, net trades are in \( \langle \hat{A} \rangle \) and gradients are orthogonal to \( \langle \hat{A} \rangle \) as stated in the following proposition. (Note that \( \hat{A} \) and \( \hat{\omega}_i \) depend on \( q^* \) and thus have equilibrium values \( \hat{A}^* \) and \( \hat{\omega}_i^* \).)

**Proposition 3** Fix the asset structure \( a \). At a stock market equilibrium \((q^*, x^*, \theta^*)\),

\[ x_i^* - \hat{\omega}_i^* \in \langle \hat{A}^* \rangle \quad \text{and} \quad Du_i(x_i^*) \in \langle \hat{A}^* \rangle^\perp \]

**Proof:** Immediate from the first order conditions of \( i \)'s optimization program. \( \square \)
A change of actions in firm \( j \) impacts the consumer through his consumption: when firm \( j \) changes actions from \( a_j \) to \( a'_j \), the utility of consumer \( i \) changes from \( u_i(x_i) \) to \( u_i(x_i + \theta_{ij}(a'_j - a_j)) \).

A marginal change of actions \( da_j \in \mathbb{R}^{S+1} \) in firm \( j \) changes the utility of consumer \( i \) by approximately \( \theta_{ij} Du_i(x_i) \cdot da_j \). As a consequence, the individual valuation vector of consumer \( i \) is given by the gradient, aka the vector of present values.

Observation 5 The individual valuation vector of consumer \( i \) takes the form

\[
\nabla_{ij}(x_i, a) = \theta_{ij} Du_i(x_i).
\]

Markets are incomplete when the return matrix \( A_1 \) has rank smaller than \( S \): some transfers across states of nature at date 1 are impossible. In the present setup, we assume that there is no redundant asset, hence incompleteness comes from the fact that there are not enough assets available: \( J < S \). The return matrix \( A_1 \) has rank \( J \). Therefore \( \dim \langle \hat{A} \rangle = J \) and \( \dim \langle \hat{A} \rangle^\perp = S + 1 - J \).

According to Proposition 3, at equilibrium consumers can use different present value vectors to value future income streams: the conditions \( \nabla_i(x^*_i, a) \in \langle \hat{A} \rangle^\perp \) allows for the normalized individual valuation vectors to be different two by two. It can even be proven to hold for a generic set of initial endowments.

Proposition 4 Fix the asset structure \( a \). Assume there are at least 2 consumers (\( I \geq 2 \)), and that market are incomplete (\( J < S \)). Then for almost all collection of initial endowments \((\omega_i)\), the normalized present value vectors of consumers are distinct

\[
\nabla_i(x^*_i, a) \neq \nabla_i(x^*_{i'}, a) \text{ if } i \neq i'
\]
at each equilibrium \((q^*, x^*, \theta^*)\) of the economy.

Proof: Magill & Quinzii (1996) Theorem 11.6. \( \Box \)

As a consequence, equilibria are in general not Pareto optimal. And if production plans are chosen endogenously, they are not even constrained Pareto optimal.

5 More general equilibrium: Reciprocal aggregation in assemblies of shareholders

As illustrated in the previous section, in case markets fail, if shareholders stick to their initial, primitive characteristics, they typically disagree on how to run the firm. Then a collective decision has to be taken, based on some aggregation mechanism. There is an extensive literature on the subject, where groups of decision makers vote,\(^{20}\) bargain\(^{21}\) or agree to maximize some ad


\(^{21}\)See Britz et al. (2013).
hoc objective function. Some pieces of this literature depart from the neoclassical tradition in the sense that the firm is not represented as a monolithic decision maker. Whether a stable outcome (equilibrium) exists is of course a central question; and the characteristics and properties of this outcome, in particular its potential Pareto optimality, is also at stake.

We build here on this literature, and depart even further from the neoclassical tradition in the sense that not only the firm, but also the individual is not represented as a monolithic, immutable decision maker. We assume that some aggregation mechanism occurs at the individual level, which abides to the Strong Unanimity Principle. This entails that individuals evolve from their primitive characteristics. This evolution is illustrated in Section 6 in the case of production externalities by showing how the aggregation may lead the individuals to rebalance their initial portfolios of shares, even though there is no uncertainty in the economy. It is then illustrated in Section 7 in the case of incomplete financial markets, by showing how the aggregation may lead the individuals to revise some parameters of their utility function.

The major finding is that this reciprocal aggregation process leads to an efficient use of economic resources. With reciprocal aggregation, the Strong Unanimity Principle fixes market failures.

The primitive characteristics defining an individual agent is represented by the array \( \chi_i = (u_i, \omega_i, \delta_i) \). In the standard general equilibrium model, when there is no uncertainty, trading leads individuals to exchange goods, so that \((u_i, \delta_i)\) are stable, and only initial resources \( \omega_i \) are traded against a new bundle \( x_i \). Then a general equilibrium vector, where both consumption and production are endogenized, is an array \((p^*, x^*, a^*)\). When there is uncertainty, trading leads individuals to furthermore rebalance their portfolios, and initial portfolios \( \delta_i \) are traded against final ones \( \Theta_i \); so that only \( u_i \) are stable. A general equilibrium vector is then an array: \((p^*, x^*, \Theta^*, a^*)\). We open the possibility to go beyond this point, and have also that some parameters of \( u_i \) are ‘transacted’ (e.g., the beliefs), hence \( u_i \) is not stable (or if tastes are stable, beliefs might not be).

For that we need a comprehensive notation, and the array \((p^*, \chi^*, a^*)\) describes a general equilibrium vector. We introduce a notion of general equilibrium where individual and collective choices rest on three pillars: individual optimization, market clearing and reciprocal aggregation of shareholders’ and firms’ valuations satisfying the Strong Unanimity Principle.

The following definition builds on Definition 4 (assertions (i) and (ii)), Assumption (SUP) (assertion (iii)) and Definition 3 (assertion (iv)).

**Definition 7** A **reciprocal aggregation equilibrium** is a vector of prices, characteristics and global actions \((p^*, \chi^*, a^*)\) such that:

(i) and (ii): given the global action vector \(a^*\), \((p^*, \chi^*)\) is a market equilibrium;

(iii): given \((p^*, a^*)\), \(\chi^*\) is stable with respect to the strong unanimity (of boards) principle;

(iv): given \((p^*, \chi^*)\), \(a^*\) is stable with respect to the strong unanimity (of shareholders) principle.

Let us immediately underline the obvious fact that, given Equations (8), under perfect

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markets Arrow-Debreu equilibria are reciprocal aggregation equilibria.

The unanimity of board principle necessitates that the aggregation of collective preferences occurring at the individual level be compatible with optimization. In the case of production externalities, it only requires that markets to trade shares are active, even though there is no uncertainty in the economy and therefore shares of firms are technically redundant (and priced by no-arbitrage). Section 6 below shows how the Strong Unanimity Principle may lead individual agents to trade shares, giving rise to an efficient internalization of production externalities.

In the case of incomplete markets, the Strong Unanimity Principle may lead individual agents to revise some parameters of their utility function. Section 7 shows how updating subjective beliefs, in a way that is compatible with the standard competitive analysis, may lead to an efficient allocation of resources and risks even though markets are incomplete.

6 When reciprocal aggregation leads to efficient internalization

Let us first underline a very special feature of production externalities: in assessing both their economic impact and political consequences, it is enough to look at the portfolio distribution.

6.1 From valuations vectors to portfolios

Given \((p, a)\), denote

\[
\Phi_j = \begin{pmatrix}
D_a f_1(a)^T p & \cdots & D_a f_J(a)^T p
\end{pmatrix}
\]

the \(d \times J\) matrix whose column vectors are the vectors that measure how a (marginal) change of actions in firm \(j\) changes the profits of other firms. A direct consequence of Proposition 10 (in the appendix) is that, given a price vector \(p\), a portfolio \(\delta\), and actions \(a_{-j}\), the problem of maximizing the dividend \(\sum_{j'} \delta_{j'} p \cdot f_j(a_j, a_{-j})\) and that of maximizing value with respect to the valuation vector \(\nabla_j = \Phi_j \delta\) are equivalent. Hence with a slight abuse of language, if \(a_j = \arg\max \left\{ \sum_{j'} \delta_{j'} p \cdot f_j(a_j', a_{-j}) \mid a_j' \in \mathcal{A}_j \right\}\), then it will be said that \(a_j\) is optimal for the portfolio \(\delta\), or alternatively that portfolio \(\delta\) supports actions \(a_j\).

Of course, in case \(\Phi_j\) is not onto, there might exist, for a fixed \(a_j\), a continuum of non-collinear portfolios \(\delta\) for which \(a_j\) is optimal. In case \(\Phi_j\) has rank \(J\), \(\delta\) is unique up to scalar multiplication.

This observation is useful since, instead of looking at the \(J\) distributions of individual valuation vectors \((\nabla_{ij})_{i \in \mathcal{I}}\) for \(1 \leq j \leq J\), one can just look at the distribution of portfolio \((\delta_i)_{i \in \mathcal{I}}\). It allows us to study stability of actions in all firms at once.

Consider a reciprocal aggregation equilibrium \((p^*, \chi^*, a^*)\). For firm \(j\), there exits a portfolio \(\theta_j^*\) supporting \(a_j^*\), i.e., such that \(\nabla_j^* = \Phi_j \theta_j^*\). This portfolio comes out of the aggregation of shareholders’ preferences. It represents the way through which firm \(j\)’s board proposes to aggregate externalities. There is one such supporting portfolio in all boards.

Applying the supporting portfolio \(\theta_j^*\) to firm \(j\) gives rise to a valuation vector \(\nabla_{jj'}^* = \Phi_j \theta_j^*\), interpreted by individual shareholders of firm \(j\) as conveying the compelling expertise of board
for assets, and actions
Hence of internal consistency of choice at the individual level, these portfolios must be the same for all portfolios of shares
initial portfolio.
The latter, which underlies the equilibrium allocation
is the portfolio through which shareholder
rebalanced.
Proof: Given that \( a_j \) (* superscripts are abandoned in the proof for the sake of lightness of notation) is stable with respect to the unanimity (of shareholders) principle, Proposition 1 entails that \( \forall j, \exists \lambda_j \in \mathbb{R}^{\left| \mathcal{I}_j \right|} \) such that \( \nabla_j = \sum_{i\in\mathcal{I}_j} \lambda_{ji} \nabla_{ij} \).
But \( \sum_{i\in\mathcal{I}_j} \lambda_{ji} \nabla_{ij} = \sum_{i\in\mathcal{I}_j} \lambda_{ji} \Phi_j \theta_i = \Phi_j \sum_{i\in\mathcal{I}_j} \lambda_{ji} \theta_i \). Therefore \( \theta_j = \sum_{i\in\mathcal{I}_j} \lambda_{ji} \theta_i \) supports actions \( a_j \). This yields (i) with the ad hoc rescaling of \( \lambda_j \).
Next, given the unanimity (of boards) principle, Proposition 2 entails: \( \forall i, \forall j \in \mathcal{J}_i, \exists \xi_{ij} \in \mathbb{R}^{\left| \mathcal{I}_i \right|} \) such that \( \nabla_{ij} = \sum_{j'\in\mathcal{J}_i} \xi_{ijj'} \nabla_{j'j} \).
But \( \sum_{j'\in\mathcal{J}_i} \xi_{ijj'} \nabla_{j'j} = \sum_{j'\in\mathcal{J}_i} \xi_{ijj'} \Phi_{j'} \theta_{j'} = \Phi_{j'} \sum_{j'\in\mathcal{J}_i} \xi_{ijj'} \theta_{j'} \). Therefore \( \theta_{ij} = \sum_{j'\in\mathcal{J}_i} \xi_{ijj'} \theta_{j'} \) is the portfolio through which shareholder \( i \) assesses the value of actions in firm \( j \). For reasons of internal consistency of choice at the individual level, these portfolios must be the same for all \( j \). Hence \( \xi_{ijj'} = \xi_{ijj} \) for all \( j \). This yields (ii) with the ad hoc rescaling of \( \xi_{ij} \).

The aggregation of collective preferences at the individual level give rise to an ‘equilibrium’ portfolio. The latter, which underlies the equilibrium allocation \( x_j^* \), is typically different from the initial portfolio \( \delta_i \). Hence consistency of individual choice necessitates that portfolio be actually rebalanced.

### 6.2 Rebalancing portfolios for internal consistency of choice

Assume there are markets to trade shares. Given a price vector \( p \) for commodities, a price vector \( q \) for assets, and actions \( a \) in firms, a consumer, with endowments \( \omega_i \in \mathbb{R}^J \) and initial (prior) portfolios of shares \( \delta_i \in \mathbb{R}^J \), chooses optimal consumption in the following budget set, which builds on (9):

\[
B_i(p, q, a, \omega_i, \delta_i) = \left\{ x_i \mid p \cdot x_i = p \cdot \omega_i + q \cdot (\delta_i - \theta_i) + \sum_j \theta_{ij} p \cdot f_j(a) \right\}.
\]  

(12)

It is unavoidable that the (posterior) portfolio \( \theta_i \) be chosen to maximize the wealth \( p \cdot \omega_i + q \cdot \delta_i + \sum_j \theta_{ij} (p \cdot f_j(a) - q_j) \). But then obviously no-arbitrage conditions impose

\[
q_j = p \cdot f_j(a),
\]

(13)

so that wealth is independent of \( \theta_i \), and the budget set (12) boils down to the budget set (9). In absence of aggregation at the individual level, the markets for shares are obviously redundant. We assume here that consumers trade assets, at the no-arbitrage prices, for the sake of internal
consistency of choice. The new portfolio $\theta_i$ is chosen to support the individual valuation vectors $\nabla_{ij}$'s of the consumer, and the allocation $x_i$: consumer $i$ chooses $\theta_i$ such that $\nabla_{ij} = \Phi_j \theta_i$. And at equilibrium, markets clear: $\sum_i \theta_i = 1$.

6.3 Competitive price perceptions and internal consistency

With budget set (12) consumer $i$ behaves competitively with respect to both prices $p$ and $q$. This corresponds to standard individual rationality for the asset prices $q$ if $i$ is not a dominant shareholder: he perceives to be too small to influence actions in firms, hence, given equations (13), he perceives to be too small to influence asset prices $q$.

On the one hand, given these competitive price perceptions, individual rationality commands that, on the basis budget set (12), consumer $i$ wants actions in firms to be maximized with respect to his posterior portfolio $\theta_i$, once the latter is fixed.

On the other hand, these competitive price perceptions exclude that the posterior portfolio $\theta_i$ be chosen to maximize wealth and thus utility, because wealth is independent of the latter. But we insist that optimization is not the only process of rationality that occurs at the individual level. There are other principles at play. In absence of transaction costs, it is natural to assume that rebalancing occurs also for internal consistency of choice.

The latter assumption is an inner, personal equilibrium condition. It certainly carries an ethical dimension with respect to oneself.\textsuperscript{23}

Of course, a question immediately arises: that of the hierarchy of rationale, when they are conflicting. In the present case, there is no conflict given the competitive price perceptions. Hence the results would survive the assumption that rationale occur in lexicographic order: individual agents first maximize wealth (or utility); and in case of neutral choices with respect to wealth, they search to be internally consistent.

6.4 Existence and optimality of reciprocal aggregation equilibrium

Existence is trivial, as stated in the following observation. We start from the notion of market equilibrium of Definition 4, and endogenize production plans in such a way that all firms maximize joint profits, giving rise to a standard Walrasian equilibrium with perfect internalization of externalities. We prove that a reciprocal aggregation equilibrium can be associated with this Walrasian equilibrium.

\textbf{Observation 6} For any Walrasian equilibrium $(p^*, x^*, a^*)$ with perfectly efficient internalization of externalities (all firms maximize the joint profit), there exists $\theta^* = (\theta^*_i)_{i \in I}$ such that $(q^*, \chi^*, a^*)$ is a reciprocal aggregation equilibrium, with $\chi^* = (x^*, \theta^*)$. Existence of Walrasian equilibrium follows from standard argument.

\textsuperscript{23}See the discussion in Section 8.5, in particular Bovens (1992)'s characterization of ‘character planning’.
Proof: Fix \( \theta_j = 1_j \) for all \( j \), \( q_j = p \cdot f_j(a) \) and choose

\[
a_j = \arg \max \left\{ \sum_{j'} p \cdot f_j'(a'_j, a_{-j}) \mid a'_j \in A_j \right\},
\]

i.e. all firms maximize the joint profit. Existence of a market equilibrium (for fixed actions) \((p^*, x^*)\) follows from the standard Walrasian model (see, e.g., Balasko, 2011).

Fix \( \theta_i = t_i 1_j \) for all \( i \), with \( \sum_i t_i = 1 \) and \( t_i > 0 \). Then in every firm \( j \), all consumers have collinear valuation vectors: \( \nabla_{ij} = t_i \Phi_j 1_j \), and moreover they all agree on the choice of \( a_j \), so that \( a^* \) is stable with respect to the strong unanimity (of shareholders) principle.

Reciprocally, since all boards optimize production according to the market portfolio, and so do the shareholders, the market equilibrium is stable with respect to the strong unanimity (of boards) principle. \( \square \)

But the main finding is that all reciprocal aggregation equilibria are efficient.

**Theorem 2** For an open and dense set of endowment vectors \((\omega_1, \ldots, \omega_I)\) and portfolios of shares \((\delta_1, \ldots, \delta_I)\) with full measure, all reciprocal aggregation equilibria are Pareto optimal.

Proof: Define \( \Xi \) the \( I \times J \) matrix with entries \((\xi_{ij})\), with \( \xi_{ij} = 0 \) if \( j \notin J_i \); \( \Lambda \) the \( J \times I \) matrix with entries \((\lambda_{ji})\), with \( \lambda_{ji} = 0 \) if \( i \notin I_j \); \( \Theta_J \) the \( J \times J \) square matrix whose \( j \)-th row is \( \bar{\theta}_j^T \); \( \Theta_I \) the \( I \times I \) matrix whose \( i \)-th row is \( \bar{\theta}_i^T \). Of course, \( \Lambda \) and \( \Xi \) are stochastic matrices.

Then (i) and (ii) in Proposition 5 read in matricial form:

\[
\Theta_J = \Lambda \Theta_I \quad \text{and} \quad \Theta_I = \Xi \Theta_J.
\]

Moreover, for an open and dense set of endowment vectors \((\omega_1, \ldots, \omega_I)\) and portfolios of shares \((\delta_1, \ldots, \delta_I)\) with full measure, the stochastic product matrices \( \Lambda \Xi \) and \( \Xi \Lambda \) are strongly connected. \( \square \)

We turn now to the case of incomplete markets and show under which condition reciprocal aggregation can restore Pareto optimality.

### 7 When reciprocal aggregation leads to efficient allocation of risks in incomplete markets

Consider the special case where the utility functions are state-by-state additively separable, with (possibly state-dependent) Bernoulli utility functions \( u_s^\pi, \ s \in \{1, \ldots, S\} \):

\[
u_i(x_i) = \sum_{s=0}^{S} \pi_i^s u_i^s(x_i^s),
\]

where, with some abuse of language to which we come back in Section 8, the vector \( \pi_i \in \mathbb{k}^{S+1} \) will be dubbed *beliefs* in the sequel. The parameter \( \pi_i^0 \) represents preference for present consumption...
and is more a question of taste than belief. It is easy to escape the abuse of language by considering a slight adaptation of the present model where there is no consumption at date zero - the analysis unfolds in exactly the same way.

Consumer i is Bayesian and characterized by (prior) vector of subjective probabilities, \( \pi_{i1} = \left( \frac{\pi_s^i}{1-\pi_s^i} \right)_{s \in \{1, \ldots, S\}} \), over the states of nature at date 1.

Then individual i’s valuation vector (11) is collinear to:

\[
\Delta \mathbf{\varphi}(x_i) = \begin{pmatrix}
\pi_0^i u_0^i(x_0^i), & \cdots, & \pi_S^i u_S^i(x_S^i)
\end{pmatrix}
\]

7.1 Competitive price perceptions and aggregation of beliefs

At a market equilibrium, the vector \( \mathbf{\pi}(x_i) = \begin{pmatrix}
\pi_0^i u_0^i(x_0^i), & \cdots, & \pi_S^i u_S^i(x_S^i)
\end{pmatrix} \in \mathbb{R}^{S+1} \) can be interpreted as the vector of shadow prices of individual i. It measures the value of one unit of the good, in each state of nature, independently of its probability, given as per marginal utility.

Define the \( \circ \)-product on \( \mathbb{R}^{S+1} \) as:

\[
\pi_i \circ p_i(x_i) = \begin{pmatrix}
\pi_0^i u_0^i(x_0^i), & \cdots, & \pi_S^i u_S^i(x_S^i)
\end{pmatrix}.
\]

For all \((i, s)\), the \( S + 1 \) vectors \( e_i^s (x_i) = (0, \cdots, 0, u_s^i(x_s^i), 0, \cdots, 0) \) form a basis of \( \mathbb{R}^{S+1} \) since \( u_s^i(x_s^i) \neq 0 \). In this basis, \( Du_i(x_i) \) has coordinates \( \pi_i \).

Obviously, there exists a unique \( \pi_{ij} \in \mathbb{k}^{S+1} \) such that, up to normalization, \( \nabla_j \) has coordinates \( \pi_{ij} \) in agent i’s basis:

\[
\nabla_j \text{ is collinear to } \pi_{ij} \circ p_i(x_i).
\]

In the spirit of the competitive analysis, we assume that an individual agent does not have access to any information about the marginal utilities of other agents. Moreover collectives are not endowed with utility. So given a valuation vector \( \nabla_j \) in firm \( j \in J_i \), the individual agent i uses his own vector of shadow prices to assess the beliefs associated to \( \nabla_j \). Therefore, the coordinates \( \pi_{ij} \) can be interpreted as the beliefs of firm j according to individual i’s shadow price vector. This is an extension, and a consequence, of the concept of competitive price perceptions of Grossman & Hart (1979).

Assumption (CPP) - Competitive Price Perceptions: Consumer i associates belief \( \pi(i) \) to a valuation vector \( \nabla \) according to his shadow price vector \( p_i \), i.e., \( \pi(i) \) is such that \( \nabla \) is collinear to \( \pi(i) \circ p_i(x_i) \).

As a corollary, the aggregation taking place at the individual level can be interpreted as an aggregation of beliefs.

Proposition 6 At a reciprocal aggregation equilibrium, the strong unanimity (of boards) principle holds if and only if

\[
\forall i, \exists \xi_i \in \mathbb{k}^{J_i} \text{ such that } \pi_i = \sum_{j \in J_i} \xi_{ij} \pi_{ij}.
\]
Proof: Immediate from Proposition 2 and Equations (14). □

The aggregation, at the individual level, of the boards’ beliefs resembles the averaging of utilities in social choice theory, and Assumption (SUP) is reminiscent of a unanimity condition à la Harsanyi (1955).24

Proposition 6 makes precise what it may take to assume aggregation at the individual level. In the present setup, it assumes two things: (i) that prior beliefs \( \pi_{i1} \) be updated according to Assumption (SUP); (ii) that prior preference for present consumption \( \pi_{i0} \) be updated according to Assumption (SUP). Assumption (ii) is clearly the more problematic one, as it deals with updating tastes more than mere beliefs. This point is discussed in Section 8. Assumption (i) is much less deviant with respect to the standard rational choice theory, since it merely asserts that agents update prior subjective beliefs according to some (weighted) expertise that they recognize to various experts, here boards of firms. It happens to be that the entailed updating of beliefs parallels that suggested in the DeGroot (1974) model.

7.2 Existence and optimality of reciprocal aggregation equilibrium

Consider the notion of welfare weighted equilibrium, a special case of the notion of Pareto partnership equilibrium (see Definition 31.1 in Magill & Quinzii, 1996). It is derived from the notion of stock market equilibrium of Definition 6, where the asset structure is endogenized in such a way that production plans are all optimal with respect to the same valuation vector, which is moreover taken to average the shareholders equilibrium valuation vectors.

**Definition 8** Fix the welfare weights \( \alpha = (\alpha_i)_{i \in I} \in \mathbb{R}^I \). The welfare weighted equilibrium \( (q^*, x^*, \theta^*, a^*) \) associated with \( \alpha \) is such that

(i) and (ii) given \( a^* \), \( (q^*, x^*, \theta^*) \) is a market equilibrium for fixed actions \( a^* \);

(iii) given \( (q^*, x^*, \theta^*) \), for all \( j \in J \), \( a_j^* = \arg \max \{ \bar{\nabla}^* \cdot a_j \mid a_j \in A_j \} \), with \( \bar{\nabla}^* = \sum_{i \in I} \alpha_i \pi_i \circ p_i(x_i^*) \).

A reciprocal aggregation equilibrium can easily be constructed from any welfare weighted equilibrium.

**Proposition 7** For any welfare weighted equilibrium \( (q^*, x^*, \theta^*, a^*) \), there exists equilibrium beliefs \( \pi^* = (\pi_{i1}^*)_{i \in I} \) such that \( (q^*, x^*, \theta^*) \) is a reciprocal aggregation equilibrium, with \( \chi^* = (\pi^*, x^*, \theta^*) \).

**Proof:** Only Assertions (i), (iii) and (iv) in Definition 7 have to be checked. (The market clearing equation in Assertion (ii) is the same in Definition 7 and 8.)

At a welfare weighted equilibrium, we know from Proposition 3 that for all \( i \), \( \pi_i \circ p_i(x_i^*) \in \langle \hat{A}^* \rangle \). Hence \( \bar{\nabla}^* \in \langle \hat{A}^* \rangle \).

24Such derivations are restricted to the case in which all experts share the same utility function, and they require some richness conditions (see Hylland and Zeckhauser (1979) and Mongin (1995) for impossibility results with simultaneous aggregation of utilities and probabilities). This condition is insured here by the assumption of competitive price perceptions.
For all $i$, define the equilibrium beliefs $\pi^*_i$ such that $\pi^*_i \circ p_i(x^*_i)$ is collinear to $\bar{\nabla}^*$. Then for all $i$, $\pi^*_i \circ p_i(x^*_i) \in \langle \hat{A}^* \rangle^{\perp}$, therefore $(q^*, x^*, \theta^*)$ is also a market equilibrium for (the fixed actions $a^*$ and) utility functions with equilibrium beliefs $\pi^*_i$. Hence Assertion $(i)$ holds.

Trivially Assertion $(iv)$ holds true: $\exists a_j \in A_j$ such that $\pi^*_i \circ p_i(x^*_i) \cdot a_j \geq \pi^*_i \circ p_i(x^*_i) \cdot a^*_j$ (with at least one $>$, for e.g., agent $i$) for all $i \in I_j$. Indeed, since $\bar{\nabla}^*$ is collinear to $\pi^*_i \circ p_i(x^*_i)$, if there were such an $a_j$, then we would have $\bar{\nabla}^* \cdot a_j > \bar{\nabla}^* \cdot a^*_j$, which is excluded.

The same, at $(q^*, x^*, a^*)$, since all aggregated equilibrium gradients $\pi^*_i \circ p_i(x^*_i)$ are collinear to the (common to all firms) valuation vector $\bar{\nabla}^*$, the strong unanimity (of boards) principle holds. □

Existence of welfare weighted equilibrium follows from standard arguments.

**Proposition 8** For every economy $\{(u_i, \omega_i, \delta_i)_{i \in I}, (g_j)_{j \in J}\}$ and welfare weights $\alpha \in \mathbb{R}^I$, there exists a welfare weighted equilibrium associated to $\alpha$. Hence there exists a reciprocal aggregation equilibrium for every economy.

**Proof:** The first assertion follows from Magill & Quinzii (1996). The second is a corollary of Proposition 7. □

The main finding is that all reciprocal aggregation equilibria are efficient.

**Theorem 3** For an open and dense set of endowment vectors $(\omega_1, \ldots, \omega_I)$ and portfolios of shares $(\delta_1, \ldots, \delta_I)$ with full measure, all reciprocal aggregation equilibria are Pareto optimal.

**Proof:** Pareto optimality holds if both productive efficiency and distributive efficiency hold. Given an asset structure, a necessary and sufficient condition for distributive efficiency is the collinearity of all equilibrium individual valuation (present value) vectors (see, e.g., Magill & Quinzii, 1996). This holds for reciprocal aggregation equilibria, with aggregated valuation vectors. Indeed, through the same argument as in Theorem 2, and with the notation following from Proposition 2, one has both

$$\bar{\nabla}_I = \Xi \bar{\nabla}_J \quad \text{and} \quad \bar{\nabla}_J = \Lambda \bar{\nabla}_I$$

Hence: $\bar{\nabla}_J = \Lambda \Xi \bar{\nabla}_J$, and reciprocally, $\bar{\nabla}_I = \Xi \Lambda \bar{\nabla}_I$. For an open and dense set of endowment vectors $(\omega_1, \ldots, \omega_I)$ and portfolios of shares $(\delta_1, \ldots, \delta_I)$ with full measure, the product matrices $\Lambda \Xi$ and $\Xi \Lambda$ are stochastic and strongly connected.

Moreover, since production plans are all optimized through the same valuation vector, collinear to individual valuation vectors, the asset structure is productive efficient. □

**8 Discussion**

We discuss in this section what it takes to assume reciprocal aggregation based on the unanimity principle. More precisely, we focus of the shifts in valuations that is required to support aggregation at the individual level.
In the case of production externalities, shifts in valuations come from trading for the sake of internal consistency of choice. This does not require any shift in preferences, neither through beliefs nor tastes. In the case of incomplete markets, aggregation at the individual level necessitates that not only beliefs (subjective probability vectors \( \pi_{it} \)'s) are up-dated, but also tastes (preference for present consumption \( \pi_{it}^0 \)'s).

Although the so-called 'economic approach' has been described by Becker (1976) as the 'combined assumptions of maximizing behavior, market equilibrium and stable preferences', the standard rational choice theory widely accepts preference changes due to Bayesian information learning. Preferences evolving with experience goods is another instance of well-accepted information-driven preference change. But shifts in tastes is much less commonly accepted.

Before we argue for the interest of considering such shifts, let us frame the broader field of the debate.

### 8.1 How stable are preferences?

Going on reading Becker (1976), the reluctance to admit preference changes is based on two reasons. First, "economists have little to contribute, especially in recent times, to the understanding of how preference get formed." Second, "the assumption of stable preferences [...] prevents the analyst from succumbing to the temptation of simply postulating the required shift in preferences to ‘explain’ all apparent contradiction to his predictions" (p. 5). As he insists "the economic approach does not take refuge in assertions about irrationality [...] or convenient ad hoc shift in values (i.e., preferences)" (p. 7). A more benevolent way to put it is that the assumption of exogenously fixed preferences is parsimonious (Dietrich & List, 2011).

Fortunately the mentality, and the achievements, of the economic profession has considerably evolved since the seventies, often thanks to the inspiration provided by other disciplines, from the social sciences and beyond. Becker is famous for having extended the economic approach to a wide range of objects from, mainly, sociology. By a swing of the pendulum, approaches from sociology, anthropology or even biology have been fruitfully applied to economic objects, and to the study of preferences in particular. The recent extensive use of social networks to better understand preference formation, authoritatively surveyed in Jackson (2008), is an eloquent illustration; this approach borrows the main concepts of network analysis from a long sociological tradition. Another illustration is the literature on the transmission of cultural traits (see, e.g., Bowles, 1998, or Bisin & Verdier, 2001) which borrows tools from evolutionary biology.

As for the second reason put forward by Becker to ban shifts in individual preferences, namely preventing the analyst from succumbing to the temptation of postulating the ‘required’ shifts, it must be observed that the latter are not banned because they lack descriptive power, but because they might be ad hoc. In other words it might well be true that people shift valuations, but it is not ‘fair game’ to resort to this kinds of explanation.

We object to this view for two sets of reasons. Firstly, assuming stable preferences is just not convincing on a descriptive basis. There is a vast evidence from the literature on behavioral economics, and an endless list of pitfalls and bias that contradict the latter assumption. Secondly, it totally discards the normative interest of economic modelling. What if it were ‘rational’ for
individual to shift valuations? What if, on top of being rational, it would enhance social welfare? In that sense, the model does not aim at making predictions, and the theorist is less interested in explaining why reality contradicts his model than in figuring out what it would take for the economy to do better.

We briefly review these two sets of reasons in the sequel.

8.2 Do individuals shift preferences?

The assumption of stability of individual preferences, although a parsimonious one, in first approach, to describe and explain economic phenomena, seems to be contradicted by lots of evidence from the literature on behavioral economics. A first instance, even though it is disputed (Plott & Zeiler, 2005), is the pervasiveness of the ‘endowment effect’ (see Kahneman et al., 1991). The fact that one gives higher value to some goods at the very point where one starts owning them is a clear shift in preferences; it enhances individual welfare by mechanically creating value; and therefore it enhances social welfare from a utilitarian perspective.

And by the way, it raises the issue of another duality, that of preferences and goods: what is the primal set, and what is the dual? The field of industrial organization has for long identified the phenomenon of ‘experience goods’; and consumer theory the phenomenon of ‘habit formation’ (axiomatized by Rozen, 2010). For more than forty years, the notion of endogenous change of tastes has been studied (see von Weizsäcker, 1971, Hammond, 1976, Pollack, 1978). In this early literature, change of tastes is triggered by past consumption, but the subsequent literature on ‘reference-dependent preferences’ (Kahneman & Tversky, 1979) has extended possible causes of change to rational expectations about future variables (see, e.g., Köszegi & Rabin, 2006).

There are furthermore attempts to account for preference shifts driven by non-informational factors. Even in Becker (1976)’s eyes, there seems to be some room for that.25 Lancaster (1966a, 1966b) theory of characteristics launched the interest in the field, Dietrich & List (2011) enriched the approach.

Returning back to behavioral puzzles, ‘time inconsistency’ is also a challenge for the stability of individual preferences. Becker himself (1976, 1996) is a huge contributor to that question with his work on acquisition of taste, and changes of preferences over time based on past behavior and experiences. More recent evolutionary studies predict how cardinal properties of hedonic utility adapt to the decision environment (see, e.g., Netzer, 2009).

Another challenge to the doctrine of stable preferences is the apparent tendency of decision maker to choose the ‘default option’. In the context of the present paper, the ‘default option bias’ is an argument in favor of the unanimity principle: If some production policy is bound to be chosen because of unanimous support by firms, then it acquires de facto the status of default option. Last example, once an option is bound to be imposed on us by the force of unanimity

25He writes: "The preferences that are assumed to be stable do not refer to market goods and services [...] but to underlying objects of choice that are produced by each household using market goods and services, their own time, and other inputs. These underlying preferences are defined over fundamental aspects of life, such as health, prestige, sensual pleasure, benevolence, or envy, that do not always bear a stable relation with market goods and services" (p. 5)
of the collective preferences of the groups we choose to stick with, subscribing to this option is the most direct way to avoid ‘cognitive dissonance’ in the future. We come back to that soon.

Assuming stability of individual preferences is not credible when looking at the individual facing social and political interactions. The individual is subject to many influences, acts in many different capacities, is at the node of many social connections, and therefore we strongly believe that some aggregation is bound to occur in oneself on a continuous basis that impact one’s preferences. This belief, shared by many scholars, is at the origin of the prolific economic literature on social, or interdependent, preferences. Many aspects of the latter are reviewed in Benhabib et al. (2011).

More directly relevant to the present model is the theory of deliberation-induced preference change (e.g., Miller, 1992, Knight & Johnson, 1994, Dryzek and List, 2003). For many scholars (following Habermas, 1984) the central concern of politics is to change preferences, rather than aggregate them. And the goal of political interaction is to forge a unanimous, rational consensus rather than optimally compromise between diverging, inflexible opinions or interests.

Our model is compatible with both views. The unanimity of shareholders principle is compatible both with pure voting without deliberation, as well as deliberation converging toward a unique, common valuation vectors, consensual with the shareholders’ initial valuations, i.e. in their convex hull. As for the unanimity of boards principle, it is compatible with both views too: it can be seen as an optimal compromise between multiple selves,26 as well as an intimate deliberation weighing various considerations for a better judgement; both mechanisms occurring in the heart of the considered individual. In any case, we believe that aggregation and deliberation are complementary, and probably difficult to disentangle in the real world. The more entanglement, the richer the notion of reciprocal aggregation.

8.3 Is it irrational to shift one’s preferences?

Why should it be irrational to shift preferences? Of course, rationality being mainly defined in terms of the properties of the preference relation – e.g., completeness and transitivity of the preorder – it looks like an impossible task to rigorously define potential rationality of shifts in preferences. One needs a concept of meta-rationality, or define a meta-preference.

As for defining meta-rationality, Gilboa (2011) proposes a useful and, we believe, convincing approach. Suppose that theoretical analysis proposes an axiom of rationality that is refuted by some decision maker in some particular instance. Once the decision maker is explained how he refuted the axiom, the question is: does he feel embarrassed to the point of making sure to abide by the axiom in the future? If yes, then the early decision can be considered irrational; otherwise it may not. If we are to follow this route, then any shift in preferences enhancing individual welfare could be quite rational. ‘If I cannot get what I like, I’d better like what I get’ sounds indeed like pure wisdom. The same, ex-post rationalization by adjusting some parameters of the preferences, or extending the set of possible preferences and follow a ‘max-max’ approach over preferences and goods are seemingly rational moves along this line.

26 Bismarck said: “Faust complained that he had two souls in his breast. I have a whole squabbling crowd. It goes on as in a republic.” (cited in Steedman & Krause, 1985).
When facing externalities, and in absence of any control on them, e.g., because there are no coasian markets to trade on these externalities, or because they involve too high transaction costs, it makes a lot of sense to decide to adjust one’s preferences over these externalities. When your neighbor’s dog keeps barking, and it deeply annoys you, it might be more rational to work hard stopping to care about it (and become indifferent), rather than move to another home, or rather than engage into a costly litigation to obtain that damages be fixed. Our brain actually helps us getting rid of familiar noises and odors, by just stopping to pick up the information.

We are back here to the duality between preferences and goods. If optimization behavior is a pillar of the economic approach, it makes lots of sense, when markets fail, and no adjustment of the quantity consumed is possible, to open extra spaces of freedom over which to optimize. In a pure cardinalist approach, it is trivial to note that, if the problem of solving: \( \arg \max \{ u(x) \mid x \in B \} \) is void because \( B = \{ \bar{x} \} \) is a singleton, one reaches a higher individual welfare by opening some possibilities of preference shifts: \( u \in U \), and solve: \( \arg \max \{ u(\bar{x}) \mid u \in U \} \) (of course neutralizing plain affine transformations). Adaptive preferences have such a flavor.

The concept of adaptive preferences has a long history (Elster, 1983, Sen, 1985, von Weizsäcker, 2013). It is often used to describe how people living in great deprivation tend to get used to it. Some even end up being happy about their lot. This is why tenants of the capability approach (Sen, 1985, 1999, Nussbaum, 2003, 2004) discard utility as a tool for measuring welfare. Although we appeal to the concept of adaptive preferences, we consider that this criticism does not apply to our setup. First and foremost because of the freedom of choice that characterizes group affiliation: our individuals are autonomous investors. Moreover we are dealing here with goods, services and investment opportunities, not with deep values about life and society. When we contemplate updating beliefs, or revising preference for present consumption, or, why not, changing ratios of marginal utilities, we do not necessarily mean a re-initialisation of the active, let alone cognitive and sensitive, software of the individual. Our objects are meant to be more practical.

We now turn toward the elements of the debate more directly and precisely linked with our model.

### 8.4 Updating beliefs and adapting tastes

There is nothing revolutionary in Equations (15), as far as beliefs \( \pi_{i1} \) are concerned. Given the assumption of competitive price perception, and considering boards as experts, they mean mere Bayesian up-dating of subjective probability. But the adapting of the preference \( \pi_{i0} \) for present consumption is more problematic, as it is more a matter of taste than belief. Nevertheless we can follow Elster (1983) and build a formal parallel between beliefs and tastes.

In his search for a broad theory of individual and collective rationality, Elster appeals to the notion of autonomy to be for desires what judgement is for beliefs:28 "[...] autonomous
desires are desires that have been deliberately chosen, acquired, or modified — either by an act of will or by a process of character planning”. Both for beliefs and desires, a crucial condition for substantive rationality is the absence of distortions and illusions.

We considered that judgements are substantively rational beliefs, not in the sense that they are grounded in available evidence, but in the sense that they purged from idiosyncracies by the operation of SUP (Proposition 1). In that sense, boards are collective ‘persons well known for their judgement’.

29 They have their own rules and regulations, so are typically autonomous, in the etymological sense. To paraphrase Elster, they typically are in control of the processes whereby their tastes are formed, or at least not in the grip of the individual preference-formation processes. Would there remain scoria of idiosyncratic distortions and illusions in the aggregate, these scoria would be washed away in the backward loop toward the individual level — cf. Diagram (1) — by the (second) operation of SUP.

According to Dworkin (1988), p. 20: "Autonomy is conceived of as a second-order capacity of persons to reflect critically upon their first-order preferences, desires, wishes, and so forth and the capacity to accept or attempt to change these in light of higher-order preferences and values." The idea of higher-order preferences is also present in Elster (1983) when he distinguishes ‘sour grapes preferences’, shaped by drives, from ‘character planning preferences’, shaped by meta-preferences. Our individual do not have a meta-preference. But the backward looping individual aggregation mechanism $S$ can be conceived of as a ‘second-order capacity to reflect critically upon one’s first-order preferences’, with potentially a great deal of reflexivity as we saw. And when an investor, or a partner, joins an assembly or a board, there is a will, based on self-interest, to invest in common, which creates a ‘higher-order value’, which we dubbed affectio societatis.

On top of that, behind autonomy, there is necessarily freedom. And here individuals exercise their freedom of choice when joining a board (buying shares in the firm, or partnering). They do it through their own rules (optimization of risk bearing) hence in sheer autonomy. And, by the way, they can discard the underlying $\pi^0_j$ of firm $j$ by just short-selling firm $j$’s stock. If they do not, it is that something is good to take in the beliefs and tastes $\pi_j$ that the investor (competitively) perceived that firm $j$ has. And $\pi_j$ comes as a block, with its inner consistency. You cannot cherrypick among its components.

The question remains to which point adapting one’s taste $\pi^0_j$ is not a token of conformism, or imitation, however autonomous. Beyond recalling the traditional argument of sociology, we

\begin{footnote}{in the grip of processes with which they do not identify themselves\(\) (p. 21).}
\end{footnote}

\begin{footnote}{This idea underlies the Condorcet Jury Theorem, and the ‘wisdom of the crowds’ (Jackson & Golub, 2012).}
\end{footnote}

\begin{footnote}{Colburn (2011) characterizes autonomous preference change by the absence of covert influence. No influence is covert in our concept of reciprocal aggregation.}
\end{footnote}

\begin{footnote}{A quote of Simmel (1955) discards the idea that through the operation of a dual aggregation mechanism $S$ an individual agent is bound to lose its core identity: "It is true that external and internal conflicts arise through the multiplicity of group-affiliations, which threaten the individual with psychological tensions or even a schizophrenic break. But it is also true that multiple group-affiliations can strengthen the individual and reenforce the integration of his personality. Conflicting and integrating tendencies are mutually reinforcing. Conflicting tendencies can arise just because the individual has a core of inner unity. The ego can become more clearly conscious of this unity, the more he is confronted with the task of reconciling within himself a diversity of group-interests. [...] These conflicts may induce the individual to make external adjustments, but also to assert himself}
\end{footnote}
want here to raise four points. First, individual \( i \) does not take, at least directly, into account the preference parameters of other individuals, but that of the groups he affiliates to. It is not imitation of individual peers. And if it is conformism, it is of the sort of ‘conformity to the party line’, and we insist: a party that one has freely chosen to join, not conformity to norms imposed upon the individual by the accident of birth. Second, and to carry the metaphor further, one is affiliated to potentially many different parties, and synthetizes all their respective lines; moreover, and most importantly, SUP entails adapting tastes only if there is a massive alignment in the form of unanimous agreement of all the parties one joins. Third, we should not forget Observation 3 that, up to Assumption (PE), the result obtains with potentially lots of reflexivity in the individual aggregation process, in such a way that it allows an indifference to expand into a preference, but does not suggest a reversal of preferences.\(^{32}\)

Fourth, and this is the main argument, from an individual perspective it is rational to shift tastes. This is toward this rationality that we turn now.

### 8.5 Amor fati preferences

Following Observation 2, with a slight abuse of language, firms can be said to have an induced (linear) preference with respect to which they optimize, given by the linear operator

\[
\succeq_{a_j} : a_j' \rightarrow \nabla_j(a_j) \cdot a_j'
\]

where \( \nabla_j(a_j) \in \mathbb{k}^d \) is collinear to \( Dg_j(a_j) \). This preference is therefore a function of the chosen production plan \( a_j \). It is a simple instance of a reference-dependent preference where the reference point is the current production plan, or, alternatively, the rational expectation about the future production plan. It is remarkable that:

\[
\forall a_j, a_j' \in \partial A_j, a_j \succeq_{a_j} a_j'.
\]

Therefore, logically these preferences are adaptive on the frontier of the production set (see von Weizsäcker, 2013).

**Definition 9** An induced preference \( (\succeq_{\_}) \) is adaptive on a possibility set \( \mathcal{P} \) if

\[
\forall a, b \in \mathcal{P}, a \succeq b \Rightarrow a \succeq a b \text{ and } a \succ b \Rightarrow a \succ a b.
\]

Here adaptiveness comes trivially as a direct logical consequence of the fact that, on the possibility frontier \( \partial A_j \), \( a_j \) is the best possible production plan according to \( (\succeq_{a_j}) \). Paying our tribute to Nietzsche, let us dub amor fati this last property.\(^{33}\) ‘Faire de nécessité vertu’ is compatible with freedom. Paradoxically, adapting one’s preferences is a ‘strategy of liberation’.

\(^{32}\)A condition for autonomy for preferences, according to Elster (1983), p. 131.

\(^{33}\)We do not want to enter too deeply into the subtle distinctions between ‘sour grapes’ and ‘character planning’ preferences, and the grey area (Bovens, 1992) in-between. But amor fati (suggested in Elster, 1983) is clearly meant to be much closer to the latter than to the former.
The rather extreme Stoic doctrine even characterizes freedom in terms of the strength to embrace the ineluctable.\(^34\)

**Definition 10** An induced preference \((\succeq_i)\) is amor fati on a possibility set \(P\) if
\[
\forall a, b \in P, a \succeq a b.
\]

The same, given a fixed global production plan \(a\), at a market equilibrium \((p^*, x^*, a)\) consumer \(i\) is endowed with an induced preference \((\geq_{i,(p^*, x^*, a)})\) on alternative production plans; it is defined by the equilibrium valuation vector \(\nabla_i(p^*, x^*, a)\).\(^35\)

**Proposition 9** Consider a reciprocal aggregation equilibrium \((p^*, \chi^*, a^*)\). When it is Pareto-optimal, the induced preference \((\geq_{i,(p^*, \chi^*, a^*)})\) is amor fati on all production possibility frontiers, hence adaptive.

**Proof:** Pareto optimality implies equality of all valuation vectors: \(\forall j, \nabla_i(p^*, \chi^*, a^*) = \nabla_j(a^*_j)\); therefore, \(\forall a_j \in \partial A_j, a^*_j \geq_{i,(p^*, \chi^*, a^*)} a_j\), which remains true a fortiori when \(a^*_j \geq_{i,(p^*, x^*, a)} a_j\).

Can we go beyond this ordinal consistency, and say anything about the cardinal consistency? Suppose a shareholder perceives that all firms in which he takes shares invest according to a preference parameter \(\pi^0\) in a range way below his own preference \(\pi^0\). According to his tastes, all firms over-invest at date 0. He wishes he could fin dap o l i c yo p t i m i z i a ( p e r c e i v e d ) \(\pi^0\) above his, but he cannot. He is like the fox of the fable.\(^36\) An intentional adjustment of his own \(\pi^0\) to a point where he finds more optimal the way firms transfer resources across dates enhances his affectio societatis and reduces tension, both intrapersonal (dissonance) and interpersonal (deliberation). It so improves his well-being, but not as measured by utility. Could it also improve his (cardinal) utility level?

In the context of production externalities (Section 6), rebalancing portfolios is neutral in terms of wealth, and does not impact the utility function, even though it impacts the reference-dependent preference \((\geq_{i,(p^*, \chi^*, a^*)})\) on production possibility frontiers. Hence amor fati changes of preferences are cardinally neutral.

The context of incomplete markets (Section 7) is richer in that respect. Denote \(u^*_i(x_i) = \sum_{s=0}^S \pi^*_i u^*_i(x^*_i)\) the equilibrium utility function with shifted tastes and beliefs. To be completed.

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\(^34\)Goldman (1972) writes: *"The Stoics, and Spinoza as well, recommended that one forms one’s desires to accord with what can realistically be expected to happen in any case; they regarded freedom as conformity of events with actual desires, or rather, as conformity of desires with events"* (p. 223).

\(^35\)In his theory to sort the irrational ‘sour grapes’ out of the rational ‘character planning’ preferences, Bovens (1992) characterizes the latter as a more involved project in which preferences are adjusted, *"as well as the reasons for the ranking at hand"*. Rebalancing portfolio for internal consistency of choice (in Section 6.2), or updating beliefs and adapting tastes (in Section 7.1), is doing exactly that.

\(^36\)For Elster (1983) ‘sour grapes’ downgrade the consumption plans stemming from unavailable policies, and ‘character planning’ upgrades the consumption plans stemming from available policies. The difference is ambiguous in a general equilibrium perspective because of price normalization.
9 Concluding comments

Great accomplishments are achieved by the market when it works perfectly. Existence and optimality of economic equilibrium are the epitome of such great achievements. And the general equilibrium theory provides profound, sharp and elegant arguments to advocate for the power of the invisible hand.

That is why free trade seems irreplaceable. And in many ways, the invisible hand is invincible. But the market cannot offer what it does not have. And when it offers no vehicles to trade externalities, or cannot provide all desired insurance services... the market only partially succeeds. In the context of the publicly traded firm, the market only partially succeeds at driving the shareholders, and a fortiori the stakeholders, to agree on how to run the firm. We show here that when the general equilibrium is extended to include reciprocal aggregation mechanisms to supplement the market mechanism, and give a hand to the invisible hand, then assuming only the strong unanimity principle ensures that markets never fail.

How demanding is this supplementary mechanism? In the extreme world of perfect markets, the invisible hand drives individuals, and therefore collectives, to unanimous agreement about the relative values of goods and services. Therefore aggregation at both collective and individual levels is immediate, and demands strictly no effort whatsoever. But the more heterogenous the valuations, the more demanding is aggregation in terms of skill, energy, wisdom. This is very intuitive when a collective preference has to be finalized within a group. It is not less intuitive when an individual has to synthesize many heterogenous valuations in his heart of hearts. Hence, we argue that the more market succeeds, the less demanding are the aggregation mechanisms, especially in terms of revision of primitive characteristics at individual levels. We finally confess that our notion of reciprocal aggregation equilibrium loses of its descriptive power in case of severe market failure.

10 References


Appendix

To make the analysis complete, we need to make precise the link between the fact that consumer \( i \) prefers actions \( a'_{ij} \) to \( a_j \) in firm \( j \) and the fact that he better values \( a'_{ij} \) through \( \nabla ij(p) \).

The following proposition deals with the case of externalities in production.

**Proposition 10** Fix \((p, a)\). One has, for \( a'_{ij} \neq a_j \in A_j \) and consumer \( i \):

\[
\sum_{j'} \delta_{ij'} p \cdot f_{ij'}(a'_{ij}, a_{-j}) > \sum_{j'} \delta_{ij'} p \cdot f_{ij'}(a_j, a_{-j}) \Rightarrow \nabla ij(p, a) \cdot a'_{ij} > \nabla ij(p, a) \cdot a_j. \quad (16)
\]

Reciprocally, there exists \( t_{ij} \in [0,1] \) such that

\[
\nabla ij(p, a) \cdot a'_{ij} > \nabla ij(p, a) \cdot a_j \Rightarrow \sum_{j'} \delta_{ij'} p \cdot f_{ij'}(a'_{ij}(t), a_{-j}) > \sum_{j'} \delta_{ij'} p \cdot f_{ij'}(a_j, a_{-j}), \text{ } \forall \text{ } t \in [0, t_{ij}]
\]

where \( a'_{ij}(t) = (1 - t)a_j + ta'_{ij} \).

**Proof:** Define the real function \( H(t) = \sum_{j'} \delta_{ij'} p \cdot f_{ij'}(a'_{ij}(t), a_{-j}) \). Thanks to Assumption (A.4) - the concavity of \( f_j - H \) is obviously concave since prices are positive.
Now, the left hand side of Assertion (16) reads: \( H(1) > H(0) \); and, applying Equation (??), the right hand side reads that its derivative at 0 is strictly positive: \( H'(0) > 0 \).

Obviously, by concavity of \( H \), \( H(1) > H(0) \Rightarrow H'(0) > 0 \). Reciprocally, if \( H'(0) > 0 \), then by continuity there exists \( t_{ij} \in ]0, 1[ \) such that \( H(t) > H(0) \) for all \( t \in ]0, t_{ij}[ \) which proves the second assertion of the proposition. \( \Box \)

The following proposition deals with the case of incomplete financial markets. Let us make precise the link between the fact that consumer \( i \) prefers actions \( a_j' \) to \( a_j \) in firm \( j \) and the fact that he better values \( a_j' \) through \( \nabla_{ij} \).

**Proposition 11** Fix \( (x, a) \). One has, for \( a_j' \neq a_j \in A_j \) and consumer \( i \) such that \( \theta_{ij} \neq 0 \):

\[
  u_i(x_i + \theta_{ij}(a_j' - a_j)) \geq u_i(x_i) \implies \nabla_{ij}(x_i, a) \cdot a_j' > \nabla_{ij}(x_i, a) \cdot a_j.
\]

Reciprocally, there exists \( t_{ij} \in ]0, 1[ \) such that

\[
  \nabla_{ij}(x_i, a) \cdot a_j' > \nabla_{ij}(x_i, a) \cdot a_j \implies u_i(x_i + t\theta_{ij}(a_j' - a_j)) > u_i(x_i) \text{ for all } t \in ]0, t_{ij}[.
\]

**Proof:** Immediate consequence of the strict quasi-concavity of \( u_i \). \( \Box \)