A Theory of Liquidity and Risk Management
Based on the Inalienability of Risky Human Capital*

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November 28, 2014

Abstract

We analyze a dynamic optimal financial contracting problem in continuous time with risky cash flows between a risk-averse entrepreneur and risk-neutral investors. Two fundamental constraints on the contracting parties are that: 1) the entrepreneur cannot alienate his human capital, and 2) investors have limited liability protection. Given that human capital is risky, the entrepreneur’s inability to commit his human capital to the firm generates significant distortions for corporate investment and consumption risk-sharing. We show that the optimal contracting problem boils down to a corporate liquidity and risk-management problem (implementable via futures and insurance contracts) for the firm. Our analysis thus provides new foundations for liquidity and risk management policies that firms routinely pursue in practice.

*We thank Hengjie Ai, Marco Bassetto, Michael Brennan, Wei Cui, Peter DeMarzo, Darrell Duffie, Lars Hansen, Arvind Krishnamurthy, Guy Laroque, Jianjun Miao, Tom Sargent, Suresh Sundaresan, René Stulz, Jeff Zwiebel, and seminar participants at AFR Summer Institute, Caltech, Cheung Kong Graduate School of Business (CKGSB), Columbia Finance Free Lunch, University of Hong Kong, National University of Singapore (NUS), Ohio State University (OSU), Sargent SRG Group, Summer Institute of Finance Conference (2014), Shanghai University of Finance & Economics (SUFE), Stanford Business School, University of British Columbia, University College London, and University of Oxford for helpful comments. First draft: 2012.

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1 Introduction

Neither an entrepreneur in need of funding, nor anyone else for that matter, can legally agree to enslave himself to a firm in exchange for financing by outside investors. This simple observation has led Hart and Moore (1994) to formulate a theory of corporate external financial constraints stemming from the inalienability of human capital. In a stylized model of a firm with a single fixed project producing deterministic cash flows over a finite time interval, in which both entrepreneur and investors are risk neutral and the entrepreneur’s human capital is certain, they show that there is a continuum of optimal debt contracts involving more or less rapid debt repayment paths. They also show that there is a unique optimal debt contract when the entrepreneur and investors have different discount rates. They argue that their framework provides a new foundation for a theory of corporate debt as well as a theory of debt maturity, and that their model “does not have room for equity per se” [pp 865].

In this paper we generalize the framework in Hart and Moore (1994) along several important dimensions: first, we introduce risky human capital and cash flows; second, we let the entrepreneur be risk averse; third, we consider an infinitely-lived firm with ongoing investment and consumption; and, fourth we add a limited liability or commitment constraint for investors. In this significantly more realistic and intricate but still tractable framework we derive the entrepreneur’s optimal financing, investment and consumption policy, and show how the firm’s optimal financial contract can be implemented using replicating portfolios of standard liquidity and risk management instruments such as cash, credit line, futures, and insurance contracts.

A first obvious reason for considering this more involved framework is to explore how the Hart and Moore theory of debt based on the inalienability of human capital generalizes and how the introduction of risky human capital modifies the theory. But, more importantly, our framework reveals that Hart and Moore’s focus on the notion of a firm’s “debt capacity” is misleading. As it turns out, this is not the most relevant metric for the firm’s optimal financial policy when human capital is risky. Rather, we show that the two key instruments summarizing the firm’s financial policy are the firm’s liquidity buffer or “financial slack” and the firm’s hedging position or “risk management”. That is, inalienability of risky human
capital is not a foundation for a theory of debt capacity, but rather a foundation for a theory of corporate liquidity and risk management. There are hints of the relevance of corporate liquidity in Hart and Moore's discussion of their theory\(^1\), however they do not emphasize the importance of this variable. Also, as a result of the absence of any risk in their framework they overlook the importance of the firm's hedging policy.

For convenience we introduce risk in the form of shocks to the capital stock, which affect the profitability of investment. Most importantly, these shocks induce risky inalienable human capital and introduce a stochastic dynamic participation constraint for the entrepreneur. That is, whether the entrepreneur is willing to stay with the firm now depends on the history of realized capital shocks. When there is a positive shock, the entrepreneur's human capital is higher and she must receive a greater promised compensation to be induced to stay. But the entrepreneur is averse to risk and has a preference for smooth consumption. These two opposing forces give rise to a novel dynamic optimal contracting problem between the infinitely-lived risk-averse entrepreneur and the fully diversified (or risk-neutral) investors.

A key step in our analysis is to show that the optimal long-term contracting problem between investors and the entrepreneur can be reduced to a recursive formulation with a single key state variable \(w\), the entrepreneur's promised certainty equivalent wealth \(W\) under the optimal contract scaled by the firm’s capital stock \(K\). The optimal recursive contract then specifies three state-contingent policy functions: i) the entrepreneur’s consumption-capital ratio \(c(w)\); ii) the firm’s investment-capital ratio \(i(w)\), and; iii) the firm’s risk exposure \(x(w)\) or hedging policy. This contract maximizes investors’ payoff while providing insurance to the entrepreneur and retaining her. The optimal contract thus involves a particular form of the well-known tradeoff between risk sharing and incentives in a model of capital accumulation and limited commitment. Here the entrepreneur’s inalienability of human capital constraint at each point in time is in effect her incentive constraint. She needs to be incentivized to stay rather than deploy her human capital elsewhere.

If the entrepreneur were able to alienate her human capital, the optimal contract would simply provide her with a constant flow of consumption and shield her from any risk. Under

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\(^1\)For example, on pages 864–865 they wrote: “There is some evidence that firms borrow more than they strictly need to cover the cost of their investment projects, in order to provide themselves with a “financial cushion.” This fits in with our prediction in Proposition 2 about the nature of the slowest equilibrium repayment path; indeed, it is true of most paths.”
this contract the firm’s investment policy reduces to the standard ones prescribed by the $q$-theoretic models under the Modigliani-Miller (MM) assumption of perfect capital markets. But with inalienable human capital the entrepreneur must be prevented from leaving. To retain the entrepreneur in the states of the world where the entrepreneur may find her outside option to be greater than her promised certainty equivalent wealth $W$, the optimal contract must promise her sufficiently high $w$ thus exposing her to risk.

Following the characterization of the optimal dynamic corporate policy $(c(w), i(w), x(w))$ we proceed with the implementation of this policy in terms of familiar standard dynamic financing securities. In particular, we show that the optimal contract can be implemented by delegating control over the firm to the entrepreneur in exchange for a credit line with stochastic liquidity $S$ and an endogenously determined stochastic limit $\mathcal{S}$. The effective key state variable for this implementation problem is $s = S/K$, the ratio between $S$ and the firm’s contemporaneous capital stock $K$. The entrepreneur then maximizes her life-time utility by optimally choosing her consumption-capital ratio $c(s)$, investment-capital ratio $i(s)$, and hedge-capital ratio $\phi(s)$ as a function of her savings $s$. In other words, the optimal contract under risky inalienable human capital can be implemented via a credit line combined with optimal cash management and dynamic hedging policies.

The optimal contract provides the entrepreneur with a (locally) deterministic consumption stream as long as the capital stock does not grow too large. When the capital stock increases as a result of investment or positive shocks to the point where the entrepreneur’s inalienability of human capital constraint may be violated the contract provides a higher consumption stream to the entrepreneur. As long as investors can perfectly commit to an optimal stochastic credit-line limit $\mathcal{S}$ (what we refer to as the one-sided commitment problem), the entrepreneur’s consumption and wealth are positively correlated with the capital stock under the optimal contract, and the firm will generally underinvest relative to the first-best MM benchmark of fully alienable human capital.

In the two-sided commitment problem, where a limited liability constraint for investors must also hold, we obtain further striking results. The firm may now overinvest and the entrepreneur may overconsume (compared with the first-best benchmark). The intuition is as follows. In order to make sure that investors do not have incentives to default on their promised future utility for the entrepreneur, the entrepreneur’s scaled promised wealth $w$
cannot be too high otherwise the investors will end up with negative valuations for the firm. As a result, the entrepreneur needs to substantially increase investment and consumption in order to satisfy the investors’ limited-liability participation constraint.

**Related literature.** Our paper provides foundations for a dynamic theory of liquidity and risk management based on risky inalienable human capital. As such it is obviously related to the early important contributions on corporate risk management by Stulz (1984), Smith and Stulz (1985) and Froot, Scharfstein, and Stein (1993). Unlike our setup, they consider a static framework with exogenously given financial frictions to show how corporate cash and risk management can create value by relaxing these financial constraints.

Our paper is also evidently related to the corporate security design literature, which seeks to provide foundations for the existence of corporate financial constraints, and for the optimal external financing by corporations through debt or credit lines. This literature can be divided into three separate strands. The first approach provides foundations for external debt financing in a static optimal contracting framework with either asymmetric information and costly monitoring (Townsend, 1979, and Gale and Hellwig, 1985) or moral hazard (Innes, 1990, and Holmstrom and Tirole, 1997).

The second more dynamic optimal contracting formulation derives external debt and credit lines as optimal financial contracts in environments where not all cash flows generated by the firm are observable or verifiable.²

The third approach which is closely related to the second provides foundations for debt financing based on the inalienability of human capital (Hart and Moore, 1994, 1998). Harris and Holmstrom (1982) is an early important paper that generates non-decreasing consumption profile in a model where workers are unable to commit to long-term contracts. Berk, Stanton, and Zechner (2010) incorporate capital structure and human capital bankruptcy costs into Harris and Holmstrom (1982). Rampini and Viswanathan (2010, 2013) develop a model of corporate risk management building on similar contracting frictions. A key result in their model is that hedging may not be an optimal policy for firms with limited capital

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that they can pledge as collateral. For such firms hedging demand, in effect, competes for limited collateral with investment demand. They show that for growth firms the return on investment may be so high that it crowds out hedging demand. Li, Whited, and Wu (2014) structurally estimate optimal contracting problems with limited commitment along the line of Rampini and Viswanathan (2013) providing empirical evidence in support of these class of models.

The latter two approaches are often grouped together because they yield closely related results and the formal frameworks are almost indistinguishable under the assumption of risk-neutral preferences for the entrepreneur and investors. However, as our analysis with risk-averse preferences for the entrepreneur makes clear, the two frameworks are different, with the models based on non-contractible cash flows imposing dynamic incentive constraints that restrict the set of incentive compatible financial contracts, while the models based on inalienable human capital only impose (dynamic) participation constraints for the entrepreneur. With the exception of Gale and Hellwig (1985) the corporate security design literature makes the simplifying assumption that the contracting parties are risk neutral. By allowing for risk-averse entrepreneurs, we not only generalize the results of this literature on the optimality of debt and credit lines, but we are able to account for the fundamental role of corporate savings and risk management (via futures, options or other commonly used derivatives), and also to provide micro-foundations for executive compensation contracts.

A closely related paper to ours is by Ai and Li (2013) that analyzes a similar contracting framework to study corporate investment and managerial compensation but with very different economic motivation and focus. We show that liquidity and risk management optimally implements the optimal contracting solution and also we characterize the dynamics of optimal corporate liquidity and risk management. Additionally, we incorporate stochastic productivity shocks and focus on the implications due to the inalienability of risky human capital.

Our paper also contributes to the macroeconomics literature that studies the implications of dynamic agency on firms’ investment and financing decisions. Albuquerque and Hopenhayn (2004), Quadrini (2004) and Clementi and Hopenhayn (2006) study firms’ financing and investment decisions under a limited commitment or inalienability of human capital assumption similar to ours. Lorenzoni and Walentin (2007) study Tobin’s Q and
investment under a similar limited commitment assumption. Finally, Grochulski and Zhang (2011) consider a risk sharing problem under limited commitment.\footnote{Green (1987), Thomas and Worrall (1990), Marcet and Marimon (1992), Kehoe and Levine (1993) and Kocherlakota (1996) are important early contributions on optimal contracting under limited commitment. Miao and Zhang (2013) develop a duality-based solution method for limited commitment problems. See Ljungqvist and Sargent (2004) Part V for a textbook treatment on these class of models widely used in macro.}

Our financial implementation of the optimal financial contract is also related to the portfolio choice literature featuring illiquid productive assets and under-diversified investors in an incomplete-markets setting. Building on Merton’s intertemporal portfolio choice framework, Wang, Wang, and Yang (2012) study a risk-averse entrepreneur’s optimal consumption-savings decision, portfolio choice, and capital accumulation when facing uninsurable idiosyncratic capital and productivity risks. Unlike Wang, Wang, and Yang (2012), our model features optimal liquidity and risk management policies that arise endogenously from an underlying financial contracting problem.

Our framework also provides a micro-foundation for the dynamic corporate savings models that take external financing costs as exogenously given. Hennessy and Whited (2005, 2007), Riddick and Whited (2009), and Eisfeldt and Muir (2014) study corporate investment and savings in a model with financial constraints. Bolton, Chen, and Wang (2011, 2013) study the optimal investment, asset sales, corporate savings, and risk management policies for a firm that faces external financing costs. It is remarkable that although these models are substantially simpler and more stylized the general results on the importance of corporate liquidity and risk management are broadly similar to those derived in our paper based on more primitive assumptions. Conceptually, our paper shows that to determine the dynamics of optimal corporate investment, a critical variable in addition to the marginal value of capital (marginal $q$) is the firm’s marginal value of liquidity. Indeed, we establish that optimal investment is determined by the ratio of marginal $q$ and the marginal value of liquidity, which reflects the tightness of external financing constraints.\footnote{Pinkowitz and Williamson (2004), Faulkender and Wang (2006), Pinkowitz, Stulz, and Williamson (2006), Dittmar and Mahrt-Smith (2007), and Bolton, Schaller, and Wang (2014) empirically measure the marginal value of cash.} Our model thus shares a similar focus on the marginal value of liquidity as Bolton, Chen, and Wang (2011, 2013) and Wang, Wang, and Yang (2012).
2 The model

We consider an optimal long-term contracting problem with limited commitment to participate between an infinitely-lived risk-neutral investor (the principal) and a financially constrained, infinitely-lived, risk-averse entrepreneur (the agent). The entrepreneur requires funding from the investor to finance a proprietary business idea for a growth venture that we represent as a production function and a capital accumulation process. We begin by describing the production technology and the preferences of the entrepreneur and investor before formulating the dynamic optimal contracting problem between the two agents.

2.1 Capital Accumulation and Production Technology

We adopt the stochastic capital accumulation specification used in Cox, Ingersoll, and Ross (1985), Jones and Manuelli (2005), and Barro (2009), among others. Thus, denoting by $I_t$ the gross investment of the entrepreneurial firm, the firm’s capital stock $K_t$ is assumed to accumulate as follows:

$$dK_t = (I_t - \delta K_t)dt + \sigma_K K_t dZ_t,$$

(1)

where $\delta \geq 0$ is the expected rate of depreciation, $Z$ is a standard Brownian motion, and $\sigma_K$ is the volatility of the capital depreciation shock. The firm’s capital stock can be interpreted as either tangible capital (property, plant and equipment), firm-specific intangible capital (patents, know-how, brand value, and organizational capital), or any combination of these.

Production requires combining the entrepreneur’s inalienable human capital with the firm’s production technology. When the two are united the firm’s output or revenues are given by $A_tK_t$, where $K_t$ is the firm’s capital stock and $\{A_t; t \geq 0\}$ is a stochastic productivity shock. To keep the analysis simple, we model $A_t$ as a two-state Markov regime-switching process, where $A_t \in \{A^L, A^H\}$ with $0 < A^L < A^H$, and $\lambda_n$ is the transition intensity out of state $n = L$ or $H$ to the other state.\(^5\) In other words, given a current value of $A_t = A^L$ the firm’s productivity changes to $A^H$ with probability $\lambda_L dt$, and if $A_t = A^H$ the firm’s productivity changes to $A^L$ with probability $\lambda_H dt$ in the time interval $(t, t + dt)$. The

\(^5\) Piskorski and Tchistyi (2007) consider a model of mortgage design in which they use a Markov-switching process to describe interest rates. DeMarzo, Fishman, He, and Wang (2012) use a Markov-switching process to model the persistent productivity shock.
productivity process \(\{A_t; t \geq 0\}\) is observable to both the investor and entrepreneur, and is therefore contractible.

Investment involves both a direct purchase cost and adjustment cost, so that the firm’s cash flows (after capital expenditures) are given by:

\[
Y_t = A_tK_t - I_t - G(I_t, K_t),
\]

where the price of the investment good is normalized to unity and \(G(I, K)\) is the standard adjustment cost function in the \(q\)-theory of investment. Importantly, \(Y_t\) can be negative, which means that the investor would be financing investment \(I_t\) and associated adjustment costs \(G\) from other sources than just current realized revenue \(A_tK_t\). We follow the \(q\)-theory literature and assume that the firm’s adjustment cost \(G(I, K)\) is homogeneous of degree one in \(I\) and \(K\), so that \(G(I, K)\) takes the following homogeneous form:

\[
G(I, K) = g(i)K,
\]

where \(i = I/K\) denotes the firm’s investment-capital ratio and \(g(i)\) is an increasing and convex function. As Hayashi (1982) has first shown, with this homogeneity property Tobin’s average and marginal \(q\) are equal under perfect capital markets.\(^6\) However, as we will show, under limited commitment to participate an endogenous wedge between Tobin’s average and marginal \(q\) will emerge in our model.\(^7\)

Hart and Moore (1994) is a special case of our model when we set: \(i\) \(\sigma_K = 0\) so that there are no shocks to the capital stock; \(ii\) \(\delta = 0\), so that the initial capital does not depreciate; \(iii\) \(I_t = 0\), so that there is no endogenous capital accumulation; \(iv\) \(A_t = A > 0\), so that there are no shocks to earnings; and \(v\) \(t \in [0, T]\), with \(T < \infty\), so that the horizon is finite. In other words, our framework adds to the basic Hart and Moore (1994) setup an endogenous capital accumulation process and shocks to both productivity and capital. Our goal is to

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\(^6\)Lucas and Prescott (1971) analyze dynamic investment decisions with convex adjustment costs, though they do not explicitly link their results to marginal or average \(q\). Abel and Eberly (1994) extend Hayashi (1982) to a stochastic environment and a more general specification of adjustment costs.

\(^7\)An endogenous wedge between Tobin’s average and marginal \(q\) also arises in cash-based optimal financing and investment models such as Bolton, Chen, and Wang (2011) and optimal contracting models such as DeMarzo, Fishman, He, and Wang (2012).
explore the interactions of productivity shocks \(\{A_t; t \geq 0\}\) and capital shocks \(\{Z_t; t \geq 0\}\) on corporate investment/asset sales, financial slack, risk management, and managerial compensation, when the entrepreneur contracts with investors under limited commitment to participate.

### 2.2 Preferences

We also generalize the Hart and Moore (1994) setup by introducing risk aversion for the entrepreneur. We assume that the infinitely-lived entrepreneur has a standard time-additive separable expected utility function over expected positive consumption flows \(\{C_t; t \geq 0\}\) given by:

\[
V_t = \mathbb{E}_t \left[ \int_t^\infty \zeta e^{-\zeta(v-t)}U(C_v)dv \right],
\]

where \(\zeta > 0\) is the entrepreneur’s subjective discount rate, \(U(C)\) is an increasing and concave function, and \(\mathbb{E}_t[\cdot]\) is the time-\(t\) conditional expectation. We further assume that the entrepreneur is risk averse and in addition that \(U(C)\) takes the standard iso-elastic constant-relative-risk-averse (CRRA) utility form:

\[
U(C) = \frac{C^{1-\gamma}}{1-\gamma},
\]

where \(\gamma > 0\) is the coefficient of relative risk aversion. Note that since we have normalized the value function with the constant \(\zeta\) in (4), so that the utility flow is \(\zeta U(C)\) as is standard in dynamic contracting models.\(^8\)

### 2.3 The Entrepreneur’s Outside Option

The entrepreneur’s human capital is inalienable and she can at any time leave the firm. When the entrepreneur exits she obtains an outside payoff of \(\hat{V}_n(K_t)\) (in utils) in state \(n \in \{L,H\}\). In other words, \(\hat{V}_n(K_t)\) is the entrepreneur’s endogenous outside option: it

\(^8\)We can generalize these preferences to allow for a coefficient of relative risk aversion that is different from the inverse of the elasticity of intertemporal substitution, à la Epstein and Zin (1989). Indeed, as Epstein-Zin preferences are homothetic, allowing for such preferences in our model will not increase the dimensionality of the optimization problem.
depends on both accumulated capital $K_t$ and the firm’s productivity $A^n$ at the moment of exit. The entrepreneur’s participation constraint due to the inalienability of human capital at each point in time $t$ is therefore given by:

$$V_t \geq \hat{V}_n(K_t), \quad t \geq 0.$$  \hfill (6)

This inalienability of human capital constraint can be interpreted in several ways.

1. A first interpretation is that when she quits the entrepreneur can start a new firm whose size is a fraction $\alpha \in (0, 1)$ of the on-going firm’s capital stock and find a new investor to finance her venture. We assume that the production function remains the same at the new firm as in the existing firm. In this narrative, there is no misappropriation involved and the outside option simply reflects the market value of the entrepreneur’s accumulated human capital. The key insight here is that the entrepreneur’s outside option offers her a larger fraction of a smaller firm upon exit, which the incumbent financier has to take into account in the optimal contract.

2. A second interpretation is that when she quits the entrepreneur puts her human capital to use in a competing firm and obtains a maximum payoff of $\hat{V}_n(K_t)$.

3. A third interpretation is that the entrepreneur may abscond a fraction with $\alpha \in (0, 1)$ of the on-going firm’s capital stock and start a new firm.

4. A fourth common interpretation is that the entrepreneur appropriates the capital stock and continues operating in autarky. She then forgoes intertemporal consumption-smoothing opportunities.\(^9\)

In our analysis, we will adhere to the first interpretation. We discuss the alternative interpretations in Section 9.

### 2.4 The Contracting Problem

We assume that the output process $Y_t$ is publicly observable and verifiable. In addition, we assume that the entrepreneur cannot privately save, as is standard in the literature on

\(^9\)This interpretation is commonly used in the international macro literature. See Bulow and Rogoff (1989).
dynamic moral hazard (see Bolton and Dewatripont, 2005 chapter 10). Without loss of
generality we assume that the investor has all the bargaining power. The contracting game
begins at time 0 with the investor making a take-it-or-leave-it long-term contract offer to
the entrepreneur. The contract specifies:

1. funding for an investment process \( I = \{I_t; t \geq 0\} \), and

2. a consumption allocation \( C = \{C_t; t \geq 0\} \) to the entrepreneur, in return for the firm’s
   operating income \( \{Y_t; t \geq 0\} \).

The investment and consumption processes depend on the entire history of productivity
shocks \( \{A_t; t \geq 0\} \), capital stock \( \{K_t; t \geq 0\} \), and output \( \{Y_t; t \geq 0\} \).

There are two limited-commitment frictions: the entrepreneur cannot commit to contin-
uing under the contract in perpetuity, and the investor is protected by limited liability so
that he cannot commit to a long-term contract that yields negative net present value at some
point in time. Specifically, at any time \( t \) investors cannot commit to \textit{ex post} negative net
present value (NPV) projects, which imposes the following constraints on the contracting
problem:

\[
F_t \equiv \mathbb{E}_t \left[ \int_{t}^{\infty} e^{-r(v-t)}(Y_v - C_v)dv \right] \geq 0. \tag{7}
\]

Under these limited-commitment frictions a long-term contract induces an optimal stop-
ning problem akin to optimally exercising a perpetual American option for both entrepreneur
and investor. However, given that the entrepreneur’s human capital is always most efficiently
employed in the firm, the optimal contract will be such that neither will exercise their options
in equilibrium.

At the moment of contracting at time 0 the entrepreneur also has a reservation utility
\( V_0 \), so that the optimal contract must also satisfy the constraint:

\[
V_0 \geq \tilde{V}_0. \tag{8}
\]

Without loss of generality, we let \( \tilde{V}_0 \geq \check{V}_n(K_0) \) for \( n \in \{L, H\} \). The investor’s problem at
time 0 is thus to choose dynamic investment \( I_t \) and consumption \( C_t \) to maximize the time-0
discounted value of cash flows,

\[
\max_{I,C} \quad \mathbb{E}_0 \left[ \int_0^\infty e^{-rt}(Y_t - C_t)dt \right], \quad (9)
\]

subject to the capital accumulation process (1), the production function (2), the entrepreneur’s limited-commitment constraints (6) at all \(t\), the investor’s limited-liability condition (7) at all \(t\), and the entrepreneur’s time-0 participation constraint (8).

The participation (8) constraint is always binding under the optimal contract. Otherwise, the investor can always increase his value by lowering the agent’s consumption and still satisfy all other constraints. However, the entrepreneur’s limited-commitment constraints (6) and the investors’ limited-liability constraints (7) will often not bind as the investor dynamically trades off the benefits of providing the entrepreneur with risk-sharing/consumption smoothing and the benefits of extracting higher contingent payments from the firm.

3 The Full-Commitment First-Best Benchmark

Before fully characterizing the optimal contract, we characterize the optimal outcome under full commitments by both investors and the entrepreneur. In that case our contracting problem reduces to the standard neoclassical setting of Hayashi (1982) with stochastic productivity. The risk-neutral investor simply buys off the entire venture from the risk-averse entrepreneur at time 0 for the reservation utility \(V_0\) and takes on all the output risk. The investor then maximizes the present discounted value of the venture’s cash flows with respect to \(I\).

Given the stationarity of the economic environment and the homogeneity of the production technology with respect to \(K\), there is an optimal productivity-dependent investment-capital ratio \(i_n = I/K\) in state \(n \in \{L, H\}\) that maximizes the present value of the venture. The following proposition summarizes the main results under full commitment.

**Proposition 1** In each state \(n \in \{L, H\}\), the firm’s value \(Q_n^{FB}(K)\) is proportional to its
capital stock $K - Q^F_n(K) = q^F_n K$ where $q^F_n$ is Tobin’s $q$ in state $n$. In state $H$, $q^H$ solves:

$$(r + \delta) q^F_H = \max_i \left( A^H - i - g(i) \right) + \lambda_H \left( q^F_L - q^F_H \right),$$

(10)

and the maximand for (10), denoted by $i^F_H$, is the first-best investment-capital ratio.

Homogeneity implies that return and present value relations hold for both the whole firm and for each unit of capital $K$. The first term on the right side of (10), $A^H - i - g(i)$, is the firm’s unit cash flow, and the second term, $\lambda_H \left( q^F_L - q^F_H \right)$, is the expected unit capital gain over the interval of time $dt$. At the optimum, the expected rate of return on capital is given by the sum of the discount rate $r$ and the expected deprecation rate of capital $\delta$, explaining the left side of (10). A similar (and symmetric) valuation equation holds for $q^F_L$.

Note that $q^F_n$ is the familiar Tobin average $q$, which under constant returns to scale is also the marginal value of capital, often referred to as marginal $q$. Adjustment costs create a wedge between the value of installed capital and newly purchased capital, so that that $q^F \neq 1$ in general.

We can also express Tobin’s $q$ via the first-order optimality condition for investment:

$$q^F_n = 1 + g'(i^F_n), \quad n = L, H,$$

(11)

which states that marginal $q$ is equal to the marginal cost of investing, $1 + g'(i)$, at the optimum investment level $i^F_n$. By jointly solving (10) and (11) and the similar two equations for state $L$, we obtain the values for $q^F_n$ and $i^F_n$, where $n \in \{L, H\}$.

In the full-commitment problem, the entrepreneur is perfectly insured and obtains a deterministic consumption stream that is independent of the firm’s investment dynamics:

$$C_t = C_0 e^{-(\zeta - r)t/\gamma}, \quad t \geq 0.$$  

(12)

To the extent that the investor and entrepreneur have different discount rates, $\zeta \neq r$, the optimal contract will be structured so that they can trade consumption intertemporally with each other. Specifically, consumption changes exponentially at a rate $-(\zeta - r)/\gamma$ per unit of time, where $1/\gamma$ should be interpreted as the elasticity of intertemporal substitution,
which is the inverse of the coefficient of relative risk aversion for standard CRRA utility functions. Thus, depending on the sign of \((\zeta - r)\) the entrepreneur’s consumption may grow or decline deterministically over time. It is only when the investor and the entrepreneur are equally impatient \((\zeta = r)\) that the entrepreneur’s consumption is constant over time under the optimal full-commitment contract.

To complete the solution of the full-commitment case, it remains to explicitly solve for the initial consumption \(C_0\). Note first that for a given level of the entrepreneur’s utility \(V\), we can calculate the corresponding certainty equivalent wealth (CEW) by inverting the expression \(V(W) = U(bW)\) and obtain:

\[
W = U^{-1}(V)/b, \tag{13}
\]

where \(U^{-1}(\cdot)\) is the inverse function of the CRRA utility (5) and \(b\) is a normalization constant given by: \(^{10}\)

\[
b = \zeta \left[ \frac{1}{\gamma} - \frac{r}{\zeta} \left( \frac{1}{\gamma} - 1 \right) \right]^\frac{1}{\gamma - 1}. \tag{14}
\]

Because the entrepreneur’s participation constraint (6) at time 0 is binding the initial CEW \(W_0\) must satisfy: \(W_0 = U^{-1}(V_0)/b\), so that the entrepreneur’s initial consumption \(C_0\) is proportional to \(W_0\):

\[
C_0 = \chi W_0 = \left( \frac{\zeta}{b} \right)^\frac{1}{\gamma} \left( (1 - \gamma)V_0 \right)^\frac{1}{1-\gamma}, \tag{15}
\]

where \(\chi\) is the marginal propensity to consume (MPC) given by

\[
\chi = b^{1-\frac{1}{\gamma}}\zeta^\frac{1}{\gamma} = r + \gamma^{-1}(\zeta - r). \tag{16}
\]

In the Appendix, we show that the entrepreneur’s utility process, denoted by \(V_t^{FB}\) is then given by:

\[
V_t^{FB} = U(bW_t) = U(bC_t/\chi) \sim V_0 e^{-((\zeta - r)(1-\gamma)t)/\gamma}, \tag{17}
\]

where \(U(\cdot)\) is given by (5). For the special case where \(\zeta = r\), the entrepreneur’s utility under the first-best case is time-invariant, \(V_t^{FB} = U(C_0) = V_0\), as consumption growth is zero at

\(^{10}\)As a special case, when \(\gamma = 1\), we have \(b = \zeta e^{-\frac{r - \zeta}{\zeta}}\).
all times.

Given that the entrepreneur’s outside option value $\hat{V}_n(K_t)$ grows stochastically with $K$, the entrepreneur’s participation constraint will then be violated at some point, which is why the first-best contract characterized above is not feasible under limited-commitment constraints. In summary, under perfect capital markets, the first-best investment-capital ratio $i^{FB}$ depends on the current state $n \in \{L, H\}$ but is independent of capital shocks. Moreover, the investor perfectly insures the entrepreneur’s consumption. As we will show next, the entrepreneur’s inability to fully commit to the venture indefinitely and the investors’ limited liability constraint significantly alters this solution.

4 Optimal Dynamic Contracting

The first-best outcome is not achievable when the entrepreneur cannot commit her human capital. Under the first-best investment policy the firm’s capital stock $K_t$ grows over time (in expectation). When it reaches the cut-off values $\overline{K}^H_t$ and $\overline{K}^L_t$ for which $\hat{V}_n(K_t) > V_t^{FB}$ when $K_t > \overline{K}^H_t$ in state $H$, and when $K_t > \overline{K}^L_t$ in state $L$ (where $V_t^{FB}$ is given by (17)), the entrepreneur’s participation constraint will be violated and the entrepreneur will walk away. To prevent such an outcome the investor writes a second-best contract where he commits to a consumption flow $\{C_t : t \geq 0\}$ for the entrepreneur such that $V_t \geq \hat{V}_n(K_t)$ at all times $t$ in both states $H$ and $L$. Since $\hat{V}_n(K_t)$ is a stochastic process, this second-best contract will inevitably expose the entrepreneur to consumption risk. Accordingly, the optimal dynamic contracting problem under limited commitment involves a specific form of the classic agency tradeoff between insurance of the agent’s consumption risk and incentive provision for the agent to stay with the firm. An important difference from the standard dynamic moral hazard problem, however, is that the entrepreneur’s and investors’ dynamic participation constraints often will not bind. The reason is that if the contract were to always hold the entrepreneur down to her participation constraint or to hold investors to their participation constraint then the entrepreneur’s promised consumption would be inefficiently volatile.
4.1 Formulating the optimal recursive contracting problem

The second-best dynamic contracting problem includes: i) a contingent investment plan \( \{I_t; t \geq 0\} \), and ii) consumption promises \( \{C_t; t \geq 0\} \) to the entrepreneur that maximize the present value of the firm for investors. As is well known (see e.g. DeMarzo and Sannikov, 2006), an important simplification of the contracting problem is to summarize the entire history of the contract in the entrepreneur’s promised utility \( V_t \) conditional on the history up to time \( t \). Under the optimal contract the dynamics of the agent’s promised utility can then be written in the recursive form below. The sum of the agent’s utility flow \( \zeta U(C_{t-})dt \) and change in promised utility \( dV_t \) has the expected value \( \zeta V_t dt \), or:

\[
E_t [\zeta U(C_{t-})dt + dV_t] = \zeta V_t dt. \tag{18}
\]

To see why equation (18) must hold, note first that we construct a stochastic process for the agent’s utility \( \{\hat{U}_t, t \geq 0\} \) as follows:

\[
\hat{U}_t = \int_0^t e^{-\zeta v} \zeta U(C_v)dv + e^{-\zeta t} V_t = E_t \left[ \int_0^\infty \zeta e^{-\zeta v} U(C_v)dv \right]. \tag{19}
\]

Second, assuming that standard integrability conditions hold (which we verify later) we know that \( \{\hat{U}_t; t \geq 0\} \) is a martingale: \( E_s[\hat{U}_t] = \hat{U}_t \) for all \( s \) and \( t \) such that \( s > t \). Third, applying Ito’s formula to the marginal process \( \hat{U} \) given in (19), and using the property that a martingale’s drift is zero, we then obtain (18). In other words, delivering a marginal unit of consumption to the entrepreneur lowers his promised utility \( V \) by reducing its drift \( \zeta V_t \) by \( \zeta U(C_{t-}) \), and hence we have the following equivalent representation of (18):

\[
E_t [dV_t] = \zeta (V_t - U(C_{t-})) dt. \tag{20}
\]

Next, given that there are two shocks—the capital shock (via the Brownian motion \( Z \)) and the productivity shock (via the two-state Markov chain)—we may write the stochastic differential equation (SDE) for \( dV \) implied by (18) as the sum of: i) the expected change (i.e., drift) term \( E_{t-}[dV_t] \); ii) a martingale term driven by the Brownian motion \( Z \); and iii) a martingale term driven by the productivity shock. Accordingly, if we denote by \( N_t \) the
cumulative number of productivity changes up to time \( t \), and adopt the convention that the current productivity state at time \( t−1 \) is \( H \), we may write the dynamics of the entrepreneur’s promised utility process \( V \) as follows:

\[
dV_t = \zeta(V_t - U(C_t))dt + x_t V_t dZ_t + \Gamma_H(V_{t-}, A^H)(dN_t - \lambda_H dt),
\]

where \( \{x_t; t \geq 0\} \) controls the diffusion volatility of the entrepreneur’s promised utility \( V \), and \( \Gamma_H(V_{t-}, A^H) \) controls the endogenous adjustment of promised utility \( V \) conditional on the change of productivity from \( A^H \) to \( A^L \). That is:

1. the first term on the right side of (21) is the expected change of \( dV_t \) as implied by (18),
2. the second term is the unexpected change due to capital shock \( Z \), and
3. the last term captures the mean-zero unexpected component of \( dV_t \) due to the change of productivity. Indeed, given that \( \lambda_H \) is the probability per unit of time of a productivity switch from \( A^H \) to \( A^L \), the expected value of \( (dN_t - \lambda_H dt) \) is zero.

Finally, we can write investors’ objective as a value function \( F(K_t, V_t, A^n) \) with three state variables: 

i) the entrepreneur’s promised utility \( V \); ii) the venture’s capital stock \( K \); and, iii) the state of productivity \( n \in \{L, H\} \).

The optimal contract then specifies dynamic investment \( I \), consumption \( C \), risk exposure \( x \), and insurance adjustment of promised utility \( \Gamma_n \), to solve the following optimization problem,

\[
F(K_t, V_t, A^n) = \max_{C_t, I_t, x_t, \Gamma_n} \mathbb{E}_t \left[ \int_t^\infty e^{-r(v-t)}(Y_v - C_v)dv \right],
\]

subject to the entrepreneurs’ interim participation constraints (6), the investors’ limited-liability conditions (7), and the entrepreneur’s initial participation constraint (8).

To solve this optimization problem we next proceed to the characterization of the investor’s optimization problem in the interior region and then we describe the boundary conditions.
The interior region. For expositional simplicity, suppose that the current state is $H$. Then, the following Hamilton-Jacobi-Bellman (HJB) equation holds:

$$rF(K, V, A^H) = \max_{C, I, x, \Gamma_H} \left\{ Y - C + (I - \delta K)F_K + \sigma^2_K K^2 F_{KK}/2 \right.$$

$$+ [\zeta(V - U(C)) - \lambda_H \Gamma_H] F_V + \frac{(xV)^2}{2} F_{VV} + \sigma_K xKF_{VK}$$

$$+ \lambda_H [F(K, V + \Gamma_H, A^L) - F(K, V, A^H)] \right\}. \quad (23)$$

The right-hand side of (23) gives the expected change of the investor’s value function $F(K, V, A^H)$. The first term is the venture’s flow profit $(Y - C)$ for the investor, which can be negative. In this case, the investor is financing operating losses; The second term reflects the expected change of the investor’s value $F(K, V, A^H)$ resulting from the expected (net) capital accumulation $(I - \delta K)$; The third term represents the expected change in the investor’s value resulting from the volatility of the capital shock; The fourth and fifth terms in turn reflect the change in investor’s value from the drift and volatility of the entrepreneur’s promised utility $V$; The sixth term captures how the investor’s value is affected by the (perfect) correlation between $K$ and $V$; Finally, the last term captures the effect of the persistent productivity shock on the value function. Importantly, in addition to the direct effect on the investor’s value $F$, the productivity switch from $A^H$ to $A^L$ also has an indirect effect on the investor’s value $F$ due to the endogenous adjustment of the entrepreneur’s promised utility from $V$ to $V + \Gamma_H$. As investors earn the rate of return $r$ at all times, the sum of all terms on the right side of (23) must equal $rF(K, V, A^H)$, which is given on the left-hand side of (23).

Differentiating the right-hand side of (23) with respect to $C, I$, and $x$ we then obtain the following first-order conditions (FOCs):

$$\zeta U'(C^*) = -\frac{1}{F_V(K, V, A^H)}, \quad \text{(24)}$$

$$F_K(K, V, A^H) = 1 + G_I(I^*, K), \quad \text{and} \quad \text{(25)}$$

$$x^* = -\frac{\sigma_KKF_{VK}}{VF_{VV}(K, V, A^H)}. \quad \text{(26)}$$

As there is only one exogenous diffusion shock in the model, $V$ and $K$ are locally perfectly correlated.
FOC (24) characterizes the entrepreneur’s optimal consumption $C^*$, which must equalize the entrepreneur’s marginal utility of consumption $\zeta U'(C^*)$ with $-1/F_V$, which is positive as $F_V < 0$. Multiplying (24) through by $-F_V$, we observe that at the optimum the agent’s normalized marginal utility of consumption, $-F_V \zeta U'(C)$, has to equal unity, the risk-neutral investor’s marginal cost of providing a unit of consumption.

FOC (25) characterizes investors’ optimal investment decision, which is obtained when the marginal benefit of investing, $F_K(K, V, A^n)$, is equal to the marginal cost of investing, $1 + G_I(I, K)$.

FOC (26) characterizes the optimal exposure of the entrepreneur’s promised utility $V$ to the shock $Z$. As we show later, $x$ is closely tied to the firm’s optimal risk management policy.

Finally, we turn to the optimal choice of $\Gamma_H$, the discrete change in the entrepreneur’s promised utility contingent on the change of productivity from $H$ to $L$. Whenever feasible, the optimal contract equates investors’ marginal cost of delivering compensation just before and after the productivity change, so that:

$$F_V(K, V + \Gamma^*_H, A^L) = F_V(K, V, A^H),$$

which is the FOC with respect to $\Gamma_H$ implied by (23). Note that the second-order condition (SOC) with respect to $\Gamma_H$ is given by $F_{VV}(K, V + \Gamma^*_H, A^L) < 0$ which implies that $F$ is concave in $V$ at $\Gamma^*_H$. The condition (27) only holds when neither the entrepreneur’s limited-commitment constraint nor the investor’s limited-liability constraint bind. When either constraint binds, we will have inequalities rather than equalities for the FOC with respect to $\Gamma_H$. We return to the corner-solution case later with a more detailed discussion.

Next we turn to the boundary conditions, where either the entrepreneur’s participation constraint or the investors’ limited liability constraint bind.

**Boundary conditions.** Consider first the endogenous boundary condition for the promised utility to the entrepreneur $\overline{V}_n(K)$ at which the investor’s limited-liability condition binds. When the investor is indifferent between continuing with a promise to the entrepreneur of
The entrepreneur’s value function when quitting is given by:
\[ \tilde{V}_n(K_t) = V_n(\alpha K_t), \] (29)
where \( V_n(\cdot) \) is given by (28). Note that here we assume that the new investor breaks even earning zero profit. Thus, the lower endogenous boundary condition for \( V \) is:
\[ V_t = V_n(\alpha K_t). \] (30)

In summary, with the entrepreneur’s limited participation and the investor’s limited liability constraints, the entrepreneur’s utility \( V_t \) must satisfy the conditions:
\[ V_n(\alpha K_t) \leq V_t \leq V_n(K_t). \] (31)
The HJB equation (23), the FOCs (24), (25), (26), (27), and the boundary conditions (31) then jointly characterize the solution for the second-best optimal contract.

4.2 The Entrepreneur’s Promised Certainty Equivalent Wealth \( W \)

How do we link the entrepreneur’s promised utility \( V \), the key state variable characterizing the optimal contract, to variables that are empirically measurable? As we will show, it is possible to formulate the optimal contract as a liquidity and hedging policy for the firm that
can be implemented using standard financial instruments. The corporate liquidity policy can be implemented through a combination of retained earnings and a line of credit commitment by investors. And the hedging policy can be implemented using a combination of futures positions and contingent convertible claims held by investors.

A helpful simplification towards the contracting formulation in terms of corporate liquidity and risk management is to express the entrepreneur’s promised utility in units of consumption rather than utils. This involves mapping the promised utility $V$ into the promised (certainty-equivalent) wealth $W$, defined as the solution to the equation $U(bW) = V$, and transforming the investor’s value function $F(K, V, A^n)$ in terms of $V$ into the value function $P(K, W, A^n)$ in terms of $W$ using the following identity:

$$P(K, W, A^n) \equiv F(K, U(bW), A^n) = F(K, V, A^n), \quad n \in \{L, H\}. \quad (32)$$

As is shown in the appendix, we can reformulate the HJB equation for $F(K, V, A^n)$ (with the corresponding FOCs for $C, I, x, \Gamma_H$, and boundary conditions) into an equivalent HJB equation for $P(K, W, A^n)$ with associated FOCs and boundary conditions by using the identity in (32) and applying Itô’s formula to $P(K, W, A^n)$.

5 Implementation: Liquidity and Risk Management

Having characterized the optimal contract in terms of the entrepreneur’s promised certainty-equivalent wealth $W$, we show next how to implement the optimal contract via commonly used financial instruments. As we have emphasized, the entrepreneur’s limited commitment and investors’ limited liability constraints naturally give rise to dynamic corporate risk management and liquidity management policies for the entrepreneur, which are implemented via standard financial instruments.\(^{12}\)

Liquidity management. Consider first the entrepreneur’s liquidity management problem. We endow the entrepreneur with a bank account and let $S_t$ denote the account’s time-$t$

\(^{12}\)It is well known that implementation is not unique. To simplify the exposition we focus on one intuitive implementation and later discuss alternative ways of implementing the dynamic optimal contract.
balance. This balance \( S_t \) becomes the relevant state variable in the implementation problem. Naturally \( S_t < 0 \) corresponds to a draw-down on a line of credit (LOC) granted by the bank to the entrepreneur. In the implementation problem the entrepreneur can borrow on this LOC at the risk-free rate \( r \) up to a maximal value of \( \bar{D}_n(K_t) \), which we refer to as the endogenously determined liquiditiy capacity of the firm in state \( n \). This borrowing limit is derived to ensure that the entrepreneur does not walk away from the firm in an attempt to evade her debt obligations. Then, as long as the entrepreneur remains with the firm, the firm’s debt is risk free and hence can be financed at the risk-free rate. The liquidity buffer \( S_t \) in the risk-free savings/credit account is the state variable for the implementation problem, but as the optimal contracting problem highlights, liquidity management alone will only provide partial insurance to the entrepreneur. To replicate the optimal contracting outcome, additional insurance instruments are needed to which we turn next.

**Risk management against capital shocks.** One instrument the entrepreneur can use to hedge the capital risk \( Z \) is a standard futures contract.\(^\text{13}\) Since investors are risk neutral the futures price involves no premium given that the futures contract payoffs have zero mean. Moreover, since profits/losses of the futures position are only subject to diffusion shocks that are instantaneously credited/debited from the entrepreneur’s bank account, there is no default risk. Hence, if the entrepreneur takes a unit long position in the futures contract she adds \( \sigma_K dZ_t \) to her capital risk exposure \( Z \). More generally, the entrepreneur can take any futures position \( \phi_t K_t \) to hedge the firm’s risk exposure to the capital risk \( Z \). By choosing \( \phi_t K_t \) optimally the entrepreneur can thus obtain further consumption insurance.

**Insurance against productivity shocks.** Finally, the entrepreneur can take out a contingent claim to hedge the risk with respect to changes in the productivity state. Suppose that the current productivity state is \( H \). If the entrepreneur takes a unit long position in the contingent claim she pays an insurance premium \( \lambda_H \) per unit of time and receives a unit payment from the insurer when the state switches from \( H \) to \( L \). Given that insurers are risk neutral, the actuarially fair premium per unit of time for this insurance is then \( \lambda_H \).

\(^{13}\)Bolton, Chen, and Wang (2011) analyze the optimal corporate risk management for a financially constrained firm. In that model, they also analyze the dynamic futures trading strategies but their model is not a dynamic contracting framework.
Let \( \pi_H(S, A^H)K \) denote the entrepreneur’s demand for this insurance contract in state \( H \). She then pays a total insurance premium \( \pi_H(S, A^H)K\lambda_H \) per unit of time and receives a lump-sum payment \( \pi_H(S, A^H)K \) when the state switches from \( H \) to \( L \) (i.e. when \( dN_t = 1 \)). Therefore, the total stochastic exposure of this contingent claim is \( \pi_H(S, A^H)K_t(dN_t - \lambda_H dt) \) where \( dN_t \in \{1, 0\} \).

**Liquidity dynamics.** With these three financial instruments the dynamic evolution of the entrepreneur’s savings balance, denoted by \( S_t \), evolves as follows in state \( H \):

\[
dS_t = (rS_t + Y_t - C_t)dt + \phi_t K_t \sigma_K dZ_t + \pi_H(S, A^H)K_t(dN_t - \lambda_H dt),
\]

as long as the limit on the LOC granted by the bank is not violated:

\[
S_t \geq -D_H(K_t).
\]

The first term in (33), \( rS_t + Y_t - C_t \), is simply the sum of the firm’s interest income \( rS_t \) and net operating earnings \( Y_t - C_t \). In the absence of any risk management and hedging, \( rS_t + Y_t - C_t \) is simply the rate at which the entrepreneur saves or draws on the LOC at the risk-free rate \( r \). The second term \( \phi_t K_t \sigma_K dZ_t \) in (33) is the cost of hedging the capital shock \( Z \) via the futures position \( \phi_t K_t \). The third term, \( \pi_H(S, A^H)K_t(dN_t - \lambda_H dt) \), captures the effect of the insurance contract against productivity changes.

**The entrepreneur’s optimization problem.** The implementation problem can now be formulated as the following dynamic optimization problem: In each period the entrepreneur optimally chooses consumption \( C_t \), investment \( I_t \), futures position \( \phi_t K_t \) and insurance demands, \( \pi_H K_t \) and \( \pi_L K_t \), to maximize her utility function given in (4)-(5), subject to the liquidity accumulation dynamics (33) and the endogenous borrowing limits (34) \( D_n(K) \). The latter limits are obtained from the boundary conditions that ensure that the entrepreneur’s participation constraints are always satisfied.

This dual optimization problem for the entrepreneur is then equivalent to the primal problem for the investor in (23) if and only if the borrowing limits in the entrepreneur’s

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\(^{14}\)We will later extend the model to incorporate a stochastic discount factor (SDF) capturing risk premia.
problem $\mathcal{D}_n(K_t)$ are such that:

$$
\mathcal{D}_n(K) = P(K, W_n, A^n), \quad n \in \{L, H\},
$$

(35)

where $P(K, W_n, A^n)$ is the investors’ value when the entrepreneur’s limited-commitment constraint binds, that is, when $W_n = W_n$. Accordingly, we characterize the implementation solution for the dual problem by first solving the investors’ problem in (23), and then imposing the constraint in (35).

Guided by the observation in the full-commitment case that the value function of the entrepreneur inherits the CRRA form of the entrepreneur’s utility function, we make the educated guess (and later verify) that the entrepreneur’s value function $J(K, S, A^n)$ takes the form:

$$
J(K, S, A^n) = \left(\frac{bM(K, S, A^n)}{1 - \gamma}\right)^{1-\gamma}, \quad n = L, H,
$$

(36)

where $M(K, S, A^n)$ is the entrepreneur’s certainty equivalent wealth and the normalization constant $b$ is given by $U(bM) = J$. In the Appendix, we provide the Hamilton-Jacobi-Bellman (HJB) equation that characterizes $M(K, S, A^n)$ together with the corresponding boundary conditions.

To summarize, the primal optimal contracting problem gives rise to the investor’s value function $F(K, V, A^n)$, with the promised utility to the entrepreneur $V$ as the key state variable. By expressing the entrepreneur’s promised utility in units of consumption rather than utils, the investor’s value function can be rewritten in terms of the entrepreneur’s promised certainty-equivalent wealth $W$: $P(K, W, A^n)$. The dual problem for the entrepreneur gives rise to the entrepreneur’s value function $J(K, S, A^n)$, with $S = -P(K, W, A^n)$ as the key state variable. Or, again expressing the entrepreneur’s value in units of consumption, the entrepreneur’s value function is her certainty equivalent wealth $M(K, S, A^n)$ and the relevant state variable is her savings $S = -P$. Again, the key attraction of the dual formulation is that it frames the optimal financial contracting problem in terms of a more operational liquidity and risk management problem for the firm.

To simplify the exposition of the key economic mechanism in our model, we next analyze the case with capital (diffusion) risk only, which is a special case with $A^L = A^H = A$. 

24
6 The Case with no Productivity Shocks

In this section we simplify the problem by assuming that the firm’s productivity is constant, \( A^L = A^H = A \), so that the only shock to cash flows is the diffusion shock \( Z \).

Given that the firm has a constant returns to scale production technology, and that investment adjustment costs and the entrepreneur’s outside option are homogenous in \( K \), we may transform the two-dimensional PDE formulations of our Hamilton-Jacobi-Bellman (HJB) equations into analytically more tractable one-dimensional ODE formulations by dividing both sides of the HJB equations by \( K \) and thus expressing all our variables per unit of capital. That is, we show that the investors’ value function \( P(K, W, A) \) and the entrepreneur’s certainty equivalent wealth \( M(K, S) \) are homogeneous of degree one in \( K \), so that:

\[
P(K, W) = p(w) \cdot K,
\]

where \( w = W/K \) is the entrepreneur’s certainty-equivalent wealth scaled by the firm’s capital stock \( K \), and \( p(w) \) is the scaled value function of investors, and

\[
M(K, S) = m(s) \cdot K,
\]

(37)

where \( s = S/K \) is the entrepreneur’s savings \( S \) scaled by the firm’s capital stock \( K \), and \( m(s) \) is the scaled promised (certainty equivalent) wealth.\(^{15}\) The other variables are also scaled by \( K \), so that \( c(s) \) is the consumption-capital ratio, \( i(s) \) the investment-capital ratio, and \( \phi(s) \) the hedge ratio. In the interior region we then have:

\[
ds_t = \mu^*(s_t)dt + \sigma^*(s_t)dZ_t,
\]

(38)

\(^{15}\)Wang, Wang, and Yang (2012) solve an entrepreneur’s optimal consumption-savings, business investment, and portfolio choice problem with endogenous entry and exit decisions. By exploiting homogeneity, they derive the optimal investment policy in a \( q \)-theoretic context with incomplete markets. In our model, we optimally implement the solution of the optimal contacting problem.
where the drift and volatility processes \( \mu^s(\cdot) \) and \( \sigma^s(\cdot) \) for \( s \) are given by

\[
\begin{align*}
\mu^s(s) &= (A - i(s) - g(i(s)) - c(s)) + (r + \delta - i(s))s - \sigma_K^2 \sigma^s(s), \\
\sigma^s(s) &= (\phi(s) - s)\sigma_K.
\end{align*}
\] (39) (40)

Note from (40) that the volatility of savings can be controlled by the futures position \( \phi(s) \). In particular, by setting \( \phi(s) = s \) the entrepreneur can make sure that he faces no risk with respect to the growth of savings \( \mu^s(s) \).

We begin by characterizing the solution of the one-sided commitment problem, when only the entrepreneur cannot commit to always stay with the firm.

**The one-sided limited-commitment case.** The following proposition summarizes the solution when only the entrepreneur is unable to commit to the firm.

**Proposition 2** In the region where \( s > \underline{s} \), the entrepreneur’s scaled promised wealth \( m(s) \) solves the equation:

\[
0 = \max_{i(s)} \left[ \frac{m(s)}{1 - \gamma} \left[ \gamma \chi \left( m'(s) \right)^{\frac{\gamma - 1}{\gamma}} - \zeta \right] - \delta m(s) + [(r + \delta) s + A] m'(s) \\
+ i(s)(m(s) - (s + 1)m'(s)) - g(i(s))m'(s) - \frac{\gamma \sigma_K^2}{2} \frac{m(s)^2 m''(s)}{m(s)m''(s) - \gamma m'(s)^2},
\]

subject to the following boundary conditions:

\[
\begin{align*}
\lim_{s \to \infty} m(s) &= q^{FB} + s, \\
\quad m(\underline{s}) &= \alpha m(0), \\
\lim_{s \to \underline{s}} \sigma^s(s) &= 0 \quad \text{and} \quad \lim_{s \to \underline{s}} \mu^s(s) \geq 0.
\end{align*}
\] (42) (43) (44)

The ODE given by (41) characterizes the entrepreneur’s scaled promised wealth \( m(s) \) in the interior region \( s > \underline{s} \). Note that as the entrepreneur’s savings become infinitely large the entrepreneur’s promised wealth must be equal to the first-best value of investment and savings. At that point the entrepreneur’s inability to commit no longer affects firm value, as the entrepreneur’s self insurance is sufficient to achieve the first-best resource.
allocation outcome. In this limit the marginal value of liquidity is simply unity as a financially unconstrained entrepreneur does not pay a premium for liquid assets, and the entrepreneur values a unit of capital $K$ at its first-best maximal value $q^{FB}$.

At the other endogenous boundary $s$, where the entrepreneur runs out of liquidity, the entrepreneur’s promised wealth $m(s)$ equals $\alpha m(0)$, the entrepreneur’s certainty-equivalent wealth per unit of capital under the outside option.

Finally, the third condition (44) ensures that the entrepreneur does not quit for sure as $s$ approaches $s$. This condition ensures that the volatility of $s$ evaluated at $s$ is zero and that the drift $\mu^s(s)$ is weakly positive, so as to guarantee that the constraint $s \geq s$ is satisfied at all times and that the entrepreneur will not run out of liquidity.

Remark: Note that for these scaled variables the primal and dual optimization problems are linked as follows: $\underline{s} = -p(w) = -d = -D(K)/K$.

The two-sided limited-commitment case. When a limited liability condition for investors must also be satisfied, the solution above for the one-sided commitment case must be modified to include a different condition at the upper boundary in Proposition 2. The investors’ limited liability condition then implies that the upper boundary is $s = 0$ rather than the natural limiting boundary $s \to \infty$. We thus replace condition (42) with the following condition:

$$\lim_{s \to 0} \sigma^s(s) = 0 \quad \text{and} \quad \lim_{s \to 0} \mu^s(s) \leq 0.$$ 

Again, at the upper boundary $s = 0$, the volatility $\sigma^s(\cdot)$ has to be zero and the drift needs to be weakly negative to pull $s$ to the interior, thus ensuring that $s$ will not violate the constraint $s \leq 0$.

6.1 Parameter Choices and Calibration

Our model with no productivity shocks is parsimonious with only eight parameters. Three parameters essential for the contracting tradeoff between risk sharing and limited commitment are the entrepreneur’s coefficient of relative risk aversion $\gamma$, the volatility of the capital shocks $\sigma_K$, and the parameter measuring the degree of human capital inalienability $\alpha$. The other five parameters (the risk-free rate $r$, the entrepreneur’s discount rate $\zeta$, the depreci-
ation rate $\delta$, the adjustment cost $\theta$, and the productivity parameter $A$) are basic to any dynamic model with investment. We choose plausible parameter values to highlight the model’s mechanism and main insights.

Thus, we take the widely used value for the coefficient of relative risk aversion, $\gamma = 2$; the annual risk-free interest rate $r = 5\%$; and, the entrepreneur’s annual subjective discount rate set to equal to the risk-free rate, $\zeta = r = 5\%$. As for investment, we rely on the parameter findings suggested by Eberly, Rebelo, and Vincent (2009): we set the annual productivity $A$ at 20% and the annual volatility of capital shocks at $\sigma_K = 20\%$. While our model is equally tractable for any homogeneous adjustment cost function $g(i)$, we choose the following quadratic adjustment cost function for illustrational simplicity,

$$g(i) = \frac{\theta i^2}{2},$$

(45)

as we then have explicit formulas for Tobin’s $q$ and the optimal investment-capital ratio $i$ in the first-best MM benchmark:

$$q^{FB} = 1 + \theta i^{FB}, \text{ and } i^{FB} = r + \delta - \sqrt{(r + \delta)^2 - 2A - (r + \delta)}.$$  

(46)

Fitting the first-best values of $q^{FB}$ and $i^{FB}$ to the sample averages, we set the adjustment cost parameter at $\theta = 2$ and the (expected) annual depreciation rate for capital stock at $\delta = 12.5\%$. These parameters imply a $q^{FB} = 1.2$ and an annual investment-capital ratio of $i^{FB} = 0.1$.

Finally, we choose the fraction of capital stock that the entrepreneur may divert, $\alpha$ to be 0.4, broadly in line with empirical estimates. The parameter values for our baseline case are summarized in Table 1. Note that all parameter values are annualized when applicable.

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16See Li, Whited, and Wu (2014) for the empirical estimates of $\alpha$. The averages are 1.2 for Tobin’s $q$ and 0.1 for the investment-capital ratio, respectively, for the sample used by Eberly, Rebelo, and Vincent (2009). The imputed value for the adjustment cost parameter $\theta$ is 2 broadly in the range of estimates used in the literature. See Hall (2004), Riddick and Whited (2009), and Eberly, Rebelo, and Vincent (2009).
Table 1: Summary of Parameters

This table summarizes the parameter values for the baseline model with $A^H = A^L$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-free rate</td>
<td>$r$</td>
<td>5%</td>
</tr>
<tr>
<td>The entrepreneur’s discount rate</td>
<td>$\zeta$</td>
<td>5%</td>
</tr>
<tr>
<td>The entrepreneur’s relative risk Aversion</td>
<td>$\gamma$</td>
<td>2</td>
</tr>
<tr>
<td>Capital depreciation rate</td>
<td>$\delta$</td>
<td>12.5%</td>
</tr>
<tr>
<td>Volatility of capital depreciation shock</td>
<td>$\sigma_K$</td>
<td>20%</td>
</tr>
<tr>
<td>Quadratic adjustment cost parameter</td>
<td>$\theta$</td>
<td>2</td>
</tr>
<tr>
<td>Firm’s productivity</td>
<td>$A$</td>
<td>20%</td>
</tr>
<tr>
<td>Inalienability of human capital parameter</td>
<td>$\alpha$</td>
<td>40%</td>
</tr>
</tbody>
</table>

6.2 Promised Wealth $W$ and Financial Slack $S$

For our scaled value functions the primal contracting problem and the dual implementation formulation are linked as follows:

$$s = -p(w) \text{ and } w = m(s),$$

(47)

where $p(w)$ is the scaled investors’ value as a function of promised scaled wealth $w$ in the contracting problem and $m(s)$ is the entrepreneur’s scaled certainty equivalent wealth as a function of scaled liquidity $s$. Thus, financial slack $s$ for the entrepreneur is the payoff the investor is giving up through the promised wealth $w$ to the entrepreneur. More formally, (47) implies that the composition of $-p$ and $m$, denoted by $-p \circ m$, yields the identity function: $-p(m(s)) = s$.

Scaled promised wealth $w$ and scaled investors’ value $p(w)$. Figure 1 plots the investor’s scaled value $p(w)$ and the sensitivity of the value to changes in promised wealth $p'(w) = P_W$ in Panels A and B respectively. A first observation is that $p(w)$ is decreasing in $w$ for both the one-sided and two-sided limited-commitment cases. In other words, the higher the entrepreneur’s promised certainty equivalent wealth $w$, the lower the investors’ value $p(w)$. Moreover, as $w$ increases, the entrepreneur becomes less constrained and the
Figure 1: **Investors’ scaled value** $p(w)$ and the investors’ marginal value of $w$, $p'(w)$, as functions of the entrepreneur’s scaled promised wealth $w$. The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, $p(w)$ is decreasing and concave in $w$. For the one-sided case, $w \geq w = 0.479$. For the two-sided case, $0.4 = w \leq w \leq 0.988$. The dotted line depicts the first-best MM results: $p(w) = q^{FB} - w$ and the sensitivity $p'(w) = -1$.

A second observation in the one-sided limited-commitment problem (for the entrepreneur) is that $p(w)$ approaches $q^{FB} - w$, and $p'(w) \to -1$, as $w \to \infty$. That is, the first-best payoff obtains when the entrepreneur is unconstrained. However, the entrepreneur’s inability to fully commit not to walk away *ex post* imposes a lower bound $w$ on $w$. For our parameter values we have, $w \geq w = 0.479$.

In the two-sided limited-commitment case $w$ lies between two boundaries: $w = 0.40$ and $w = 0.988$. The upper boundary $w$ is then determined by the investors’ limited liability condition: $p(w) = 0$. Note that the left boundary in the two-sided case, $w = 0.40$, is lower than in the one-sided case, $w = 0.479$. The reason is that by restricting the support of $w$ to be lower than $w = 0.988$, the additional limited liability constraint also reduces the value of the entrepreneur’s outside opportunity so that the entrepreneur can be induced to stay for lower values of promised wealth $w$. Note also that while $p'(w) \geq -1$ holds in the one-sided case, $p'(w)$ can be less than $-1$ in the two-sided commitment case. This reflects the fact that the benefit from an increase in $w$ for the entrepreneur may not be sufficient to offset...
the cost to the investor (due to the increased likelihood that the investors’ limited liability constraint may bind in the future), implying that $p'(w) + 1 < 0$.

Note finally that despite being risk neutral, the investor effectively behaves in a risk-averse manner due to the entrepreneur’s limited commitment and the investors’ limited liability constraints. This is reflected in the concavity of the investors’ scaled value function $p(w)$. This concavity property is an important difference of the limited commitment problem relative to the neoclassical problem, where volatility has no effect on firm value.

Figure 2: The entrepreneur’s scaled certainty equivalent wealth $m(s)$ and marginal (certainty equivalent) wealth of $s$, $m'(s)$, as functions of scaled liquidity $s$. The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, $m(s)$ is increasing and concave. For the one-sided case, $s \geq \underline{s} = -\bar{d} = -0.692$. For the two-sided case, $-0.738 = -\bar{d} = \underline{s} \leq s \leq 0$. The dotted line depicts the first-best MM results: $m(s) = q^{FB} + s$ and the sensitivity $m'(s) = 1$.

Scaled liquidity $s$ and the entrepreneur’s scaled certainty-equivalent wealth $m(s)$.

Figure 2 plots the entrepreneur’s wealth $m(s)$ conditional on the firm’s financial slack $s$ and the marginal value of savings $m'(s)$ in Panels A and B respectively. As one might expect $m(s)$ is increasing in $s$ in both the one-sided and two-sided limited-commitment cases: The higher the firm’s financial slack $s$ the less the entrepreneur is likely to want to walk away and hence the higher the value of $m(s)$ in the long-term bilateral relationship. Moreover, as $s$ increases the entrepreneur is less constrained so that the marginal value of savings $m'(s)$
decreases \( m''(s) < 0 \).

In the one-sided limited-commitment case, the entrepreneur’s scaled wealth \( m(s) \) approaches \( q^{FB} + s \) and \( m'(s) \to 1 \) as \( s \to \infty \).\(^{17}\) And the entrepreneur’s LOC limit, or in other words, her risk-free debt capacity \( s = -p(w) = -\overline{d} = -\overline{D}(K)/K \) is given by \(-0.692\).

In contrast, in the two-sided limited-commitment case, \( s \) lies between \( \underline{s} = -0.738 \) and \( \overline{s} = 0 \), so that the entrepreneur has a larger LOC limit of \( \overline{d} = 0.738 \). But this comes at the expense of lower promised utility, which translates into no corporate savings in this implementation (\( \overline{s} = 0 \)). Indeed, if we had \( \overline{s} > 0 \) the investors’ value would be strictly negative violating the investors’ limited-liability condition. In sum, the additional investor limited-liability condition limits the entrepreneur’s self savings capacity, which in turn increases the entrepreneur’s demand for relying on an LOC. Remarkably, here a firm with a larger debt capacity is not necessarily less constrained and may have a lower value!

While \( m'(s) \geq 1 \) holds for the one-sided case, \( m'(s) \) can be less than 1 in the two-sided limited-commitment case. This is again due to the fact that in the two-sided case the benefit of relaxing financial constraints for the entrepreneur with an increase in \( s \) may not be sufficient to offset the cost to the investor (due to a shorter distance investors’ limited-liability constraint) implying that \( m'(s) < 1 \) in the region \(-0.708 < s \leq 0 \). We next analyze the optimal policy rules.

### 6.3 Investment, Consumption, Liquidity and Risk Management

We first analyze the firm’s investment decisions, then the entrepreneur’s consumption, and finally corporate liquidity and risk management.

#### 6.3.1 Investment, marginal \( q \), and marginal value of liquidity \( m'(s) \).

We can simplify the FOC for investment to:

\[
1 + q'(i(s)) = \frac{J_K}{J_S} = \frac{M_K}{M_S} = \frac{m(s) - sm'(s)}{m'(s)},
\]

\(^{17}\)See Wang, Wang, and Yang (2012) for similar conditions in a model with exogenously-specified incomplete-markets model of entrepreneurship.
where the first equality is the investment FOC, the second equality follows from the definition of the value function in (36), and the last equality follows from the homogeneity property of $M(K, S)$ in $K$.

Under perfect capital markets the entrepreneur’s certainty equivalent wealth $M(K, S) = m(s) \cdot K = (q^{FB} + s) \cdot K$ and the marginal value of liquidity is $M_S = 1$ at all times. Hence in this case, the FOC (48) specializes to the classical Hayashi condition for optimal investment, where the marginal cost of investing $1 + \theta i(s)$ equals marginal $q$.

Under limited commitment, $M_S \neq 1$ in general and the FOC (48) then states that the marginal cost of investing (on the left-hand side) equals the ratio between (a) marginal $q$, measured by $M_K$, and (b) the marginal value of liquidity measured by $M_S$. While financing does not matter in the classical $q$ theory of investment, here financing matters and $M_S$ measures the (endogenous) marginal cost of financing generated by limited commitment constraints.

Figure 3 illustrates the effect of limited commitment on marginal $q$ and optimal investment $i(s)$. The dotted lines in Panels A and B of Figure 3 give the first-best $q^{FB} = 1.2$ and $i^{FB} = 0.1$, respectively. In the one-sided limited-commitment case, the investment-capital ratio $i(s)$ is lower than the first-best benchmark $i^{FB} = 0.1$ for all $s$, and increases from 0.02 to $i^{FB} = 0.1$ as $s$ increases from the left boundary $s = -0.692$ towards $\infty$. This is to be expected: increasing financial slack mitigates the severity of under-investment for a financially constrained firm.

Note however that, surprisingly, marginal $q$ (that is, $M_K$) decreases with $s$ from 1.43 to 1.197 in the credit region $s < 0$. What is the intuition? When the firm is financing its investment via credit at the margin (when $S < 0$), increasing $K$ moves a negative-valued $s$ closer to the origin thus mitigating financial constraints, which is an additional benefit of accumulating $K$. Formally, this result follows from $dM_K/ds = -sm''(s) < 0$ when $s < 0$ and $m(s)$ is concave.

But why does a high marginal-$q$ firm invest less in the credit region $s < 0$? And how do we reconcile an increasing investment function $i(s)$ with a decreasing marginal $q$ function, $M_K = m(s) - sm'(s)$ in the credit region $s < 0$? The reason is simply that in the credit region ($s < 0$) a high marginal-$q$ firm also faces a high financing cost. When $s < 0$, marginal $q$ and the marginal financing cost are perfectly correlated. And investment is determined
by the ratio between marginal $q$ and marginal financing cost as we have noted. Thus, at the left boundary $s = -d = -0.692$, marginal $q$ is 1.43 and the marginal value of liquidity $m'(s)$ is 1.38 both of which are high. Together they imply that $i(-0.692) = 0.02$, which is a very low compared with the first-best $i^{FB} = 0.10$. In contrast, in the savings region, $s > 0$, $M_K$ and $m'(s)$ are negatively correlated, which explains the shape of the optimal investment schedule $i(s)$ in this region.\footnote{See Bolton, Chen, and Wang (2011) for discussions on how cash and credit influence the behaviors of investment, marginal $q$, and marginal value of liquidity.}

Consider next the two-sided limited-commitment case. The limited liability constraint for investors in this case prevents the entrepreneur from owning the whole equity of the firm and to hold positive liquid wealth. Hence, there is only a credit region in the two-sided case: $s \leq 0$. Remarkably, in this case the firm may either under-invest or over-invest compared with the first-best benchmark. The firm under-invests when $s < -0.591$ but over-invests when $-0.591 < s \leq 0$. Whether the firm under-invests or over-invests depends on the net effects of the entrepreneur’s limited-commitment and the investors’ limited-liability

Figure 3: **Marginal $q$, $M_K = m(s) - s m'(s)$, and the investment-capital ratio $i(s)$**. The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For the one-sided case, the firm always under-invests and $i(s)$ increases with $s$. For the two-sided case, the firm may either under-invest or over-invest. For $-0.591 < s \leq 0$, the firm over-invests due to the investors’ limited-liability condition. The dotted line depicts the first-best MM results where $M_K = q^{FB} = 1.2$ and the investment-capital ratio $i(s) = i^{FB} = 0.1$. 

Consider next the two-sided limited-commitment case. The limited liability constraint for investors in this case prevents the entrepreneur from owning the whole equity of the firm and to hold positive liquid wealth. Hence, there is only a credit region in the two-sided case: $s \leq 0$. Remarkably, in this case the firm may either under-invest or over-invest compared with the first-best benchmark. The firm under-invests when $s < -0.591$ but over-invests when $-0.591 < s \leq 0$. Whether the firm under-invests or over-invests depends on the net effects of the entrepreneur’s limited-commitment and the investors’ limited-liability
constraints. For sufficiently low values of \( s \) (when the entrepreneur is deep in debt) the entrepreneur’s participation constraint matters more and hence the firm under-invests. For sufficiently high values of \( s \) the investors’ value is close to zero and hence the investors’ limited-liability constraint has a stronger influence on investment. To ensure that \( s \) will drift back into the credit region the entrepreneur needs to “save” in the form of the illiquid productive asset (by increasing \( K \)) by borrowing more. By over-investing, the firm optimally manages to keep \( s \) between \(-\overline{d}\) and 0. Having analyzed the firm’s investment, we next turn to the entrepreneur’s consumption.

### 6.3.2 Consumption

![Figure 4: Consumption-capital ratio \( c(s) \) and the MPC \( c'(s) \).](image)

The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For the one-sided case, the entrepreneur always under-consumes and \( c(s) \) increases with \( s \). For the two-sided case, the entrepreneur may either under-consume or over-consume. In our two-sided example, for \(-0.131 < s \leq 0\), the entrepreneur over-consumes due to the investors’ limited-liability condition. The dotted line depicts the first-best permanent-income results: \( c(s) = \chi(s + q^{FB}) \) and MPC \( c'(s) = \chi = 5\% \).

The entrepreneur’s optimal consumption rule \( c(s) \) is given by:

\[
c(s) = \chi m(s)^{-1/\gamma} m(s),
\]  

(49)
where $\chi$ given in (16) is the classical marginal propensity to consume (MPC) as in Ramsey. Figure 4 plots the optimal consumption-capital ratio $c(s)$, and the MPC $c'(s)$ in Panels A and B respectively. For both one-sided and two-sided limited-commitment cases, the higher the financial slack $s$ the higher is the entrepreneur’s consumption. That is, $c(s)$ is increasing in $s$, as seen in the figure.

In the one-sided limited-commitment case $m(s) \to q^{FB} + s$ and the marginal value of liquidity $m'(s) \to 1$ as $s \to \infty$, and therefore $c(s) \to \chi(q^{FB} + s)$, the permanent-income benchmark result. When $s$ is bounded, so that $-\bar{d} \leq s < \infty$, consumption $c(s)$ is strictly below the first-best permanent-income benchmark (see the dotted line) and is concave in $s$ as Panel A illustrates. Panel B shows that the MPC $c'(s)$ decreases significantly with $s$ and approaches the permanent-income benchmark $\chi = 5\%$ as $s \to \infty$. Thus, financially constrained entrepreneurs deep in debt (with $s$ close to $-\bar{d}$) have MPCs that are substantially higher than the permanent-income benchmark.

In the two-sided limited-commitment case, the entrepreneur’s consumption can be either below or above the first-best benchmark consumption rule. The entrepreneur under-consumes for $s < -0.131$ but over-consumes for $-0.131 < s \leq 0$. At $s = 0$, $c(0) = 6.41\%$, which is greater than $c^{FB}(0) = \chi q^{FB} = 6\%$. Whether the entrepreneur under-consumes or over-consumes depends on the net effects from the entrepreneur’s limited-commitment and the investors’ limited-liability constraints. For sufficiently low values of $s$ (when she is deep in debt) the entrepreneur’s participation constraint matters more and hence the entrepreneur under-consumes in order to build up $s$. For sufficiently high values of $s$ the investors’ value is nearly equal to zero and hence the investors’ limited-liability constraint has a stronger influence on the entrepreneur’s consumption. To make sure that $s$ will drift back into the credit region the entrepreneur needs a high consumption rate financed by drawing down the credit line. Finally, note that the MPC $c'(s)$ is not monotonic in $s$ due to the interactions between the entrepreneur’s limited commitment and the investors’ limited-liability constraints.

Next we turn to the firm’s optimal financial policies.
6.3.3 Hedging via Futures

Before we delve into the details of corporate liquidity and risk management, we first review the entrepreneur’s total wealth portfolio which consists of three parts: (1) a 100% equity in the underlying business; (2) a zero-value mark-to-market futures position; and (3) a liquidity asset holding in the amount of $s$ (possibly negative in which case means borrowing.)

The entrepreneur’s optimal futures position $\phi(s)$ is given by

$$
\phi(s) = \frac{sm''(s)m(s) + \gamma m'(s)(m(s) - sm'(s))}{m(s)m''(s) - \gamma m'(s)^2}. \tag{50}
$$

By construction, the only liquid risky asset in this implementation is futures, which is marked to market and has zero market value at all times, therefore, all corporate liquidity $s$ must be held in the risk-free asset. (When $s < 0$, liquidity refers to the credit borrowed by the entrepreneur from the investors.)

Figure 5 plots the futures position $\phi(s)$. First, we note that under the first-best MM benchmark, the entrepreneur is fully insured from the capital shock by taking a short futures position with a short position, $\phi(s) = -q^{FB} = -1.2$. See the dotted line for the first-best complete hedging results for the entrepreneur with $\phi(s) = -q^{FB} = -1.2$ in Figure 5.

For both limited-commitment cases, the entrepreneur takes a short position in the futures to partially hedge the equity exposure to the underlying business, in that $\phi(s) < 0$. How does $|\phi(s)|$ depend on $s$? For the one-sided limited-commitment case, as the firm becomes less constrained (i.e. as $s$ increases), the entrepreneur increases the (absolute) size of the futures hedging position measured by $|\phi(s)|$. That is, less financially constrained firms hedge more (after controlling for firm size) and in the limit as $s \to \infty$, the entrepreneur can fully diversity the idiosyncratic business risk by taking a short futures position with a size of $-q^{FB} = -1.2$ achieving the first-best MM benchmark. Rampini, Sufi, and Viswanathan (2013) document less constrained firms hedge more.

For the two-sided limited-commitment case, the entrepreneur’s futures hedging position $|\phi(s)|$ is non-monotonic in $s$ where $s$ lies between $\underline{s} = -0.738$ and $\bar{s} = 0$. By requiring $\sigma^*(s) = 0$ at both boundaries $\underline{s} = -0.738$ and 0, we have $\phi(s) = \underline{s} = -0.738$ and $\phi(0) = 0$ by using (40). Therefore, the firm optimally chooses not to hedge at $s = 0$ as it fully pays back all
Figure 5: Futures hedging position $\phi(s)$ and savings $s$. The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, the entrepreneur takes a short position in the futures to partially hedge the equity exposure to the underlying business, in that $\phi(s) < 0$. For the one-sided case, interestingly, the total exposure $|\phi(s)|$ increases with $s$. For the two-sided case, $|\phi(s)|$ is non-monotonic in $s$ due to the interaction between the entrepreneur’s limited-commitment constraint and the investors’ limited-liability condition. The dotted line depicts the first-best complete hedging results for the entrepreneur with $\phi(s) = -q^{FB} = -1.2$.

its credit. Note that the maximal amount of hedge is $|\phi| = 1.137$ at $s = -0.408$. Intuitively, in the two-sided limited-commitment case, the maximal hedge occurs at an interior value of $s$ as the entrepreneur minimizes the total costs of financial frictions due to the entrepreneur’s limited-commitment and the investors’ limited-liability frictions.

One general take-away message from this comparative analysis between the one-sided and two-sided limited-commitment cases is that the investors’ limited-liability constraint can have important implications that are very different from and sometimes opposite to those implied by the entrepreneur’s limited-commitment constraint.
6.4 Risk Management via Shorting Stocks: An Alternative

Next, we provide an alternative implementation achieving the same resource allocation as the previous implementation (based on the bank savings/credit account and futures) does. However, this implementation has very different implications on debt capacity.

For the new implementation, we introduce a risky liquid financial asset that is perfectly correlated with the shock $Z$ for capital accumulation (1) so that the entrepreneur can choose how much capital shock $Z$ to hedge. Let $dR_t$ denote the incremental return for this risky asset over time period $(t, t + dt)$. Because investors are risk neutral, by using the standard equilibrium argument, we may write down $dR_t$ as follows,

$$dR_t = rdt + \sigma_K dZ_t.$$  \hfill (51)

Without loss of generality, we choose the volatility of this new risky asset to be $\sigma_K$. By setting the volatility of this risky asset to equal to the volatility of capital accumulation process, we essentially are requiring a unit short position in the risky asset provides an instantaneous perfect hedging against capital accumulation risk.

Let $\Omega_t$ denote the entrepreneur’s investment in this new risky asset, and hence the remaining liquid wealth, $S_t - \Omega_t$, is invested in the entrepreneur’s savings account earning the risk-free asset $r$. Because the entrepreneur can costlessly and continuously rebalance between this new publicly traded risky asset and the risk-free asset, we may write the evolution for the entrepreneur’s total liquid wealth $S_t$ as follows,

$$dS_t = (r(S_t - \Omega_t) + Y_t - C_t)dt + \Omega_t(rdt + \sigma_K dZ_t)$$

$$= (rS_t + Y_t - C_t)dt + \Omega_t \sigma_K dZ_t.$$  \hfill (52)

By comparing (52) with (33) (and ignoring the jump part in (33)), it is straightforward to conclude that the stock’s position $\omega(s) = \Omega/K$ in this new implementation is the same as the futures position $\phi(s)$ in the previous implementation, i.e. $\omega(s) = \phi(s)$. Unlike futures, the entrepreneur collects the short-sale proceeds, $-\Omega$, and invests the net proceeds $S - \Omega$ (after paying down the credit usage amount) in the savings account earning interests at the risk-free rate $r$. 

Figure 6: **Optimal hedge via the risky asset** \( \omega(s) \) and savings \( s - \omega(s) \). The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, the entrepreneur takes a short position in the risky liquid asset to partially hedge the equity exposure to the underlying business, in that \( \omega(s) < 0 \), and holds a risk-free savings account with (weakly) positive balances at all times. For the one-sided case, interestingly, the total size of the “short” position \( |\omega(s)| \) increases with \( s \). For the two-sided case, \( |\omega(s)| \) is non-monotonic in \( s \) due to the interaction between the entrepreneur’s limited-commitment constraint and the investors’ limited-liability condition. The dotted line depicts the first-best MM results: \( \omega(s) = -q^{FB} = -1.2 \) and savings \( s + q^{FB} \).

Figure 6 plots the hedging position via the risky liquid asset \( \omega(s) \) and the amount of risk-free asset holdings, \( s - \omega(s) \), which earns interests at the risk-free rate \( r \) in Panels A and B, respectively. First, as we have noted, the risky asset position is the same as the futures hedging position, i.e. \( \omega(s) = \phi(s) \) because the risky asset and futures (on the risky asset) have the same risk exposures \( \sigma_K dZ \). As for the futures, the entrepreneur needs to take a short position in the risky asset whose return is given by (51) in order to partially manage the risk exposure to the underlying illiquid business project.

While the savings amount under the futures hedging is simply \( s \), the scaled savings amount in this new implementation equals \( s - \omega(s) \neq s \) as shorting \( \omega(s) \) shares of stocks (per unit of capital) generate a sales proceed in the amount of \( -\omega(s) > 0 \). Panel B of Figure 6 shows that for both cases, the entrepreneur stochastically saves, i.e., \( s - \omega(s) > 0 \). For the one-sided case, scaled savings \( s - \omega(s) \) increases from zero to \( s + q^{FB} \) in the limit as we increase liquidity \( s \) from \(-0.692 \) towards \( \infty \). For the two-sided case, risk-free savings
$s - \omega(s)$ is non-monotonic. At both left and right boundaries $s = s = -0.738$ and $s = 0$, liquid savings $s - \omega(s) = 0$ equal zero, which follows from the requirement that volatility $\sigma^s(s) = -(s-\omega(s))\sigma_K$ at the boundaries must be zero. In the interior region $-0.738 < s < 0$, the savings amount, $s - \omega(s)$, first increases and then decreases with $s$ essentially inversely tracking the non-monotonicity of the hedge ratio $\omega(s)$.

While using different securities, the two implementations share two key features in common: (1) the total corporate liquidity summarized by $s$ and (2) the total amount of risk exposures.

## 7 Persistent Productivity Shocks: Insurance

In this section, we consider the model’s general case by allowing for persistent observable shocks to the firm’s productivity. First, it is natural to assume that these productivity shocks are observable and can be contracted on. Second, because these productivity shocks are persistent naturally they will affect the firm’s investment even in the neoclassical setting as expected. We will show there is an additional interaction effect due to financing constraint and persistent productivity shocks.\(^{19}\)

We explore the interaction effect of persistent productivity shocks and the entrepreneur’s limited commitment and the consequences for investment, consumption, managerial compensation, and liquidity and risk management. As we will show, persistent productivity shocks will naturally give rise to demand for insurance against the change of productivity. Equivalently, we show that default on debt as productivity decreases from $H$ to $L$ can be a natural equilibrium outcome.

We leave the solution for the optimal contracting problem to the Appendix and focus on an intuitive financial implementation with commonly used securities.

\(^{19}\)See DeMarzo, Fishman, He, and Wang (2012) for a model of optimal investment in a $q$-theoretic context with persistent shocks and agency frictions along the line of DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006).
7.1 Implementation: Liquidity and risk management

First, by using the homogeneity property, we write the entrepreneur’s certainty equivalent wealth function in state $n \in \{L, H\}$, $M(K, S, A^n)$, as follows,

$$M(K, S, A^n) = m_n(s)K.$$  \hfill (53)

The dynamic of scaled liquidity $s$. Given the state-contingent consumption-capital ratio $c_n(s)$, the investment-capital ratio $i_n(s)$, the hedge ratio $\phi_n(s)$ and the endogenous adjustment size $\pi_n(s)$ of liquidity holding as productivity switches out of state $n$, we write the dynamic of liquidity $s$ in the interior region as follows,

$$ds_t = \mu^s_n(s_t)dt + \sigma^s_n(s_t)dZ_t + \pi_n(s)dN_t,$$ \hfill (54)

where the drift and volatility processes $\mu^s(\cdot)$ and $\sigma^s(\cdot)$ for $s$ are given by

$$\mu^s_n(s) = (A^n - \pi_n(s)\lambda_n - i_n(s) - g(i_n(s)) - c_n(s)) + (r + \delta - i_n(s))s - \sigma_K\sigma^s_n(s),$$  \hfill (55)

$$\sigma^s_n(s) = (\phi_n(s) - s)\sigma_K.$$  \hfill (56)

Here, the last term in (54), $\pi_n(s)dN_t$, captures the effect of discrete productivity change on scaled liquidity $s$. $\pi_n(s)$ is the scaled insurance position.

The one-sided limited-commitment case. The following proposition summarizes the solution for the case with only the entrepreneur’s limited-commitment problem.

**Proposition 3** In the region $s > s_H$, the scaled value $m_H(s)$ in state $H$ solves the following ODEs,

$$0 = \max_{i_H, \pi_H} \frac{m_H(s)}{1 - \gamma} \left[ \gamma \chi m_H'(s)^{\gamma - 1} - \zeta \right] - \delta m_H(s) + [(r + \delta)s + A^n - \lambda_H\pi_H] m_H'(s)$$

$$+ i_H(m_H(s) - (s + 1)m_H'(s)) - g(i_H)m_H'(s) - \frac{\gamma\sigma^2_K}{2} \frac{m_H(s)^2 m_H''(s)}{m_H(s)m_H'(s) - \gamma m_H'(s)^2}$$

$$+ \frac{\lambda_H m_H(s)}{1 - \gamma} \left( \frac{m_L(s + \pi_H)}{m_H(s)} \right)^{1-\gamma} - 1,$$ \hfill (57)
subject to the following boundary conditions:

\[
\begin{align*}
\lim_{s \to \infty} m_H(s) &= q_H^{FB} + s, \\ m_H(s_H) &= \alpha m_H(0), \\ \lim_{s \to s_H} \sigma_H^s(s) &= 0 \quad \text{and} \quad \lim_{s \to s_H} \mu_H^s(s) \geq 0.
\end{align*}
\] (58) (59) (60)

The underlying arguments are very similar to the ones for the pure-diffusion case.

The two-sided limited-commitment case. For this two-sided limited commitment problem, we simply need to modify the condition at the upper boundary in Proposition 3. Note that the upper boundary is \( s = 0 \) rather than the natural limiting boundary \( s \to \infty \) for the one-sided limited-commitment case. We thus replace condition (58) with the following conditions at the new upper boundary \( s = 0 \) under state \( H \):

\[
\begin{align*}
\lim_{s \to 0} \sigma_H^s(s) &= 0 \quad \text{and} \quad \lim_{s \to 0} \mu_H^s(s) \leq 0.
\end{align*}
\] (61)

The arguments for (61) are essentially the same as those we have laid out earlier for the pure-diffusion case.

7.2 An Example

For illustration, we consider the simplest setting where the productivity jump from \( H \) to \( L \) is permanent and irreversible, in that \( \lambda_L = 0 \). We set \( \lambda_H = 0.1 \) and choose the productivity levels to be \( A^L = 0.18 \) and \( A^H = 0.2 \). And all the other parameter values remain the same as those for the pure-diffusion case. Figure 7 plots the results. These results are broadly in line with what we have shown earlier. Importantly, note that the lower boundary for state \( H \) is to the left of that for state \( L \), which makes intuitive sense, as the entrepreneur shall be able to borrow more in a more productive state, \textit{ceteris paribus}.

Panel A of Figure 8 plots the entrepreneur’s insurance demand \( \pi_H(s) \) in state \( H \) against the productivity change from state \( H \) to \( L \). As we see for all levels of \( s \), the entrepreneur pays a positive but time-varying insurance premium \( \lambda_H \pi_H(s) \) per unit of time in state \( H \).
Figure 7: The effect of firm’s persistent productivity shocks. Panels A, B, C, and D plot the entrepreneur’s scaled certainty equivalent wealth \( m_n(s) \), marginal (certainty equivalent) wealth of \( s \), \( m'_n(s) \), the investment-capital ratio \( i_n(s) \), and optimal hedge via the risky asset \( \omega_n(s) \), as functions of scaled liquidity \( s \), respectively. Parameter values: \( A^L = 0.18 \), \( A^H = 0.2 \), \( \lambda_L = 0 \), and \( \lambda_H = 0.1 \).

to investors in order to receive a lump-sum insurance payment in the amount of \( \pi_H > 0 \) from investors at the moment when the productivity state switches from \( H \) to \( L \). By doing so, the entrepreneur equates the marginal utility before and after the productivity changes whenever feasible. Interestingly, the insurance demand \( \pi_H(s) \) is non-monotonic in \( s \) as it first increases in liquidity \( s \) and then decreases with \( s \). The intuition is as follows. For a severely constrained entrepreneur whose \( s \) is close to the left boundary \( s \), the entrepreneur has limited funds to purchase insurance. Therefore, insurance \( \pi_H(s) \) increases as \( s \) moves towards the origin turning less negative. As \( s \) becomes sufficiently close to the origin, the entrepreneur’s demand for insurance decreases for the following reasons. First, the entrepreneurial firm has more liquidity to self insurance and hence demand for additional liquidity decreases. Second,
Figure 8: The insurance position (against the change of productivity from state $H$ to $L$) $\pi_H(s)$, and liquidity $s + \pi_H(s)$, as functions of scaled liquidity $s$. Parameter values: $A^L = 0.18$, $A^H = 0.2$, $\lambda_L = 0$, and $\lambda_H = 0.1$.

the entrepreneur’s decreasing marginal utility also suggests that the entrepreneur’s demand for insurance is decreases with liquidity, *ceteris paribus*. Additionally, the investors’ limited-liability constraint requires $\pi_H(s) \leq -s$, which in turns truncates the insurance demand. For these reasons, the insurance demand $\pi_H(s)$ is non-monotonic in liquidity $s$ as shown in Panel A of Figure 8.

Panel B of Figure 8 plots the post-productivity-change liquidity level $s + \pi_H(s)$ immediately following the change of productivity from state $H$ to $L$. Note that $s + \pi_H(s)$ is increasing in $s$, which makes intuitive sense. The higher the liquidity level in the current state $H$, the higher the post-productivity-change liquidity level $s + \pi_H(s)$ (ignoring the diffusion part.) We have already commented that $s + \pi_H(s) = 0$ for sufficiently high level of $s$ because the demand for insurance reaches the constrained maximum level $\pi_H(s) = -s$, as $s \leq 0$ is required in both states $H$ and $L$ due to the investors’ limited-liability condition. For firms that are not much in debt (with $s$ sufficiently close to the origin,) the firm has sufficient liquidity and it is optimal to choose the constrained maximal insurance $\pi_H(s) = -s$ against the stochastic productivity change.
8 Deterministic Case à la Hart and Moore

The Hart and Moore (1994) model can be viewed as a special case of our model with no shocks: When $\sigma_K = 0$ and $A^L = A^H = A$. When consumption per unit of capital $c_t$ and investment per unit of capital $i_t$ are set the entrepreneur’s (scaled) liquidity $s$ then grows deterministically as follows:

$$\mu^s(s_t) \equiv \frac{ds_t}{dt} = (r + \delta - i_t)s_t + A - i_t - g(i_t) - c_t. \quad (62)$$

Let $\mu^s_{FB}(s_t)$ denote the drift $\mu^s(s_t)$ in the first-best setting where the investment-capital ratio equals $i^{FB}$ given in (46) and the consumption-capital ratio equals $c^{FB} = \chi(s_t + q^{FB})$. It is straightforward to show that we then have:

$$\mu^s_{FB}(s_t) = (\delta - i^{FB} - \gamma^{-1}(\zeta - r)) m^{FB}(s_t).$$

As the entrepreneur’s first-best scaled wealth is nonnegative, ($m^{FB}(s_t) = (s_t + q^{FB}) > 0$) it immediately follows that the drift $\mu^s_{FB}(s_t) \geq 0$ if and only if the following condition holds:

**Condition A:** $i^{FB} \leq \delta + \frac{r - \zeta}{\gamma}, \quad (63)$

where $i^{FB}$ is given by (46).

There are then two mutually exclusive cases depending on the sign of $\mu^s_{FB}(s_t)$ (whether Condition A is satisfied or not). In both cases the entrepreneur’s scaled certainty equivalent wealth $m(s)$ satisfies the following ODE in the interior region:

$$0 = \frac{m(s)}{1 - \gamma} \left[ \gamma \chi(m'(s))^{\frac{\gamma - 1}{\gamma}} - \zeta \right] - \delta m(s) + [(r + \delta)s + A] m'(s) \quad + i(s)(m(s) - (s + 1)m'(s)) - g(i(s))m'(s). \quad (64)$$

The differences in the two cases are only reflected in the boundary conditions. In the case where Condition A holds we then obtain the following solution.

**Proposition 4** When Condition A (63) is satisfied we have $\mu^s_{FB}(s_t) \geq 0$ and:
1. In the one-sided limited-commitment case the entrepreneur chooses the first-best investment and consumption policies despite being financially constrained and her wealth is \( m(s) = s + q^{FB} \);

2. In the two-sided limited-commitment case the following two conditions must hold at the lower boundary \( \underline{s} \):

\[
\begin{align*}
m(\underline{s}) & = \alpha m(0), \\
\mu^*(\underline{s}) & = 0. \quad (65)
\end{align*}
\]

And to satisfy investors’ limited liability condition the following additional boundary condition must hold:

\[ \mu^*(0) = 0. \]

Figure 9 plots the entrepreneur’s wealth \( m(s) \), marginal value of financial slack \( m'(s) \), investment-capital ratio \( i(s) \), and the drift \( \mu^*(s) \) under Condition A (63). The solid and the dashed lines correspond to the one-sided and two-sided cases respectively. The dotted line gives the first-best solution.\(^{20}\) If the first-best allocation is feasible under the optimal contract at all times then it must then be the optimal solution. Hence, the question: under what conditions is the first-best allocation feasible? It is in the one-sided limited-commitment case but only when Condition A (63) is satisfied. If \( \mu^*_{FB}(s_t) > 0 \) then \( s_t \) increases with time \( t \) under the first-best investment and consumption policies. Therefore, the entrepreneur’s limited-commitment constraint never binds, so that the first-best outcome is achieved.

In the two-sided limited-commitment case the optimal contract requires that \( s_t \leq 0 \) in order to satisfy that investors’ limited-liability condition \( \mu^*(0) = 0 \). Panel D in Figure 9 illustrates the impact of investors’ limited-commitment constraint on \( \mu^*(\cdot) \). How does the entrepreneur lower the drift \( \mu^*(\cdot) \) so as to ensure that \( \mu^*(0) = 0 \)? She does this by increasing investment and consumption to the point that \( m'(s) < 1 \) as shown in Panel B.

\(^{20}\) Technically, the solution for our deterministic case is an initial value problem rather than a boundary value problem. We may solve the problem starting from \( s = 0 \) for the two-sided case. We can easily solve for the two unknowns \( m(0) \) and \( m'(0) \) by using the two equations, the ODE (64) and \( \mu^*(0) = A - i(0) - g(i(0)) - c(0) = 0 \). Note that \( i(s) \) and \( c(s) \) are also functions of \( m(s) \) and \( m'(s) \). Once we have the value of \( m(0) \) and \( m'(0) \), we can then use the ODE (64) to solve \( m(s) \) for the entire range of \( s \) as an initial value problem.
Figure 9: The entrepreneur’s scaled certainty equivalent wealth $m(s)$, marginal value of liquidity $m'(s)$, investment-capital ratio $i(s)$ and drift $\mu^s(s)$. Parameter values: $r = 0.05, \zeta = 0.05, \gamma = 2, \delta = 0.125, \theta = 2, A = 0.2$ and $\alpha = 0.4$ implying $i^{FB} = 0.1$ and $q^{FB} = 1.2$. This figure corresponds to the case with $\mu^s_{FB}(s_t) > 0$, i.e. when Condition A is satisfied as in Proposition 4. Solutions for the first-best and the one-sided case coincide for the region $s \geq s = -q^{FB} = -1.2$. For the two-sided case, the firm over-invests in the entire admissible region $s = -0.724 \leq s \leq 0$ compared to the first-best benchmark.
In other words, on net, financial slack has a negative effect on wealth $m(s)$. Over-supply of corporate liquidity causes the entrepreneur to over-consume and over-invest, which we see from Panel C. The higher the value of $s$ the lower the marginal value of liquidity $m'(s)$ and the higher the degree of over-investment. Finally, the additional investors’ limited-liability constraint lowers firm value and reduces the amount of liquidity supplied $ex$ $ante$, which we can see from the reduction of $|s|$ from $q^{FB} = 1.2$ to 0.724.

We next consider the other case when Condition A (63) is violated.

**Proposition 5** When Condition A (63) is violated, $\mu(s) < 0$ and $m(s)$ satisfies the ODE (64) subject to following boundary conditions:

1. In the one-sided limited-commitment case the boundary conditions are given by (65) and (66).

2. In the two-sided limited-commitment case the same boundary conditions (65) and (66) must hold and only $s$ in the range $\underline{s} \leq s \leq \bar{s}$ is feasible under the optimal contract.

Figure 10 plots the entrepreneur’s wealth $m(s)$, marginal value of liquidity $m'(s)$, investment-capital ratio $i(s)$ and the drift $\mu^*(s)$ when Condition A (63) is violated. Again, the solid and the dashed lines correspond to the one-sided and two-sided cases respectively. The dotted line gives the first-best solution. For this case the only parameter change is that the productivity parameter is now $A = 0.205$ instead of $A = 0.2$. Under the new value for $A$ the first-best investment-capital ratio $i^{FB}$ increases from 0.1 to 0.15, Tobin’s $q$ increases from 1.2 to 1.3, and the drift changes from $\mu_{FB}^*(s_l) = -0.025 \times (1.2 + s_l) < 0$ to $\mu_{FB}^*(s_l) = 0.025 \times (1.3 + s_l) > 0$.

When Condition A is violated and $\mu_{FB}^*(s_l) < 0$ targeting the first-best investment and consumption allocation drains liquidity $s$ even over a small time interval. Accordingly, both current and future investment is below the first-best benchmark. The drift $\mu^*(s)$ then approaches zero as the firm reaches the endogenous left limit $\underline{s}$ where $\mu^*(\underline{s}) = 0$. In fact, the left boundary is an absorbing boundary and the firm will permanently stay at $\underline{s}$ upon reaching that point. For all other values of $s$ the drift satisfies $\mu^*(s) < 0$ given that the marginal value of liquidity $m'(s) > 1$, as we see from Panel D. Moreover, the lower is $s$ the
Figure 10: The entrepreneur’s scaled certainty equivalent wealth \( m(s) \), marginal value of liquidity \( m'(s) \), investment-capital ratio \( i(s) \) and drift \( \mu^s(s) \). Parameter values: \( r = 0.05, \zeta = 0.05, \gamma = 2, \delta = 0.125, \theta = 2, A = 0.205 \) and \( \alpha = 0.4 \) implying \( i^{FB} = 0.15 \) and \( q^{FB} = 1.3 \). This figure corresponds to the case with \( \mu^s_{FB}(s_t) < 0 \), i.e. when Condition A is violated as in Proposition 5. Solutions for the one-sided and the two-sided cases are the same in their common region \(-0.769 = s \leq s \leq 0\). For the one-sided case, \( s \leq s \). For both cases, the firm is financially constrained and optimally under-invests compared to the first-best benchmark.
more valuable is liquidity, as reflected in the higher marginal value of liquidity $m'(s)$, and the lower is investment (see Panels B and C).

In the two-sided case, perhaps surprisingly, the solution is identical to that for the one-sided case. The intuition is as follows. In the parameter region where that Condition A is violated, the drift for the one-sided case is already weakly negative for all values of $s$, as we have discussed. Therefore, introducing an investors’ participation constraint has no additional effect.

We can completely characterize the dynamics of the deterministic model for a given initial liquidity $s_0$. Figure 11 plots the dynamics of liquidity $s_t$ and investment $i_t$ with $s_0 = -0.4$ for the two-sided case. When $A = 0.20$, and Condition A (63) is satisfied, the firm always over-invests and $i_t$ increases from $i_0 = 0.103$ to $i_{26.9} = 0.12$, then stays flat at that level for all $t \geq 26.9$. Despite the over-investment, the firm’s scaled liquidity $s_t$ increases from $s_0 = -0.4$ to $s_{26.9} = 0$ and thereafter stays permanently at the origin. When $A = 0.205$, and
Condition A (63) is violated, the firm always under-invests and \( i_t \) decreases from \( i_0 = 144 \) to \( i_{43.2} = 0.127 \), then stays flat at that level for all \( t \geq 43.2 \). Investment is financed by the firm depleting its liquidity over time from \( s_0 = -0.4 \) to \( s_{43.2} = -0.769 \), at which point it stays flat permanently at \( s = -0.769 \). In sum, the firm’s investment decisions are always distorted over time and reach the most distorted levels as the limited-commitment constraints bind in the long run.

While our deterministic case shares key features with Hart and Moore (1994), it differs from the Hart and Moore in several significant ways. First, our model generates a unique repayment path for investors due to the entrepreneur’s concave utility function, while Hart and Moore have a continuum of repayment paths due to their risk-neutrality and equal discount rates assumptions. Second, our deterministic model allows for dynamic capital accumulation while Hart and Moore (1994) only have a one-shot investment decision at time 0. By analyzing investment dynamics we generate novel insights. For example, investors’ limited-liability constraint can imply that liquidity is costly, in which case over-investment is optimal in order to reduce corporate liquidity.

9 Alternative Specifications of Outside Options

The critical assumptions of our framework are \((i)\) the inalienability of human capital and \((ii)\) the investors’ inability to fund the operating losses indefinitely. For expositional simplicity, the specific constraints we have chosen are \((i)\) the entrepreneur’s ability to walk away starting a new firm with a firm whose size is \( \alpha \) fraction of the current firm, and \((ii)\) the limited-liability constraint for investors at all times. However, it is important to note that our model’s main results and key insights hold under much broader settings. We next consider \((i)\) an alternative specification that will pin down the entrepreneur’s outside option and \((ii)\) one important generalization on investors’ limited-liability constraint along the line of Hart and Moore (1994).
9.1 Autarky as the Entrepreneur’s Outside Option

Model setup. Now we provide an alternative interpretation for the entrepreneur’s outside option based on the cost of losing intertemporal consumption-smoothing opportunities. Instead of assuming that the entrepreneur can divert $\alpha$ fraction of capital stock and start afresh, we assume that the entrepreneur always has an option to freely walk away from the investors. However, by doing so, the entrepreneur will permanently lose all future borrowing, saving, and insurance possibilities by remaining in autarky, as assumed in Bulow and Rogoff (1989) and the follow-up international macro literature.

Let $\hat{J}(K_t)$ denote the entrepreneur’s value function under autarky defined as follows,

$$\hat{J}(K_t) = \max I \mathbb{E}_t \left[ \int_t^\infty \zeta e^{-\zeta(v-t)} U(Y_v)dv \right], \quad (67)$$

where the entrepreneur’s consumption is given by $C_t = Y_t = A_t K_t - I_t - G_t$. The following proposition summarizes the entrepreneur’s value function $\hat{J}(K)$ and the certainty equivalent wealth $\hat{M}(K)$ under autarky for the pure diffusion case.

**Proposition 6** Under autarky, the entrepreneur’s value function $\hat{J}(K)$ is given by

$$\hat{J}(K) = \frac{(b \hat{M}(K))^{1-\gamma}}{1-\gamma}, \quad (68)$$

where $b$ is given by (14) and $\hat{M}(K)$ is the entrepreneur’s certainty equivalent wealth given by

$$\hat{M}(K) = \hat{m}K, \quad (69)$$

where

$$\hat{m} = \frac{(\zeta(1 + g'(\hat{i}))(A - \hat{i} - g(\hat{i}))^{-\gamma})^{1-\gamma}}{b}, \quad (70)$$

and $\hat{i}$ is the optimal investment-capital ratio solving the following implicit equation:

$$\zeta = \frac{A - \hat{i} - g(\hat{i})}{1 + g'(\hat{i})} + (\hat{i} - \delta)(1 - \gamma) - \frac{\sigma^2 K \gamma (1 - \gamma)}{2}. \quad (71)$$

By following essentially the same analysis in Section 6, we conclude that the lower bound-
ary $s$ for liquidity $s$ is determined by:

$$m(s) = \hat{m}.$$  \hspace{1cm} (72)

Therefore, for both the one-sided and two-sided limited-commitment cases, we only need to replace the previous boundary condition (43) in Proposition 2 with the new condition boundary condition (72) and keep all the other conditions are kept unchanged. Intuitively, the lower boundary $s$ for this new case is determined solely by (72) independent of the upper boundary $\xi$, which is very different from the benchmark specification where the entrepreneur can divert $\alpha$ fraction of capital stock and start a new firm free of liability.

**Analysis.** Figure 12 plots the entrepreneur’s scaled certainty equivalent wealth $m(s)$, the marginal value of liquidity $m'(s)$, optimal investment-capital ratio $i(s)$, and optimal hedging position (scaled stock position) $\omega(s)$ for both one-sided and two-sided limited-commitment cases. The general patterns for all four variables remain valid. For example, for the one-sided case, the firm always under-invests and the marginal value of liquidity $m'(s)$ is always greater than one. Additionally, the degree of underinvestment weakens and the marginal value of liquidity $m'(s)$ decreases, both of which eventually approach the first-best levels $i^{FB} = 0.10$ and unity, respectively, as $s \to \infty$. Finally, the optimal hedge size, $|\phi(s)|$ also increases and approaches the first-best level $q^{FB}$ for $s \to \infty$.

### 9.2 Investors’ Alternative Use of Capital (Hart and Moore, 1994)

We now consider a more general specification for the investors’ outside option. Suppose that investors’ alternative use of capital can yield a value of $\ell K_t$ at any time $t$, where $\ell > 0$ is a constant. For example, investors may simply liquidate the asset in the market or hire a less skilled manager delivering $\ell K$ to investors. Indeed, this latter case is the assumption on the investors’ side in Hart and Moore (1994). As a result, *ex ante* investors cannot credibly commit to a long-term contract when investors’ value may fall below $\ell$ per unit of capital on the equilibrium path, effectively making the investors’ participation constraint even tighter. Vice versa, we could imagine that investors may be able to provide some collateral *ex ante* to the entrepreneur in an escrow account that can be seized by the entrepreneur should investors
Figure 12: **The case with autarky as the entrepreneur’s outside option.** Panels A, B, C, and D plot the entrepreneur’s scaled certainty equivalent wealth \( m(s) \), marginal (certainty equivalent) wealth of \( s, m'(s) \), the investment-capital ratio \( i(s) \), and optimal hedge via the risky asset \( \omega(s) \), as functions of scaled liquidity \( s \), respectively. The solid and dashed lines correspond to the one-sided and two-sided limited-commitment cases, respectively. For both cases, \( m(s) \) is increasing and concave. For the one-sided case, \( s \geq -0.764 \). For the two-sided case, \(-0.720 \leq s \leq 0\). The dotted line depicts the first-best MM results: \( m(s) = q^{FB} + s \), the sensitivity \( m'(s) = 1 \), the investment-capital ratio \( i(s) = i^{FB} = 0.1 \) and \( \omega(s) = -q^{FB} = -1.2 \).

renege *ex post* on the long-term contract. Pledging collateral in advance then relaxes the investors’ limited liability constraint.

To introduce any of these changes into the model, we only need to modify the upper boundary condition as follows:

\[
F(K, \overline{V}(K)) \geq \ell K,
\]

where \( \overline{V}(K) \) is the endogenous upper boundary. The case with \( \ell > 0 \) then corresponds to
the Hart-Moore framework, where investors have an alternative use of capital. And when \( \ell < 0 \), we may interpret \( \ell \) as the amount of personal guarantee offered by investors *ex ante* in an escrow account that can be seized by the entrepreneur should the investors renege on the contract.

10 Conclusion

Our generalization of Hart and Moore (1994) to introduce risky human capital and cash flows, risk aversion of the entrepreneur, and ongoing consumption reveals the optimality of corporate liquidity and risk management for financially constrained firms. Most of the existing corporate security design literature has confined itself to showing that debt financing and credit line commitments are optimal financial contracts. By adding risky human capital and risk aversion for the entrepreneur, two natural assumptions, we show that corporate hedging policies are also an integral part of an optimal financial contract. When productivity shocks are persistent, we find that insurance contract and/or equilibrium default by the entrepreneur on her debt obligations is part of an optimal contract.

We have thus shown that the inalienability of human capital constraint naturally gives rise to a role for corporate liquidity and risk management, dimensions that are typically absent from existing macroeconomic theories of investment under financial constraints following Kiyotaki and Moore (1997).
References


Appendices

To be added.