ON THE MAXIMUM AND MINIMUM RESPONSE TO AN IMPULSE IN SVARS

Pepe Montiel Olea (NYU) and Bulat Gafarov (PSU)

September 2, 2014
Plan for Today

→ Estimate and conduct frequentist inference on:
Plan for Today

→ Estimate and conduct frequentist inference on:

The maximum (or minimum) response of $y_{t+k}$ to a structural shock $\epsilon_t$. 
Plan for Today

→ Estimate and conduct frequentist inference on:

The maximum (or minimum) response of $y_{t+k}$ to a structural shock $\varepsilon_t$.

★ Examples:
Plan for Today

→ Estimate and conduct frequentist inference on:

The maximum (or minimum) response of $y_{t+k}$ to a structural shock $\epsilon_t$.

★ Examples:

i) What is the maximum effect of a $\sigma$-structural uncertainty shock over household spending?
Plan for Today

→ Estimate and conduct frequentist inference on:

The maximum (or minimum) response of \( y_{t+k} \) to a structural shock \( \epsilon_t \).

★ Examples:

i) What is the maximum effect of a \( \sigma \)-structural uncertainty shock over household spending?

ii) What is the maximum effect of a \( \sigma \)-structural monetary shock over \( y_{t+k} \), \( \pi_{t+k} \) or \( r_{t+k} \)?
Plan for Today

→ Estimate and conduct frequentist inference on:

The maximum (or minimum) response of $y_{t+k}$ to a structural shock $\epsilon_t$.

★ Examples:

i) What is the maximum effect of a $\sigma$-structural uncertainty shock over household spending?

ii) What is the maximum effect of a $\sigma$-structural monetary shock over $y_{t+k}, \pi_{t+k}$ or $r_{t+k}$?

iii) What is the maximum effect of a $\sigma$-structural oil shock over $y_{t+k}, \pi_{t+k}$ or $P_{t+k}$?
Main Messages of this Talk

1. The maximum and minimum are ‘point-identified’ in SVARs (even if impulse response functions are not).
Main Messages of this Talk

1. The maximum and minimum are ‘point-identified’ in SVARs
   (even if impulse response functions are not).

2. Efficient delta-method inference available for these objects.
   (max and min depend nicely on reduced-form VAR parameters)
Main Messages of this Talk

1. The maximum and minimum are ‘point-identified’ in SVARs (even if impulse response functions are not).

2. Efficient delta-method inference available for these objects. (max and min depend nicely on reduced-form VAR parameters)

3. Our inference procedure can accommodate alternative sources of identification (sign restrictions or external instruments).
Related Work

a) ‘Partially Identified’ SVARs

Faust (1998); Uhlig (2005); Moon, Schorfheide, Granziera (2013)

b) Value functions of Stochastic Programs

Faccio-Ishizuka (1990); Shapiro (1991); Santos and Fang (2014)
**Related Work**

a) ‘**Partially Identified**’ SVARs

Faust (1998); Uhlig (2005); Moon, Schorfheide, Granziera (2013)

Min/Max will give efficient estimators of the ‘identified set’.

b) **Value functions of Stochastic Programs**

Faccio-Ishizuka (1990); Shapiro (1991); Santos and Fang (2014)
Related Work

a) ‘Partially Identified’ SVARs

Faust (1998); Uhlig (2005); Moon, Schorfheide, Granziera (2013)

b) Value functions of Stochastic Programs

Faccio-Ishizuka (1990); Shapiro (1991); Santos and Fang (2014)
**Related Work**

a) ‘**Partially Identified’** SVARs

Faust (1998); Uhlig (2005); Moon, Schorfheide, Granziera (2013)

b) **Value functions of Stochastic Programs**

Faccio-Ishizuka (1990); Shapiro (1991); Santos and Fang (2014)

Min/Max will be cont. and directionally differentiable functions.
Outline

1. Notation
2. Identification and Estimation
3. Illustrative Example
4. Sign-Restricted SVARs
5. Frequentist Inference
1. Notation

2. Identification and Estimation

3. Illustrative Example

4. Sign-Restricted SVARs

5. Frequentist Inference
Outline

1. Notation

2. Identification and Estimation

3. Illustrative Example

4. Sign-Restricted SVARs

5. Frequentist Inference
Outline

1. Notation

2. Identification and Estimation

3. Illustrative Example

4. Sign-Restricted SVARs

5. Frequentist Inference
Outline

1. Notation
2. Identification and Estimation
3. Illustrative Example
4. Sign-Restricted SVARs
5. Frequentist Inference
1. Notation
SVAR(p)

⋆ Vector Autoregression for the $n$-dimensional vector $Y_t$: 
SVAR(p)

Vector Autoregression for the \(n\)-dimensional vector \(Y_t\):

\[ Y_t = A_1 Y_{t-1} + \ldots + A_p Y_{t-p} + \eta_t, \]
SVAR(P)

★ Vector Autoregression for the $n$-dimensional vector $Y_t$:

$$Y_t = A_1 Y_{t-1} + \ldots A_p Y_{t-p} + \eta_t,$$

$$A \equiv (A_1, A_2, \ldots, A_p) \quad \text{and} \quad \Sigma \equiv \mathbb{E}[\eta_t \eta_t'],$$
SVAR(p)

Vector Autoregression for the $n$-dimensional vector $Y_t$:

$$Y_t = A_1 Y_{t-1} + \ldots A_p Y_{t-p} + \eta_t,$$

$$A \equiv (A_1, A_2, \ldots, A_p) \quad \text{and} \quad \Sigma \equiv \mathbb{E}[\eta_t \eta_t^\prime].$$

Structural Model for the vector of Forecast Errors $\eta_t$: 
$\textbf{SVAR}(p)$

\begin{itemize}
  \item Vector Autoregression for the $n$-dimensional vector $Y_t$:

  $$Y_t = A_1 Y_{t-1} + \ldots A_p Y_{t-p} + \eta_t,$$

  $$A \equiv (A_1, A_2, \ldots A_p) \quad \text{and} \quad \Sigma \equiv \mathbb{E}[\eta_t \eta_t'].$$ 

  \item Structural Model for the vector of Forecast Errors $\eta_t$:

  $$\eta_t = H \varepsilon_t, \quad H \in \mathbb{R}^{n \times n}.$$
\end{itemize}
Orthogonal SVAR($p$)

Key Assumption of the Structural Model: Orthogonality

$$\mathbb{E}[\varepsilon_t \varepsilon_t'] = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \equiv D$$
ORTHOGONAL SVAR(p)

- Key Assumption of the Structural Model: Orthogonality

\[ \mathbb{E}[\varepsilon_t \varepsilon'_t] = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \equiv D \]

\[ (\eta_t = H\varepsilon_t) + \text{(Orthogonality)} \implies \]
**Orthogonal SVAR(p)**

* Key Assumption of the Structural Model: Orthogonality

\[
E[\varepsilon_t \varepsilon_t'] = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \equiv D
\]

* \((\eta_t = H\varepsilon_t) + \text{(Orthogonality)} \implies\)

\[
\Sigma \equiv E[\eta_t \eta_t'] = E[H\varepsilon_t \varepsilon_t' H'] = HDH'
\]
ORTHOGONAL SVAR(p)

★ Key Assumption of the Structural Model: Orthogonality

\[ \mathbb{E}[\varepsilon_t \varepsilon_t'] = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2) \equiv D \]

★ \((\eta_t = H\varepsilon_t) + \text{(Orthogonality)} \implies \]

\[ \Sigma \equiv \mathbb{E}[\eta_t \eta_t'] = \mathbb{E}[H\varepsilon_t \varepsilon_t' H'] = HDH' \]

★ The ‘structural’ parameter \(HD^{1/2}\) is restricted.
Quick Observation

Orthogonal SVAR(p) are partially identified
(no need of sign restrictions)
STRUCTURAL IMPULSE RESPONSE FUNCTION

★ Moving-Average Representation of the SVAR
STRUCTURAL IMPULSE RESPONSE FUNCTION

Moving-Average Representation of the SVAR

\[ Y_t = \sum_{k=0}^{\infty} C_k(A)H\epsilon_{t-k}, \]
Moving-Average Representation of the SVAR

\[ Y_t = \sum_{k=0}^{\infty} C_k(A) H\varepsilon_{t-k}, \]

\( C_k(A) \) is a nonlinear transformation of \( A \).
**Structural Impulse Response Function**

★ Moving-Average Representation of the SVAR

\[ Y_t = \sum_{k=0}^{\infty} C_k(A)H\varepsilon_{t-k}, \]

\( C_k(A) \) is a nonlinear transformation of \( A \).

★ Structural Impulse Response Functions (variable \( i \), shock \( j \))
Structural Impulse Response Function

★ Moving-Average Representation of the SVAR

\[ Y_t = \sum_{k=0}^{\infty} C_k(A)H\varepsilon_{t-k}, \]

\( C_k(A) \) is a nonlinear transformation of \( A \).

★ Structural Impulse Response Functions (variable \( i \), shock \( j \))

\[ \frac{\partial Y_{i,t+k}}{\partial \varepsilon_{jt}} \cdot \sigma_j \equiv \text{IRF}_{k,ij}(A, \Sigma) = e_i' C_k(A)He_j \sigma_j \]
Introduction  Notation  Identification and Estimation  Example  Sign Restrictions  Inference

**Structural Impulse Response Function**

★ Moving-Average Representation of the SVAR

\[ Y_t = \sum_{k=0}^{\infty} C_k(A)H\varepsilon_{t-k}, \]

\(C_k(A)\) is a nonlinear transformation of \(A\).

★ Structural Impulse Response Functions (variable \(i\), shock \(j\))

\[
\frac{\partial Y_{i,t+k}}{\partial \varepsilon_{jt}} \cdot \sigma_j \equiv \text{IRF}_{k,ij}(A, \Sigma) = e_i'C_k(A)He_j\sigma_j
\]

\(e_i\) is the \(i\)-th column of \(I_n\). So \(He_j\) is \(H_j\), the \(j\)-th column of \(H\).
2. Identification and Estimation
Main Message of this Section:

The minimum and maximum response are ‘point-identified’
(even if the structural IRF is not)
THE STRUCTURAL $\text{IRF}_{k,ij}$ IS PARTIALLY IDENTIFIED

* The reduced-form parameters
The structural IRF $k_{ij}$ is partially identified

- The reduced-form parameters

$(A, \Sigma)$
The structural IRF_{k,ij} is partially identified

- The reduced-form parameters
  
  \[(A, \Sigma)\]

  are compatible with any
THE STRUCTURAL $\text{IRF}_{k,ij}$ IS PARTIALLY IDENTIFIED

* The reduced-form parameters

$$(A, \Sigma)$$

are compatible with any

$$\text{IRF}_{k,ij} \equiv e'_i C_k(A) H_j \sigma_j$$
The structural \( \text{IRF}_{k,ij} \) is partially identified

- The reduced-form parameters
  \[
  (A, \Sigma)
  \]
  are compatible with any
  \[
  \text{IRF}_{k,ij} \equiv e_i' C_k(A) H_j \sigma_j
  \]
  such that
  \[
  H_j \sigma_j \text{ satisfies } HDH' = \Sigma
  \]
The structural IRF \( F_{k,ij} \) is partially identified

- The reduced-form parameters \((A, \Sigma)\)

  are compatible with any

  \[
  \text{IRF}_{k,ij} \equiv e_i'C_k(A)H_j\sigma_j
  \]

  such that

  \[
  H_j\sigma_j \quad \text{satisfies} \quad HDH' = \Sigma
  \]

- However, ...
THE MINIMUM RESPONSE IN THE POPULATION IS IDENTIFIED

★ Consider the mathematical program

\[
\min_{H_j\sigma_j \in \mathbb{R}^n} e_i' C_k(A) H_j \sigma_j
\]
The minimum response in the population is identified

★ Consider the mathematical program

$$\min_{H_j\sigma_j \in \mathbb{R}^n} e_i' C_k(A) H_j \sigma_j$$

subject to the quadratic equality constraint
The minimum response in the population is identified

Consider the mathematical program

\[
\min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k(A) H_j \sigma_j
\]

subject to the quadratic equality constraint

\[HDH' = \Sigma\]
The minimum response in the population is identified

★ Consider the mathematical program

$$\min_{H_j\sigma_j \in \mathbb{R}^n} e_i' C_k(A) H_j \sigma_j$$

subject to the quadratic equality constraint

$$HDH' = \Sigma$$

★ \((A, \Sigma)\) maps to the unique value of the program above:
The minimum response in the population is identified

★ Consider the mathematical program

\[
\min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k(A) H_j \sigma_j
\]

subject to the quadratic equality constraint

\[
HDH' = \Sigma
\]

★ \((A, \Sigma)\) maps to the unique value of the program above:

\[
\underline{\nu}_k(A, \Sigma) \equiv -\sqrt{e_i' C_k(A) \Sigma C_k(A)' e_i}
\]
Assumption 0: There exist estimators $(\hat{A}, \hat{\Sigma})$ such that
**Assumption 0:** There exist estimators \((\hat{A}, \hat{\Sigma})\) such that

\[
\begin{pmatrix}
\hat{A} - A \\
\hat{\Sigma} - \Sigma
\end{pmatrix}
\xrightarrow{d} N_d(0, \Omega).
\]
**Assumption 0:** There exist estimators $(\hat{A}, \hat{\Sigma})$ such that

$$\left( \begin{array}{c} \hat{A} - A \\ \hat{\Sigma} - \Sigma \end{array} \right) \overset{d}{\rightarrow} \mathcal{N}_d(0, \Omega).$$

(no unit-roots, no singular covariance matrices)
Estimation and Inference

★ Assumption 0: There exist estimators \((\hat{A}, \hat{\Sigma})\) such that

\[
\begin{pmatrix}
\hat{A} - A \\
\hat{\Sigma} - \Sigma
\end{pmatrix} \xrightarrow{d} \mathcal{N}_d(0, \Omega).
\]

(no unit-roots, no singular covariance matrices)

★ We will consider the plug-in estimator:
Assumption 0: There exist estimators $(\hat{A}, \hat{\Sigma})$ such that

$$\begin{pmatrix} \hat{A} - A \\ \hat{\Sigma} - \Sigma \end{pmatrix} \xrightarrow{d} \mathcal{N}_d(0, \Omega).$$

(no unit-roots, no singular covariance matrices)

We will consider the plug-in estimator:

$$\nu(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A}) \hat{\Sigma} C_k(\hat{A})' e_i},$$
**Estimation and Inference**

★ ASSUMPTION 0: There exist estimators \((\hat{A}, \hat{\Sigma})\) such that

\[
\begin{pmatrix}
\hat{A} - A \\
\hat{\Sigma} - \Sigma
\end{pmatrix} \xrightarrow{d} \mathcal{N}_d(0, \Omega).
\]

(no unit-roots, no singular covariance matrices)

★ We will consider the plug-in estimator:

\[
\nu(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A}) \hat{\Sigma} C_k(\hat{A})' e_i},
\]

★ We provide conditions for consistency and study asy. dist.
**Main Idea for Asymptotic Distribution**

- The function
Main Idea for Asymptotic Distribution

★ The function

\[ \nu_k(\hat{A}, \hat{\Sigma}) = -\sqrt{e' \mathcal{C}_k(\hat{A}) \hat{\Sigma} \mathcal{C}_k(\hat{A})'} e, \]
Main Idea for Asymptotic Distribution

* The function

\[ v_k(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A})\hat{\Sigma} C_k(\hat{A})' e_i}, \]

is continuously differentiable for every \((A, \Sigma)\), except, when
Main Idea for Asymptotic Distribution

The function

\[ v_k(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A})\hat{\Sigma} C_k(\hat{A})' e_i}, \]

is continuously differentiable for every \((A, \Sigma)\), except, when

\[ C_k(A)'e_i = 0. \]
**Main Idea for Asymptotic Distribution**

* The function

\[ \nu_k(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A}) \hat{\Sigma} C_k(\hat{A})' e_i}, \]

is continuously differentiable for every \((A, \Sigma)\), except, when

\[ C_k(A)' e_i = 0. \]

* The function, however, is everywhere directionally differentiable
Main Idea for Asymptotic Distribution

⋆ The function

\[ v_k(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i^\prime C_k(\hat{A})\hat{\Sigma} C_k(\hat{A})^\prime e_i}, \]

is continuously differentiable for every \((A, \Sigma)\), except, when

\[ C_k(A)^\prime e_i = 0. \]

⋆ The function, however, is everywhere directionally differentiable

⋆ Adjusted delta-method inference is asymptotically valid.

(details later)
3. Example
Effects of Uncertainty on Household Spending

- Bivariate SVAR(12) with monthly data (01-1963: 05-2014)
**Effects of Uncertainty on Household Spending**

* Bivariate SVAR(12) with monthly data (01-1963: 05-2014)

  New one family houses sold (HSN1f-FRED)
Effects of Uncertainty on Household Spending

- Bivariate SVAR(12) with monthly data (01-1963: 05-2014)

  New one family houses sold (HSN1f-FRED)

  S&P 100 Volatility Index (VXO-Bloom’s Website)
**Effects of Uncertainty on Household Spending**

- Bivariate SVAR(12) with monthly data (01-1963: 05-2014)
  - New one family houses sold (HSN1f-FRED)
  - S&P 100 Volatility Index (VXO-Bloom’s Website)

- Logarithm of both series is HP-filtered ($\lambda = 129, 600$)
**Effects of Uncertainty on Household Spending**

- Bivariate SVAR(12) with monthly data (01-1963: 05-2014)
  - New one family houses sold (HSN1f-FRED)
  - S&P 100 Volatility Index (VXO-Bloom’s Website)

- Logarithm of both series is HP-filtered ($\lambda = 129, 600$)

- See Knotek and Khan (2011, FRB Kansas) for details
Maximum and Minimum Response of New O-F Houses sold
**CHOLESKY ESTIMATE OF STRUCTURAL IRF (UNCERTAINTY FIRST)**

Response of HSN1f to a Structural Uncertainty Shock

[Graph showing the response of HSN1f to a structural uncertainty shock over 50 months, with percent deviation from trend on the y-axis and months on the x-axis.]
4. Sign-Restricted SVARs
**TWO MODIFICATIONS TO THE BASIC IDEA**

1. **Min/Max response subject to sign restrictions on the IRFs.**
   (mathematical program with linear inequality constraints)
Two Modifications to the Basic Idea

1. Min/Max response subject to sign restrictions on the IRFs. (mathematical program with linear inequality constraints)

2. Min/Max Forecast Error Variance Decomposition. (quadratic objective function)
**Additional Complication: Estimation**

★ The mathematical program

\[
\min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}
\]
**Additional Complication: Estimation**

* The mathematical program

\[
\min_{H_j\sigma_j \in \mathbb{R}^n} e_i' C_k (\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}
\]

and \(S\) inequality constraints on the IRFs:
ADDITIONAL COMPLICATION: ESTIMATION

⋆ The mathematical program

$$\min_{Hj\sigma_j \in \mathbb{R}^n} e_i' C_k(\hat{A}) H_j \sigma_j \text{ subject to } HDH' = \hat{\Sigma}$$

and $S$ inequality constraints on the IRFs:

$$e_{im}' C_{km}(\hat{A}) H_j \sigma_j \geq 0 \quad m = 1, \ldots S$$
Additional Complication: Estimation

The mathematical program

$$\min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}$$

and \(S\) inequality constraints on the IRFs:

$$e_{im}' C_{km}(\hat{A}) H_j \sigma_j \geq 0 \quad m = 1, \ldots, S$$

need not have a closed form solution.
**Additional Complication: Estimation**

- The mathematical program

\[
\min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}
\]

and S inequality constraints on the IRFs:

\[
e_{im}' C_{km}(\hat{A}) H_j \sigma_j \geq 0 \quad m = 1, \ldots S
\]

need not have a closed form solution.

- Hence, the plug-in solution \( \nu_k \) will be numerical.
ADDITIONAL COMPLICATION: ESTIMATION

★ The mathematical program

$$\min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k (\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}$$

and $S$ inequality constraints on the IRFs:

$$e_{im}' C_{km} (\hat{A}) H_j \sigma_j \geq 0 \quad m = 1, \ldots, S$$

need not have a closed form solution.

★ Hence, the plug-in solution $\nu_k$ will be numerical

★ Standard Bayesian vs. Frequentist Approach:
ADDITIONAL COMPLICATION: ESTIMATION

★ The mathematical program

$$\min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}$$

and S inequality constraints on the IRFs:

$$e_i' C_m(\hat{A}) H_j \sigma_j \geq 0 \quad m = 1, \ldots S$$

need not have a closed form solution.

★ Hence, the plug-in solution $v_k$ will be numerical

★ Standard Bayesian vs. Frequentist Approach:

GRID SEARCH VS. NONLINEAR PROGRAMMING
Maximum and Minimum Response of New O-F Houses sold

Response of HSN1f to a Structural Uncertainty Shock

- Percent Deviation from trend
- Months
Maximum and Minimum Response s.t. \((\partial v_{x0_t}/\partial \epsilon_{t,unc}) > 0\)
5. Frequentist Inference
We are solving the mathematical program

$$
\nu_k(\hat{A}, \hat{\Sigma}) \equiv \min_{H_j\sigma_j \in \mathbb{R}^n} e_i' C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}
$$
Inference

We are solving the mathematical program

\[ v_k(\hat{A}, \hat{\Sigma}) \equiv \min_{H_j\sigma_j \in \mathbb{R}^n} e'_i C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma} \]

and subject to S inequality constraints on the IRFs:
Inference

We are solving the mathematical program

\[ \nu_k(\hat{A}, \hat{\Sigma}) \equiv \min_{H_j \sigma_j \in \mathbb{R}^n} e_i' C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma} \]

and subject to \( S \) inequality constraints on the IRFs:

\[ e_{im}' C_{km}(\hat{A}) H_j \sigma_j \geq 0 \quad m = 1, \ldots, S \]
Inference

* We are solving the mathematical program

\[
\nu_k(\hat{A}, \hat{\Sigma}) \equiv \min_{H_j \sigma_j \in \mathbb{R}^n} e'_i C_k(\hat{A}) H_j \sigma_j \quad \text{subject to} \quad HDH' = \hat{\Sigma}
\]

and subject to S inequality constraints on the IRFs:

\[
e'_i m C_{km}(\hat{A}) H_j \sigma_j \geq 0 \quad m = 1, \ldots, S
\]

* How do we construct 1-\(\alpha\)% confidence bands?
**DELTA-METHOD**

* Since we have assumed
**Delta-Method**

Since we have assumed

\[
\begin{pmatrix} A \\ \Sigma \end{pmatrix} \xrightarrow{d} Z \sim \mathcal{N}_d(0, \Omega)
\]
Since we have assumed
\[
\begin{pmatrix} \hat{A} \\ \hat{\Sigma} \end{pmatrix} \xrightarrow{d} Z \sim \mathcal{N}_d(0, \Omega)
\]
and we are interested in \( v_k(\hat{A}, \hat{\Sigma}) \) \ldots
Since we have assumed

\[
\begin{pmatrix} A \\ \Sigma \end{pmatrix} \xrightarrow{d} Z \sim \mathcal{N}_d(0, \Omega)
\]

and we are interested in \( v_k(\hat{A}, \hat{\Sigma}) \) ...

Can’t we just apply the delta-method to get the limit of...
Since we have assumed

\[
\begin{pmatrix} A \\ \Sigma \end{pmatrix} \xrightarrow{d} Z \sim \mathcal{N}_d(0, \Omega)
\]

and we are interested in \( v_k(\hat{A}, \hat{\Sigma}) \) ... 

Can't we just apply the delta-method to get the limit of

\[
\sqrt{T} \left( v_k(\hat{A}, \hat{\Sigma}) - v_k(A, \Sigma) \right)
\]
Δelta-MethoΔ

★ Since we have assumed

\[
\begin{pmatrix} A \\ \Sigma \end{pmatrix} \xrightarrow{d} Z \sim \mathcal{N}_d(0, \Omega)
\]

and we are interested in \( v_k(\hat{A}, \hat{\Sigma}) \ldots \)

★ Can’t we just apply the delta-method to get the limit of

\[
\sqrt{T} \left( v_k(\hat{A}, \hat{\Sigma}) - v_k(A, \Sigma) \right)
\]

★ Question: Is \( v_k(\cdot) \) differentiable in \((A, \Sigma)\)?
Envelope Theorem (Bonnans-Shapiro, Faccio-Ishizuka)

- Sign-restricted problems have a Lipschitz Cont. Value Fn
  (Enough for Consistency)
Envelope Theorem (Bonnans-Shapiro, Faccio-Ishizuka)

★ Sign-restricted problems have a Lipschitz Cont. Value Fn (Enough for Consistency)

★ Sign-restricted problems satisfy the Mangasarian-Fromowitz CQ (Lagrange Multipliers Exist)
Envelope Theorem (Bonnans-Shapiro, Faccio-Ishizuka)

- Sign-restricted problems have a Lipschitz Cont. Value Fn (Enough for Consistency)

- Sign-restricted problems satisfy the Mangasarian-Fromowitz CQ (Lagrange Multipliers Exist)

**Result**: If sign-restricted problems satisfy the LICQ and have unique minimizer or maximizer:
**Envelope Theorem (Bonnans-Shapiro, Faccio-Ishizuka)**

★ Sign-restricted problems have a Lipschitz Cont. Value Fn (Enough for Consistency)

★ Sign-restricted problems satisfy the Mangasarian-Fromowitz CQ (Lagrange Multipliers Exist)

★ **Result:** If sign-restricted problems satisfy the LICQ and have unique minimizer or maximizer:

a) Lagrange Multipliers are unique
ENVELOPE THEOREM (BONNANS-SHAPIRO, FACCIO-ISHIZUKA)

★ Sign-restricted problems have a Lipschitz Cont. Value Fn
(Enough for Consistency)

★ Sign-restricted problems satisfy the Mangasarian-Fromowitz CQ
(Lagrange Multipliers Exist)

★ RESULT: If sign-restricted problems satisfy the LICQ and have
unique minimizer or maximizer:

a) Lagrange Multipliers are unique

b) Value function is fully differentiable:
**Envelope Theorem (Bonnans-Shapiro, Faccio-Ishizuka)**

★ Sign-restricted problems have a Lipschitz Cont. Value Fn (Enough for Consistency)

★ Sign-restricted problems satisfy the Mangasarian-Fromowitz CQ (Lagrange Multipliers Exist)

★ **Result**: If sign-restricted problems satisfy the LICQ and have unique minimizer or maximizer:

a) Lagrange Multipliers are unique

b) Value function is fully differentiable:

\[ \nabla_k'(A, \Sigma) = \nabla_{(A, \Sigma)} \mathcal{L}(H_j^*(A, \Sigma), \lambda^*(A, \Sigma)), \quad \mathcal{L} : \text{Lagrangian}, \]
**Delta-Method Confidence Sets for IRFs (no sign restriction)**

Response of HSN1f to a Structural Uncertainty Shock

Percent Deviation from trend vs Months

- Graph showing the response of HSN1f to a structural uncertainty shock over 50 months.

- The y-axis represents the percent deviation from trend, ranging from -8 to 8.

- The x-axis represents the months, ranging from 0 to 50.

- The graph includes two lines, one solid and one dashed, illustrating the deviation over time.

- The solid line starts at a higher deviation and decreases more rapidly than the dashed line, which starts at a lower deviation and decreases more gradually.

- The graph provides a visual representation of how HSN1f responds to structural uncertainty shocks over time.
**Delta-Method Confidence Sets for IRFs (sign restriction)**

Response of HSNf to a Structural Uncertainty Shock

- **X-axis:** Months
- **Y-axis:** Percent Deviation from trend

---

**Inference**
**Delta-Method Confidence Sets for IRFs (no sign restriction)**

Response of VXO to a Structural Uncertainty Shock

- **X-axis:** Months
- **Y-axis:** Percent Deviation from trend
**Delta-Metho Confidence Sets for IRFs (sign restriction)**

Response of VXO to a Structural Uncertainty Shock

- **X-axis:** Months
- **Y-axis:** Percent Deviation from trend
Other things we have done

- Results for Monetary and Oil SVARs
  (Identification power of sign restrictions/external instruments)
Other things we have done

- Results for Monetary and Oil SVARs
  (Identification power of sign restrictions/external instruments)

- Monte-Carlo exercises to analyze coverage
  (we also analyze a Lagrangian Bootstrap)
OTHER THINGS WE HAVE DONE

★ Results for Monetary and Oil SVARs
   (Identification power of sign restrictions/external instruments)

★ Monte-Carlo exercises to analyze coverage
   (we also analyze a Lagrangian Bootstrap)

★ Comparison with Moon, Schorfheide, Granziera (2013)
   (our procedure is efficient at points of full differentiability)
Other things we have done

- Results for Monetary and Oil SVARs
  (Identification power of sign restrictions/external instruments)

- Monte-Carlo exercises to analyze coverage
  (we also analyze a Lagrangian Bootstrap)

- Comparison with Moon, Schorfheide, Granziera (2013)
  (our procedure is efficient at points of full differentiability)

- Relax full differentiability: directional differentiability
Conclusion: Frequentist Approach to SVARs

1. The maximum and minimum are ‘point-identified’ in SVARs (even if impulse response functions are not).
Conclusion: Frequentist Approach to SVARs

1. The maximum and minimum are ‘point-identified’ in SVARs (even if impulse response functions are not).

\[ v(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i C_k(\hat{A})\hat{\Sigma} C_k(\hat{A})' e_i} \]
Conclusion: Frequentist Approach to SVARs

1. The maximum and minimum are ‘point-identified’ in SVARs (even if impulse response functions are not).

\[ v(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A})\hat{\Sigma} C_k(\hat{A})' e_i}, \]

2. Efficient delta-method inference available for these objects.

(max and min are diff. w.r.t. reduced-form VAR parameters)
**Conclusion: Frequentist Approach to SVARs**

1. The maximum and minimum are ‘point-identified’ in SVARs (even if impulse response functions are not).

\[
v(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A}) \hat{\Sigma} C_k(\hat{A})' e_i},
\]

2. Efficient delta-method inference available for these objects.
   (max and min are diff. w.r.t. reduced-form VAR parameters)

\[
\sqrt{T} \left( v_k(\hat{A}, \hat{\Sigma}) - v_k(A, \Sigma) \right) \xrightarrow{d} v_k'(A, \Sigma) Z.
\]
CONCLUSION: FREQUENTIST APPROACH TO SVARs

1. The maximum and minimum are ‘point-identified’ in SVARs (even if impulse response functions are not).

\[ v(\hat{A}, \hat{\Sigma}) = -\sqrt{e_i' C_k(\hat{A})\hat{\Sigma} C_k(\hat{A})' e_i}, \]

2. Efficient delta-method inference available for these objects.

(max and min are diff. w.r.t. reduced-form VAR parameters)

\[ \sqrt{T} \left( v_k(\hat{A}, \hat{\Sigma}) - v_k(A, \Sigma) \right) \xrightarrow{d} v_k'(A, \Sigma) Z. \]

3. Can accommodate alternative sources of identification (sign restrictions or external instruments).
Thanks!