Frictions Lead to Sorting:

a Model of Bilateral On-the-Match Search*

Cristian Bartolucci and Ignacio Monzón†

October 16, 2014

Abstract

We present a partnership model where heterogeneous agents bargain over the gains from trade and are allowed to search on the match. Because of frictions, agents extract higher rents from more productive partners, generating an endogenous preference for high types. As individuals search while matched, more productive agents upgrade their partners more often, and therefore the equilibrium distribution becomes positively assortative. Frictions are commonly understood to hamper sorting. In contrast, our mechanism highlights the role of frictions as a driving force towards positive sorting. Our results invalidate the interpretation of positive assortative matching as evidence of complementarity since such a sorting pattern arises even with a submodular production function.

JEL Classification: C78; D83; J63; J64

Keywords: Assortative Matching; Search frictions; On-the-match Search; Bargaining; Submodularity

*We thank Ainhoa Aparicio, Melvyn Coles, Francesco Devicienti, Jan Eeckhout, Javier Fernandez-Blanco, Chris Flinn, Dino Gerardi, Toomas Hinnossar, Philipp Kircher, Igor Livshits, Fabien Postel-Vinay, David Rivers and Lones Smith for valuable comments. We thank Giorgio Martini for outstanding research assistance. Online Appendix available at https://dl.dropboxusercontent.com/u/3096559/bm-online-appendix.pdf

†Collegio Carlo Alberto, Via Real Collegio 30, 10024 Moncalieri (TO), Italy. Emails: cristian.bartolucci@carloalberto.org and ignacio@carloalberto.org.
1. Introduction

Markets with two-sided heterogeneity are prevalent. In labor markets, firms and workers typically differ in their characteristics, quality and ability. The same is true in other markets, such as the marriage market and the market for CEOs. The evidence suggests that better CEOs sort into better corporations (Parrino [1997]), that there is positive assortative mating in the marriage market (Mare [1991]), and that more productive employees work for better firms (Bartolucci and Devicienti [2013]). Traditionally, positive assortative matching has been interpreted as evidence of complementarity in the production function. In this paper we argue that, even in the absence of complementarity in production, frictions generate a simple and natural reason for positive assortative matching to arise.

In the presence of frictions higher types become more appealing. While in frictionless markets payoffs reflect individual contributions, when it takes time to find a partner the division of output becomes more even. To see this, assume that agents are infinitely impatient (or frictions infinitely strong), and therefore outside options are zero. In this simple case the gains from trade are equal to the production of the match. Under standard bargaining, both agents receive an equal share of the gains from trade, so agents receive a constant fraction of the match’s output. When frictions are strong enough, it is the total production of the match, rather than individual contributions, that shapes payoffs and preferences over partners. Production is increasing in partner’s type, so an endogenous preference for better types arises. Complementarity in production only plays a secondary role.

A preference for partners of higher type can lead to positive sorting. In fact, if the value of the match were exogenously increasing in partner type, perfect positive sorting would arise in a frictionless market (see Becker [1973] and Legros and Newman [2007]). However, this is not necessarily true in a market with frictions. When agents are not allowed to search while matched (for instance, in Shimer and Smith [2000] and Atakan [2006]) a preference for higher types does not translate into positive assortative matching. Agents are eager to upgrade partners, but cannot do so. We show that if agents in both sides of the market are allowed to replace their partners, for example by allowing on-the-
match search, a valuation increasing in partner’s type can lead to positive sorting.

Match-to-match transitions are pervasive in most developed economies. According to conservative estimates, half of all new employment relationships result from job-to-job transitions (see Fallick and Fleischman [2004]). On the firm side, Albak and Sørensen [1998] and Burgess, Lane, and Stevens [2000] present empirical evidence of replacement hiring (see Kiyotaki and Lagos [2007] for a discussion). In the market for CEOs, Parrino [1997] finds that the availability of a strong outside candidate is an important consideration in the decision to replace a poor CEO, while Murphy and Zabojnik [2006] report that a large proportion of managers were hired from another firm. Although the evidence of on-the-match search in the marriage market is less widespread, Stevenson and Wolfers [2007] find that remarriage is one of the main determinants of divorce.

We present a partnership model that includes the key elements of Shimer and Smith [2000] but allows for bilateral on-the-match search. A matched agent who finds a new partner can choose to dissolve her current match and form a new one. After a match-to-match transition, the agent bargains with her new partner without the possibility of returning to the previous one. This timing implies that the value of the match to an agent depends only on her partner’s type. Our bargaining protocol prevents agents from exploiting the presence of multiple suitors to raise their payoffs. In some markets (like the one for academic economists) counteroffers are common practice. However, this is not the norm in most labor markets (see Mortensen [2005]). In Section 5.2 we modify the bargaining protocol to allow for renegotiation and show how frictions can lead to positive sorting in this case, also without productive complementarity.

Allowing agents to search on the match adds an extra layer of difficulty to the bargaining problem: the surplus from the match depends on the bargaining outcome. Patient agents face a trade-off between per period payoff and expected duration of the match: higher wages paid to a worker on the one side reduce the firm’s per period profits, but on the other they decrease the likelihood that the worker quits. In fact, a higher wage may increase the value of the match to both worker and firm. As highlighted by Shimer [2006], bargaining sets are not necessarily convex and therefore the standard axiomatic
Nash Bargaining is not applicable in this setup.

To see why surplus may be split unevenly, consider an example with only two types of patient agents: \( x, y \in \{ \ell, h \} \), with \( \ell \) slightly less than \( h \). Assume agents produce \( f(x, y) = x + y \) if matched and zero otherwise. If \( \ell \) and \( h \) split the total surplus symmetrically, the low-type agent makes marginally more than \( \ell \) per period but is dismissed when the high-type agent finds a high-type partner. Therefore it is more convenient for the low-type agent to receive a per period payoff of \( \ell \) and get a larger expected duration of the match. The high type also benefits from that. Then, for \( \ell \approx h \), the outcome from even surplus splitting is dominated. On the other side, when the difference in types’ qualities is high, it is profitable for low types to give up match duration in exchange for higher per period payoffs. To see this, consider patient agents with \( \ell \approx 0 \). The high type only stays if she receives \( h \), which leaves the low type with a payoff close to 0. The low type is better off equalizing surplus.

We present a solution for axiomatic bargaining when both sides can leave the match if they find a preferred option. Bargaining sets do not satisfy Nash [1950]’s axioms. We show, however, that bargaining sets are compact under on-the-match search. We follow a modified version of Nash’s axioms proposed by Kaneko [1980]. Kaneko shows that when bargaining sets are compact the solution is exactly as in Nash [1950]: it selects the outcome which maximizes the product of agents’ individual surpluses.

We introduce first a model where agents are of one of two types: either low or high productivity. This simple two type model is rich enough to illustrate the trade-offs patient agents face when they can search on the match. We show that several different equilibria can arise, depending on the degree of complementarity in production, agents’ patience and the degree of frictions in the market. Each possible equilibrium induces a pattern of sorting. We fully characterize this two type model.

In our main result in this part, we provide necessary and sufficient conditions for an endogenous preference for the high type to arise. We show that a preference for the high type leads to positive assortative matching in our model. We state, as a function of the primitives, up to which degree of submodularity in production an equilibrium
with positive assortative matching exists. Moreover, we show that for some primitives, an equilibrium with positive assortative matching is the only possible one, even with a submodular production function.

Our intuition extends to the case with any (finite) number of types. We first show that as agents become impatient or frictions large, an equilibrium where agents endogenously prefer higher types arises, for any (strictly increasing) production function. Moreover, this is the only possible equilibrium. Thus, for any number of types positive assortative matching may arise, even with submodularity in production. Second, with parameter values in line with the literature, we provide numerical examples of equilibria where agents endogenously prefer higher types. Both with modular or submodular production functions there are equilibria where matching is positively assortative.

The literature on assortative matching mostly focuses on how complementarity in production affects the allocation of workers to firms. In Becker [1973]’s seminal partnership model, a supermodular production function is necessary and sufficient for positive assortative matching. This is not true in markets with frictions.¹ When it takes time to find a partner, agents are selective only if complementarity in production compensates the cost of waiting. Therefore, the conventional wisdom is that stronger frictions require stronger complementary in production for positively assortative matching to arise. The results in this paper go against this view and invalidate the interpretation of sorting as evidence of complementary in production.

The policy recommendation when sorting results from frictions differ significantly from those when sorting results from complementary. To see this, consider the linear production function case. There, positive sorting can only arise because of frictions. However, if sorting is interpreted as evidence of complementarity in production, the standard policy recommendation is to subsidize agents to wait until they find their preferred partner. Now, since production is linear, a different distribution of matches does not change the aggregate production of the economy and moreover search is costly in terms of for-

¹ There is no positive assortative matching in Shimer and Smith [2000] for modular or slightly supermodular production functions (see Section 5.1 for an in-depth discussion). In Atakan [2006], whenever the explicit cost of search is high and complementarity weak, random sorting arises in equilibrium. In Eeckhout and Kircher [2010] root-supermodularity is necessary and sufficient for positive assortative matching.
gone output. Then, such a program would be welfare detrimental.

In the next section we present the model, describe bargaining sets with on-the-match-search and we present our notion of equilibrium. In Section 3 we solve the two type case. We provide a full characterization of all equilibria in this simplified setting. Section 4 shows how our intuition extends to the case with any number of types. In Section 5 we present some extensions and variations to our model. Section 6 concludes.

2. The Model

Consider a continuous time, infinite horizon stationary economy, populated by infinitely lived, risk neutral agents. There is a unit mass population of heterogeneous agents, denoted by their fixed type $x \in X$, where $X$ is a finite list of all possible types. All types are present in equal proportion in the population.

We assume first that agents can be either of low productivity or of high productivity, so $X = \{\ell, h\}$, with $0 < \ell < h$. A match $(x, y)$ produces a flow of output $f(x, y)$. The production function is simple. Two $\ell$ agents produce $f(\ell, \ell) = 2\ell$, two $h$ agents produce $f(h, h) = 2h$ and a $\ell$-type with a $h$-type produce $f(h, \ell) = f(\ell, h) = F$. Parameter $F$ captures the degree of complementarity in production. A modular production function has $F = \ell + h$, a supermodular one has $F < \ell + h$, and $F > \ell + h$ corresponds to the submodular case. We assume that high-productivity agents are always more productive than low-productivity agents, therefore $2\ell < F < 2h$. Unmatched agents produce zero. Agents discount the future at rate $r > 0$.

Agents can be either matched or unmatched. The steady state distribution $e(x, y) : \{\ell, h\} \times \{\emptyset, \ell, h\} \rightarrow [0, \frac{1}{2}]$ specifies the number $e(x, \emptyset)$ of unmatched $x$-type agents and the number $e(x, y)$ of $x$-type agents matched to agents of type $y \in \{\ell, h\}$. Since in the population there are as many low as high productivity agents, $\sum_{y \in \{\emptyset, \ell, h\}} e(x, y) = \frac{1}{2}$ for $x \in \{\ell, h\}$.

Transitions between states occur due to both exogenous destruction and match-to-match transitions. Matches are exogenously destroyed at rate $\delta$ and meetings occur at
rate $\rho$. We allow both matched and unmatched agents to meet potential partners (who also themselves may be matched or unmatched). Any agent, regardless of type and match status, meets an agent $x \in \{\ell, h\}$ in state $y \in \{\emptyset, \ell, h\}$ at rate $\rho e(x, y)$.

In our model, each agent has to decide which partners to accept while matched to each possible partner; acceptance sets are conditional on the current partner. A decision function $d(x, y, y') : \{\ell, h\} \times \{\ell, h\} \times \{\ell, h\} \rightarrow [0, 1]$ specifies the probability that an agent of type $x$ matched to an agent of type $y$ chooses, given the chance, to switch to a partner of type $y'$.

Flow payoffs are deterministic, last for the duration of the match, and are determined through bargaining, as discussed in the next subsection. Let $\pi(x, y) : \{\ell, h\} \times \{\ell, h\} \rightarrow [0, f(x, y)]$, with $\pi(x, y) + \pi(y, x) \leq f(x, y)$, be the flow payoff agent $x$ receives when matched to agent $y$. Unmatched agents obtain a zero flow payoff.

Let $q(x, y) : \{\ell, h\} \times \{\ell, h\} \rightarrow \mathbb{R}_+$ be the rate at which $x$ finds a $y$ who is willing to form a match with her: $q(x, y) \equiv \rho \left[ e(y, \emptyset) + \sum_{x' \in \{\ell, h\}} e(y, x') d(y, x', x) \right]$. The value $V(x, \emptyset)$ for an unmatched $x$-type agent is given by

$$[r + q(x, \ell) + q(x, h)] V(x, \emptyset) = 0 + q(x, \ell) V(x, \ell) + q(x, h) V(x, h),$$

where $V(x, y)$ is the value for a $x$-type agent matched to a $y$-type, which is given by

$$\left( r + \delta + \sum_{y' \in \{\ell, h\}} d(x, y, y') q(x, y') + \sum_{x' \in \{\ell, h\}} d(y, x, x') q(y, x') \right) V(x, y) = \pi(x, y)$$

$$+ \left( \delta + \sum_{x' \in \{\ell, h\}} d(y, x, x') q(y, x') \right) V(x, \emptyset) + \sum_{y' \in \{\ell, h\}} d(x, y, y') q(x, y') V(x, y').$$

It is usually more convenient to work directly with the surplus agents obtain relative

---

2For simplicity, we assume that on-the-match search and search while unmatched are equally intensive. In equilibrium, payoffs while matched are strictly positive. Then, unmatched agents accept all partners. In Section 5.1 we allow search intensities to differ.

3Re-scaling the production function to allow for positive payoffs while unmatched leads to equivalent results.
to being unmatched. Surplus $S(x, y) : \{\ell, h\} \times \{\ell, h\} \to \mathbb{R}$ are given by

$$S(x, y) = \left( r + \delta + \sum_{x' \in \{\ell, h\}} d(y, x, x') q(y, x') \right)^{-1} \left[ \pi(x, y) \right]$$

$$+ \sum_{y' \in \{\ell, h\}} d(x, y, y') q(x, y') \left[ S(x, y') - S(x, y) \right] - \sum_{y' \in \{\ell, h\}} q(x, y') S(x, y').$$

We distinguish individual surpluses $S(x, y)$ and $S(y, x)$ from the total surplus of the match $S(x, y) + S(y, x)$, since there can be cases where total surplus is not split symmetrically.

### 2.1 Timing and Bargaining

We propose the following timing. When a matched agent finds a potential partner, she observes his type, and before bargaining, she must decide to stay or leave. If she chooses the new partner she must dissolve her current match. Therefore, when an agent bargains with her partner, she cannot exploit the existence of an alternative partner to improve her bargaining position. As a result, the outside option is always the value of the being unmatched.$^4$

When agents search on the match the bargaining set is non-standard, so we need to describe it carefully. Once agents $x$ and $y$ form a match, they bargain on how to split production. This allocation of production remains in place until the match breaks, either exogenously or endogenously. Whenever a matched agent finds a suitor offering a higher surplus, she leaves her partner. Agents cannot commit not to leave each other, and do not engage in renegotiation when an offer arrives.

When two agents bargain, they take all information from other matches as given. $S^* = \{S^*(x, y)\}_{(x, y) \in \mathcal{X}^2}$ and $q^* = \{q^*(x, y)\}_{(x, y) \in \mathcal{X}^2}$ summarize a given state of the economy.

A possible agreement $c = (\hat{\pi}, \hat{\pi})$ between $x$ and $y$ specifies both an allocation $\hat{\pi}$ and

---

$^4$The timing of match-to-match transitions follows Pissarides [1994] and several recent papers (Shimer [2006], Gautier, Teulings, and Van Vuuren [2010], and Bartolucci [2013]). In Section 5.2 we allow agents to make counteroffers à la Kiyotaki and Lagos [2007]. Positive assortative matching can also arise in this case, even without complementarity in production.
a decision function \( \hat{d} \). Let \( \hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2) \) and \( \hat{d} = \left( \{ \hat{d}_1(y') \}_{y' \in X}, \{ \hat{d}_2(x') \}_{x' \in X} \right) \), so for instance \( \hat{d}_1(y') \) denotes agent \( x \)'s decision when faced with the possibility to match a (willing) agent of type \( y' \). Of course, \( \hat{\pi}_1 + \hat{\pi}_2 \leq f(x, y) \). Taking market outcomes \( (S^*, q^*) \) as given, an agreement \( c = \left( \hat{d}, \hat{\pi} \right) \) induces surplus pair \( \hat{S}^c = \left( \hat{S}_1^c, \hat{S}_2^c \right) \) with

\[
\hat{S}_1^c = \left( r + \delta + \sum_{x' \in X} \hat{d}_2(x') q^*(y, x') \right)^{-1} \left[ \hat{\pi}_1 + \sum_{y' \in X} \hat{d}_1(y') q^*(x, y') \left[ S^*(x, y') - \hat{S}_1^c \right] - \sum_{y' \in X} q^*(x, y') S^*(x, y') \right]
\]

and \( \hat{S}_2^c \) defined accordingly.

Since there is no renegotiation or commitment, only consistent agreements can occur:

**Definition 1. Consistent Agreements.** Fix market outcomes \( (S^*, q^*) \). An agreement \( c = \left( \hat{d}, \hat{\pi} \right) \) is consistent if for all \( y' \in X \),

\[
\hat{d}_1(y') \begin{cases} 
1 & \text{if } S^*(x, y') - \hat{S}_1^c > 0 \\
[0, 1] & \text{if } S^*(x, y') - \hat{S}_1^c = 0 \\
0 & \text{if } S^*(x, y') - \hat{S}_1^c < 0
\end{cases}
\]

and the same holds for \( \hat{d}_2(x') \), for all \( x' \in X \).

With this definition in hand, we can define our bargaining sets:

**Definition 2. Bargaining Sets \( S \) under On-the-Match Search.** Fix market outcomes \( (S^*, q^*) \). Agents \( x \) and \( y \) bargain over

\[
S_{xy} = \left\{ (S_1, S_2) : \exists \text{ consistent } c \text{ with } \hat{S}_1^c = S_1 \text{ and } \hat{S}_2^c = S_2 \right\}.
\]

Bargaining sets under on-the-match search have two features that make finding a solution non-trivial. First, they may be non-convex, so Nash [1950]'s assumptions are not satisfied. Second, bargaining sets under on-the-match search may be non comprehen-
sive.\textsuperscript{5} Kaneko [1980] presents an extension of Nash [1950]'s model to allow for bargaining over non-convex (and non-comprehensive) sets. Kaneko’s version of Nash’s axioms permits set-valued decision functions.\textsuperscript{6} For the purpose of this paper, let $\mathcal{B}$ denote the class of all compact subsets $S$ of $\mathbb{R}_+^2$. A decision correspondence $\phi$ assigns to each $S \in \mathcal{B}$ a non-empty subset $\phi(S) \subset S$. Kaneko shows that a decision correspondence $\phi$ satisfies those axioms if and only if it maximizes the product of individual surpluses:

$$\phi(S) = \{ (S_1, S_2) \in S : S_1S_2 \geq S_1S_2 \text{ for all } (S_1, S_2) \in S \} \quad (2)$$

In the present model, bargaining sets $S_{xy}$ are compact. See Appendix A.1 for details. So from now on we assume that $\phi(\cdot)$ as defined in (2) is the solution to the bargaining problem.

2.2 Equilibrium

We can now characterize an equilibrium in this economy.

**Definition 3. Equilibrium with On-the-Match Search.** Take a pair of decision functions and allocations $(d^*, \pi^*)$, its induced state of the economy $(S^*, q^*)$ and its resulting bargaining sets $\{S_{xy}\}_{(x,y) \in X^2}$. We say $(d^*, \pi^*)$ is an equilibrium if for all $(x, y) \in X^2$,

1. agreements are consistent,\textsuperscript{7}

2. surpluses solve the bargaining problem: $(S^*(x,y), S^*(y,x)) \in \phi(S_{xy})$, and

3. market outcomes are robust: $S^*(x,y) > S^*(y,x) \Rightarrow \exists y' \neq y : S^*(x,y) = S^*(x,y')$.

\textsuperscript{5}$S$ is comprehensive if $0 \leq x \leq y$ and $y \in S$ implies $x \in S$. Non-comprehensiveness makes the analysis in Zhou [1997] and others unapplicable in a setup with on-the-match search. See Figure 1 in next section for an example on how bargaining sets look with on-the-match search.

\textsuperscript{6} There are three main differences between Nash’s and Kaneko’s axioms. First, Kaneko assumes strict Pareto Optimality, whereas Nash assumes a weak version. Second, the axiom of independence of irrelevant alternatives (IIA) is now: $T \subset S, \phi(S) \cap T \neq \emptyset \Rightarrow \phi(T) = \phi(S) \cap T$. This is consistent with Nash’s IIA, but it is a fairly restrictive version. Third, Kaneko assumes a weak form of continuity in the choice correspondence $\phi$.

\textsuperscript{7}For each match, define agreement $c = (\hat{d}, \hat{\pi})$ by $\hat{d} = (d^*(x,y,y'), d^*(y,x,x'))$ and $\hat{\pi} = (\pi^*(x,y), \pi^*(y,x))$. 

10
Before presenting our results, we provide a short discussion of our definition of equilibrium and its properties. First, equilibrium outcomes have some straightforward properties. For all matches, allocations exhaust production: \( \pi(x, y) + \pi(y, x) = f(x, y) \). Moreover, agents only perform match-to-match transitions if they are strictly better off after the transition: \( d(x, y, y') = 1 \{ S^*(x, y') > S^*(x, y) \} \). These results are direct consequences of the assumption of Strict Pareto Optimality in bargaining. Second, our model is symmetric in that both sides come from the same population. Thus, by construction, a low firm matched to a high worker obtains the same surplus as a low worker matched to a high firm. Third, we focus on equilibria where behavior is a function of own type and partner’s type. As a result, equilibrium outcomes with two agents of the same type are symmetric.

Conditions 1 and 2 alone are not enough to weed out some frail outcomes under on-the-match search. Several divisions of output can satisfy these two conditions for a given decision function \( d^* \). However, not all of them are robust. Consider \( (d^*, \pi^*) \) satisfying conditions 1 and 2 and leading to equal surplus splitting on match \( (x, y) \). Take an alternative \( (d^*, \pi^{**}) \) with the same decision function and a small perturbation only on match \( (x, y) \)’s payoffs. Individual surpluses change only marginally, so agreements can still be consistent. Regarding condition 2, note that under the alternative \( (d^*, \pi^{**}) \) agent \( x \) expects \( \pi^{**}(x, y) \) when matched to any type-\( y \) agent. If \( y \) offers \( x \) less than that, \( x \) breaks the match whenever she finds another type-\( y \) agent. Such an offer increases \( y \)’s flow payoff marginally while the probability that \( x \) leaves increases discretely, making both partners worse off. To sum up, once \( (\pi^{**}(x, y), \pi^{**}(y, x)) \) is expected, any small deviation from it leads to a lower surplus for both partners. This example highlights that even keeping \( d^* \) and payoffs in all other matches fixed, several divisions of production in match \( (x, y) \) can satisfy conditions 1 and 2.

We include condition 3 to rule out fragile cases like \( (d^*, \pi^{**}) \), which would not survive a positive cost of transition. If breaking a match were costly, agent \( x \) would not leave for another type-\( y \) agent when receiving slightly less than the expected \( \pi^{**}(x, y) \). So slight deviations from \( (\pi^{**}(x, y), \pi^{**}(y, x)) \) would increase the surplus of one agent while re-
ducing the surplus of the other one in the same amount (as long as these slight deviations do not make agents leave for other different types). Thus, only symmetric surplus splitting would maximize the product of individual surpluses. Our third condition states that an agent can get a higher surplus than her partner only if she is indifferent between her current partner and a partner of a different type. In Appendix A.2 we elaborate on how this is a necessary condition with positive costs of match-to-match transition.

Condition 3 in our definition of equilibrium is desirable, although not necessary for our message. We show equilibria with positive assortative matching arise, even without productive complementarity. Condition 3 restricts the set of equilibria to those robust in the sense discussed in the previous paragraph. Beyond robustness, condition 3 also provides tractability to the model. Moreover, in matches with strict preferences over partners’ type, the division of output in equilibrium leads to symmetric surplus splitting.

2.3 Assortative Matching

An equilibrium decision function \(d^*\) induces a steady state distribution of matches \(e(x, y)\). We argue that this steady state distribution can be positively assortative due to frictions.

In Becker [1973]’s frictionless market there is positive assortative matching if agents only match partners of their same type. In contrast, when it takes time to find a partner, agents may form matches with more than one type of partner; hence Shimer and Smith [2000] define sorting in terms of acceptance sets. However, if an agent can search while matched, her acceptance set depends not only on her own type, but also on her current partner’s type. Since match-to-match transitions shape the steady state distribution, a characterization of the acceptance sets of unmatched agents is not enough to describe the sorting pattern. Therefore we use the following definition proposed by Lentz [2010] to describe sorting in markets with match-to-match transitions.

**Definition 4. Positive Assortative Matching.** Take any \(x_1, x_2 \in X\) with \(x_1 > x_2\). There is positive assortative matching if the distribution of partners of \(x_1\) first order stochastically dominates the distribution of partners of \(x_2\).
3. Solution for the Two Type Case

The main insight from this paper is that frictions are a driving force towards positive sorting. Frictions generate rents, and rent splitting may induce an equilibrium preference for higher types. From now on, we say an equilibrium features hyperphily when $S^*(x, h) > S^*(x, \ell)$ for all $x \in \{\ell, h\}$. Frictions leads to sorting since hyperphily implies positive assortative matching.

**Lemma 1.** In an equilibrium with hyperphily and two types, h’s distribution of partners first order stochastically dominates \(\ell\)’s.

See Appendix A.5 for the proof.

We now present necessary and sufficient conditions for the existence of an equilibrium with hyperphily. Then, we present a complete characterization of the model. We describe all possible equilibria and the conditions for their existence. This allows us to state necessary and sufficient conditions for hyperphily to be the unique equilibrium.

3.1 An Equilibrium with Hyperphily

Under hyperphily, since no agent is indifferent between partners of different types, the total surplus of the match is split evenly in all matches (see equilibrium definition). Therefore, the equilibrium allocations are given by $\pi^*(\ell, \ell) = \ell, \pi^*(h, h) = h$, and $\pi^*(\ell, h)$ is set so that $S^*(\ell, h) = S^*(h, \ell)$.

As explained in the previous section, our definition of equilibrium requires agents’ transitions to be consistent with the surplus they obtain in each match. Moreover, we require that, for each match, no consistent agreement leads to a higher product of individual surpluses. Thus, the agreement between agents must be a global maximum in the bargaining set. This is a restrictive condition, which is not easy to check in general. We check each match step by step.

Pair $(d^*, \pi^*)$ is consistent in an equilibrium with hyperphily if the resulting surpluses satisfy

\[ S^*(h, h) > S^*(h, \ell) \quad \text{and} \quad S^*(\ell, h) > S^*(\ell, \ell). \]
We discuss next when \((d^*, \pi^*)\) solves the bargaining problem for each possible match.

**Bargaining Solution in Match \((\ell, h)\)**

Total surplus is split evenly between \(\ell\) and \(h\). An agreement leading to a higher product of individual surpluses can only exist if it also induces a larger total surplus. Since \(\ell\) does not leave the match \((\ell, h)\) under hyperphily, a larger total surplus can only be reached in the match \((\ell, h)\) if \(h\) chooses not to leave. Thus, we study consistent agreements between \(\ell\) and \(h\) where \(h\) does not leave. Let \((\hat{S}_c^\ell, \hat{S}_c^h)\) denote the surplus in some alternative agreement \(c\). \(h\) does not leave for a high agent only if \(\hat{S}_c^h \geq S^*(h, h)\).

There are three possible kinds of agreements with \(h\) staying. Either \(\ell\) always stays, or she leaves when she finds a new \(h\), or she leaves when she finds either an \(\ell\) or an \(h\). In the first kind of agreement \((c_1)\), both \(\ell\) and \(h\) choose not to leave each other. In the second one \((c_2)\), \(h\) always stays, but \(\ell\) leaves when she finds a new \(h\). In the third one \((c_3)\), \(h\) always stays, but \(\ell\) leaves when she finds any new partner. If the first kind of agreement exists, it makes both agents better off, so our original candidate is not an equilibrium. The second and third cases involve \(\ell\) obtaining a lower surplus. We need to check whether a higher product of individual surpluses is attained in these cases. To sum up, \((d^*, \pi^*)\) solves the bargaining problem in match \((\ell, h)\) if and only if Condition 1 holds.

**CONDITION 1.** Let \(c_1, c_2\) and \(c_3\) be defined as stated. No allocation generates

\[
\begin{align*}
\hat{S}_c^{c_1} &\geq S^*(h, h) \quad \text{and} \quad \hat{S}_c^{c_1} \geq S^*(\ell, h), \quad \text{or} \\
\hat{S}_c^{c_2} &\geq S^*(h, h), \quad S^*(\ell, \ell) \leq \hat{S}_c^{c_2} < S^*(\ell, h) \quad \text{and} \quad \hat{S}_c^{c_2} \hat{S}_c^{c_2} > S^*(\ell, h)S^*(h, \ell), \quad \text{or} \\
\hat{S}_c^{c_3} &\geq S^*(h, h), \quad \hat{S}_c^{c_3} < S^*(\ell, \ell) \quad \text{and} \quad \hat{S}_c^{c_3} \hat{S}_c^{c_3} > S^*(\ell, h)S^*(h, \ell).
\end{align*}
\]

Figure 1 presents two examples to illustrate how bargaining sets are built and how to verify Condition 1. As mentioned in the previous section, the trade-off between expected duration and flow payoff makes the bargaining sets non-convex. To see why, take the boundary of bargaining set \(S_{\ell h}\) in panel \(a\) in Figure 1. Consider first the point that gives \(\ell\) zero surplus and \(h\) his maximum possible surplus on \(S_{\ell h}\). At this point, \(h\) never leaves...
the match, while $\ell$ gets $\pi(\ell, h) = 0$ so she leaves for any alternative partner (either of type $\ell$ or $h$). An increase in $\pi(\ell, h)$, together with its corresponding decrease in $\pi(h, \ell)$, increases $\hat{S}_c^\ell$ in the same amount as $\hat{S}_h^c$ decreases. Thus, for small changes in flow payoffs, the boundary of the bargaining set is linear. However, consider now the point where $\hat{S}_h^c = S^*(h, h)$. A further increase in $\pi(\ell, h)$ makes $\hat{S}_h^c < S^*(h, h)$, so $h$ starts leaving whenever she finds another $h$. The expected duration of the match decreases, and although $\pi(\ell, h)$ is higher, $\hat{S}_c^\ell$ decreases discretely. It is this jump that generates a non-convexity in the bargaining set. In general, bargaining sets are non-convex in the neighborhood of agreements leading to indifference.

As Figure 1 illustrates, bargaining sets are built from potentially disjoint compact sets. In fact, agreement $(d^*, \pi^*)$ maps to a disjoint point in the bargaining set. Any marginal deviation from $\pi^*$ decreases the expected duration of the match discretely. This occurs because the partner whose flow payoff has been reduced now leaves when she finds a new partner of the same type as her current one (see the discussion on the third condition of the equilibrium definition).

Panel a shows a case where condition 1 holds, while panel b shows a case where condition 1 does not hold. The shaded are in panel a represents the bargaining set $S_{\ell h}$ (the shaded area) under hyperphily and a modular production function. The curve depicted through $(S^*(\ell, h), S^*(h, \ell))$ indicates all points attaining product $S^*(\ell, h) \times S^*(h, \ell)$. As no element in the bargaining set attains a higher product of individual surpluses, hyperphily solves the bargaining problem. Note this occurs without complementarity in production and with patient agents.

When the production function is sufficiently submodular hyperphily is no longer an equilibrium, as shown in panel b. There, an alternative consistent agreement leads to a higher product of individual surpluses and to a higher individual surplus for both agents. Agent $\ell$ receives less than half of a larger surplus in order to make her partner indifferent. Still, agent $\ell$ is better off. Therefore the first line of Condition 1 is violated.

In fact, in the example presented in panel b the second line of Condition 1 is also violated. An agreement that makes 1) $h$ indifferent to a match with another $h$ and 2) $\ell$
Figure 1: Bargaining Sets $S_{th}$

Note: $\rho = 0.1$, $r = 0.1$, $\delta = 0.05$, $\ell = 1$ and $h = 2$. In (a), $F = \ell + h$. In (b), $F = 1.6\ell + h$.

worse off than in a match to a different $h$ is also consistent and leads to a larger product of individual surpluses.

**Bargaining Solution in Match** $(\ell, \ell)$

As in match $(\ell, h)$, there are three cases to consider. In the first $(c_4)$, both agents choose not to leave each other (as in panel $b$). In the second $(c_5)$, one $\ell$ agent never leaves while the second one leaves only when finding a willing $h$. In the third $(c_6)$, one $\ell$ agent never leaves while the other one leaves when finding any willing partner. Let $(\hat{S}_1^c, \hat{S}_2^c)$ denote the surplus in an alternative contract $c$. To sum up, $(d^*, \pi^*)$ solves the bargaining problem in match $(\ell, \ell)$ if and only if Condition 2 holds.

**CONDITION 2.** Let $c_4, c_5$ and $c_6$ be defined as stated. No allocation generates

- $\hat{S}_1^{c_4} \geq S^*(\ell, h)$, or
- $\hat{S}_1^{c_5} \geq S^*(\ell, h)$, and $S^*(\ell, \ell) \leq \hat{S}_2^{c_5} < S^*(\ell, h)$, or
- $\hat{S}_1^{c_6} \geq S^*(\ell, h)$, $\hat{S}_2^{c_6} < S^*(\ell, \ell)$ and $\hat{S}_1^{c_6} \hat{S}_2^{c_6} > [S^*(\ell, \ell)]^2$.
We present again two examples to illustrate bargaining, this time on match $(\ell, \ell)$. Panels $a$ and $b$ in Figure 2 present bargaining set $S_{\ell\ell}$ with hyperphily and a modular production function. In panel $b$, types are closer: $\ell = 1.66$ and $h = 2$, whereas in panel $a$ $\ell = 1$ and $h = 2$. It is easy to see that hyperphily solves the bargaining problem in panel $a$. In panel $b$, however, an alternative agreement with both $\ell$ agents choosing not to leave each other makes them better off, so hyperphily does not solve the bargaining problem.

**Figure 2: Bargaining Sets $S_{\ell\ell}$**

Note: $\rho = 0.1$, $r = 0.1$, $\delta = 0.05$, $h = 2$ and $F = \ell + h$.

In fact, in the example presented in panel $b$, the second line in Condition 2 is also violated. An agreement that makes 1) one $\ell$ indifferent to a match with $h$ and 2) the second $\ell$ at least as well off as before is also feasible.

**Bargaining Solution in Match $(h, h)$**

There is no endogenous destruction in match $(h, h)$ and agents split the surplus evenly. Therefore, no consistent agreement leads to a higher product of individual surpluses.
Equilibrium with Hyperphily

Our first proposition summarizes the necessary and sufficient conditions for hyperphily.

**Proposition 1. Equilibrium with Hyperphily.** We characterize explicitly the set of primitives \((\ell, h, F, r, \rho, \delta)\) such that an equilibrium with hyperphily exists.

*Proof.* Equation (3), and Conditions 1 and 2 generate 8 inequalities which determine when hyperphily can be an equilibrium. Whenever Conditions 1 and 2 are satisfied, then equation (3) also is. We express Conditions 1 and 2 as explicit functions of \((\ell, h, F, r, \rho, \delta)\). We present the details in Appendix A.6. ■

Figure 3 illustrates the set of primitives \((\ell, h, F, r, \rho, \delta)\) which lead to hyperphily. The shaded areas in panels a, b, c and d represent the set of values of \(F\) consistent with an equilibrium with hyperphily as a function of the matching rate \(\rho\), the destruction rate \(\delta\), the discount rate \(r\) and the difference between \(h - \ell\) respectively.

As we see in panel a, low values of \(\rho\) allow for hyperphily even when the production function is significantly submodular. As \(\rho\) decreases, the probability that \(h\) leaves the match \((\ell, h)\) becomes lower, so compensating her to make her stay becomes less attractive. In the limit as \(\rho \to 0\), hyperphily is an equilibrium for all degrees of complementarity in the production function. On the other side, as \(\rho \to \infty\), the duration of any match with voluntary destruction approaches zero. Thus, hyperphily cannot be an equilibrium.

As we see in panel b, higher values for the destruction rate \(\delta\) make hyperphily more likely. As \(\delta\) increases, endogenous destruction becomes less relevant relative to exogenous destruction. Therefore the maximum degree of submodularity tolerated by hyperphily increases. As \(\delta \to \infty\), the duration of every match goes to zero independently of the allocation of production, so hyperphily holds for every value of the other primitives. On the other side, lower values of \(\delta\) leave less room for hyperphily. When \(\delta\) is low, there are few unmatched agents. Being unmatched becomes relatively less attractive, since it takes a long time to find a partner. However, if agents are impatient enough, as \(\delta \to 0\) there are still equilibria with hyperphily, even when the production function is submodular.

Panel c illustrates the intuition discussed early in the Introduction. As agents become
Figure 3: Existence of Equilibrium with Hyperphily

Note: In (a), $\ell = 1, h = 2, \delta = 0.05$ and $r = 0.1$. In (b), $\ell = 1, h = 2, \rho = 0.1$ and $r = 0.1$. In (c), $\ell = 1, h = 2, \delta = 0.05$ and $\rho = 0.1$. In (d), $\rho = 0.1, \delta = 0.05$ and $\ell + h = 3$, with $0 < \ell < 1.5 < h < 3$.

more impatient (higher $r$), complementarity in production becomes less important relative to rent splitting. In the limit as $r \to \infty$, hyperphily is an equilibrium for any degree of complementarity in the production function. When agents are patient, there are equilibria with hyperphily provided that the complementarity in production is not too strong.

Panel d illustrates the example discussed later in the Introduction. When the difference between types is close to zero, $\ell$ does not get much from extracting surplus from $h$. Thus, $\ell$ makes $h$ indifferent, so he does not leave for another $h$. Agreement $c_1$ leads to a higher product of surpluses in match $(\ell, h)$. As $h - \ell$ increases, hyperphily becomes an equilibrium for a larger set of $F$. Moreover, as $\ell$ approaches 0, hyperphily holds even for a significantly submodular production function.
3.2 All Possible Equilibria

We present a complete characterization of equilibria in this subsection. Depending on the value of the primitives, several different equilibria arise in our simple two type model. In principle, there could be nine different types of equilibria, each associated to a different vector \( d^* \). Table 1 shows all of them. We present necessary and sufficient conditions for the existence of all types of equilibrium. Thus, we obtain necessary and sufficient conditions for hyperphil to be the only possible equilibrium.

Positive assortative matching can arise not only with hyperphil but also with strict or weak homophily.\(^8\) Therefore, a full characterization of the model allows us to present necessary and sufficient conditions for the existence and uniqueness of an equilibrium with positive assortative matching.

Characterizing each equilibrium involves going through the same process as already performed for hyperphil. First, we select agreements that satisfy condition 3 in our equilibrium definition. Then, we verify that transitions are consistent. Lastly, for each possible match, we verify that the equilibrium agreement solves the bargaining problem.

**PROPOSITION 2. All Equilibria with Two Types.** For each possible type of equilibrium in Table 1 we characterize explicitly the set \((\ell, h, F, r, \rho, \delta)\) such that the equilibrium exists.\(^9\)

<table>
<thead>
<tr>
<th>( \ell )'s decision</th>
<th>( h )'s decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d^*(\ell, \ell, h) = 1 )</td>
<td>Hyperphil (positive sorting)</td>
</tr>
<tr>
<td>( d^*(\ell, h, \ell) = 0 )</td>
<td>Weak Heterophil</td>
</tr>
<tr>
<td>( d^*(\ell, \ell, \ell) = 0 )</td>
<td>Strict Heterophil (negative sorting)</td>
</tr>
<tr>
<td>( d^*(\ell, h, h) = 0 )</td>
<td>Weak Homophil (A) (positive sorting)</td>
</tr>
<tr>
<td>( d^*(\ell, \ell, h) = 0 )</td>
<td>Indifference (random sorting)</td>
</tr>
<tr>
<td>( d^*(\ell, h, \ell) = 1 )</td>
<td>Strict Homophil (positive sorting)</td>
</tr>
<tr>
<td>( d^*(\ell, h, h) = 0 )</td>
<td>Weak Homophil (B)</td>
</tr>
<tr>
<td>( d^*(\ell, \ell, \ell) = 0 )</td>
<td>Impossible</td>
</tr>
</tbody>
</table>

\(^8\)In the Online Appendix, for each possible equilibrium, we present closed-form solutions for densities \( e(x, y) \) and we show the sorting pattern that arises. In Table 1 we indicate if sorting is positive, negative, or random. Equilibria with weak heterophil and weak homophil (B) feature no stochastic dominance.

\(^9\)The sets are obtained analogously to those from Proposition 1. We present closed-form solutions for these sets (and how they are obtained) in the Online Appendix.
We discuss now the main results regarding other equilibria. First, note that with a supermodular production function the equilibrium cannot feature neither weak nor strict heterophily. To see this, note that $\pi^*(h, \ell) < h$ makes a $h$-type agent strictly prefer another agent of type $h$. Similarly, $\pi^*(\ell, h) < \ell$ makes an $\ell$-type agent strictly prefer another agent of type $\ell$. Then, $F \geq h + \ell$ is a necessary condition for both weak and strict heterophily. It is also straightforward to show that neither weak nor strict homophily can be equilibria with a submodular production function. Finally, only strict heterophily exists when $h$ strictly prefers $\ell$ (see Appendix A.3 for details).

Strict heterophily is the only equilibrium featuring negative assortative matching. Thus, sorting negatively only occurs with a submodular production function. Positive assortative matching occurs both with homophily and hyperphily. Random sorting only happens if both $h$ and $\ell$ are indifferent, which requires $\pi^*(\ell, h) = \ell$ and $\pi^*(h, \ell) = h$. Hence indifference, and therefore random sorting, can only happen if the production function is modular.

Figure 4 illustrates the set of primitives which lead to each possible equilibrium. Equilibria with strict heterophily or strict homophily are rare, as shown in panel c of Figure 4. In strict heterophily $h$ prefers a match with $\ell$ over a more productive match with another $h$. This can happen when $\ell$’s outside option is lower than $h$’s. Therefore, although the production of the match $(\ell, h)$ is smaller than the production of the match $(h, h)$, the total surplus of the match $(\ell, h)$ is larger than the total surplus of the match $(h, h)$. On the other hand, strict homophily requires $\ell$ to strictly prefer another $\ell$, which is demanding given that the match $(\ell, h)$ is more productive. As in the case of strict heterophily, the agent prefers a less productive match because its total surplus is larger. When $r$ or $\delta$ increase, or when $\rho$ decreases, the outside option becomes less relevant and therefore strict homophily and strict heterophily require stronger complementarity in production.

If agents can search while matched, the match duration depends on the bargaining outcome. Symmetric surplus splitting might not solve the bargaining problem, as it oc-

---

10The shaded areas in Figure 4 represent the set of values of $F$ consistent with each equilibrium as a function of the destruction rate $\delta$. In the Online Appendix we present the corresponding figures for $\rho$, $r$, and $h - \ell$. 21
curs in the cases of weak heterophily and weak homophily. In these equilibria, one agent is indifferent between partner types and takes a larger fraction of the total surplus in the match \((\ell, h)\). Uneven surplus splitting produces a larger product of surpluses because it implies a longer duration of the match and a larger total surplus. These equilibria are more likely to exist when agents care more about endogenous destruction (when \(r\) or \(\delta\) are low); or when it is easier to find partners (when \(\rho\) is large). This is shown in panel \(b\) and \(d\) of Figure 4.

As frictions vanish, the outcome does not necessarily approach that of the frictionless market in Becker [1973]. Consider the index of labor market frictions \(\kappa \equiv \frac{\rho}{\delta}\).\(^{11}\) A larger \(\kappa\) implies weaker frictions. With submodularity, one would expect perfect negative sorting in a frictionless market. In contrast, we show that hyperphily, and thus positive sorting, can arise with submodularity when \(\kappa \to \infty\). The equilibrium outcome in the limit depends on whether it is \(\rho\) or \(\delta\) what drives \(\kappa \to \infty\). On one side, if \(\kappa\) is large because \(\rho\) is large, hyperphily does not occur. On the other side, if \(\kappa\) is large because \(\delta\) is small, there are equilibria with hyperphily, even with a submodular production function. Patient enough \(\ell\)-type agents are happy to trade a shorter duration of the match for a higher allocation. Interestingly, as \(\delta \to 0\) sorting becomes perfectly positive, instead of perfectly negative as in Becker [1973].

As frictions grow, positive assortative matching becomes pervasive. When \(\delta\) and \(r\) increase, or when \(\rho\) decreases, the region where hyperphily is the unique equilibrium grows. In the next section we obtain this as a general result for any number of types. This result goes against the idea that stronger frictions require stronger complementarity in production for the equilibrium to be positively assortative.

4. The Case with \(N\) Types

In this section we study equilibrium behavior for any discrete number of types. We apply the intuition described in the introduction and developed for two types to the case of \(N\)

\(^{11}\kappa\) is used as an index of frictions in several papers. See Ridder and van den Berg [2003] for an example.
Figure 4: The Impact of Destruction Rate $\delta$

types. With a large number of types one cannot characterize equilibrium behavior for all parameter values. We address this in two ways. First, we consider the case when the future is of limited importance for agents. This case encompasses impatient agents (high value for $r$), matches with high exogenous destruction rates (high value for $\delta$), and low meeting rates (low values for $\rho$). We show that for low enough $\frac{\rho}{r+\delta}$, hyperphily is an equilibrium, and no other equilibrium exists.\textsuperscript{12} Second, we present numerical examples.

We extend the model to allow for $N$ types: $x \in X = \{1, 2, \ldots, N\}$. Types are in equal proportion in the population: $e(x, y) : X^2 \cup \{\emptyset\} \rightarrow \left[0, \frac{1}{N}\right]$ has $\sum_{y \in X \cup \emptyset} e(x, y) = \frac{1}{N}$. We assume the production function $f(x, y) : X^2 \rightarrow \mathbb{R}_+$ is strictly increasing in both variables.

\textsuperscript{12}We computed the steady state distribution under hyperphily for 1,000 values of $\kappa^{-1} \in (0, 1)$ and 1,000 values of $\kappa \in (0, 1)$ for $N = 10, 20$ and $100$. In all cases there is positive assortative matching in the steady state distribution.
All other functions are modified appropriately to allow for $N$ types.

### 4.1 When the Future is of Limited Importance

When agents care mostly about the present, it is current payoffs that matter the most. We show in Proposition 3 that this makes hyperphily an equilibrium, and in fact the only possible one. First, to show that only hyperphily can be an equilibrium, we argue that for low values of $\frac{\rho}{r+\delta}$, flow equilibrium payoffs are close to an equal split of production. We show then that when $\frac{\rho}{r+\delta}$ is low, surplus from the match depends mostly on current payoffs. Since payoffs depend on total production and production is increasing in the partner’s type, then surplus is increasing in partner type. Only hyperphily can be an equilibrium. Second, we argue that hyperphily is in fact an equilibrium. To show this, we compute individual surpluses under hyperphily, and show that for low values of $\frac{\rho}{r+\delta}$, surpluses depend mainly on production. This makes hyperphily consistent. Finally, we show that no alternative consistent agreement leads to a higher product of individual surpluses.

**Proposition 3. Hyperphily with N Types.** When $\frac{\rho}{r+\delta}$ is low enough, $(d^*,\pi^*)$ is an equilibrium if and only if it features hyperphily.

See Appendix A.4 for the proof.

### 4.2 Numerical Examples with N Types

In Section 3 we provide necessary and sufficient conditions for hyperphily in a model with two types. We then show in Section 4.1 that when there are $N$ types, as the future becomes less relevant hyperphily is the unique equilibrium. We provide now numerical examples in a model with $N$ types for parameter values in line with the literature. We show that equilibria with hyperphily and positive assortative matching arise without complementarity in production.

We solve the model by a nested fixed point algorithm. We start from a flat distribution of matches and calculate value functions for all possible matches. These first value
functions induce preferences over partner types which we use to update the steady state
distribution of matches. With the updated steady state distribution, we update the value
functions. We iterate this process until we find a fixed point for both the steady state
distribution of matches and the value functions.

We search specifically for equilibria without indifference over partners. When no
agent is indifferent, symmetric surplus splitting solves the bargaining problem in all
matches. Once we find a candidate set of value functions and distribution of matches
that solves the model, we check that the solution maximizes the product of surpluses in
all matches. To do this, for each match we evaluate all possible consistent agreements
\(c = (\hat{d}, \hat{\pi})\), given our candidate. Our candidate solves the model if it maximizes the
product of surpluses in every match.

The following example shows hyperphily for a case with \(N = 100\).

**Example 1.** *Types are uniformly distributed in a 100-point grid between 0 and 1. Production*
*is modular: \(f(x, y) = x + y\). (\(\delta = 0.05\), \(r = 0.1\) and \(\rho = 0.1\)).*

In Example 1, \(S^*(x, y)\) is increasing in partner type, so hyperphily is an equilibrium.
Panel a in Figure 5 shows the probability distribution function \(e(x, y)\) and panel b shows
the cumulative distribution function \(E(x, y) = \sum_{\tilde{y} \in X \cup \{\emptyset\}, \tilde{y} < y} e(x, \tilde{y})\). Sorting is defined in
terms of stochastic dominance, so panel b is informative on sorting patterns. The strictly
increasing contour lines of \(E(x, y)\) show that the cumulative distribution of partners is
decreasing in own type, which implies positive assortative matching.

A similar result holds with a slightly submodular production function. However, if
either \(\delta\) or \(r\) decrease enough or if \(\rho\) increases enough, symmetric surplus splitting does
not maximize the product of surpluses in some matches. Take Example 1 and double
the search intensity (so \(\rho = 0.2\)). Now, hyperphily does not maximize the product of
individual surpluses in matches where \(|x - y|\) is large. For a given set of parameters,
if there is at least one match where the agreement from hyperphily does not solve the
bargaining problem, then hyperphily is not an equilibrium. However, this does not imply
that matching is not positively assortative. As in the case of two types with weak and
strict homophily, there may be other equilibria with positive assortative matching.
Note: Panel \(a\) presents the density of matches \(e(x,y)\). Panel \(b\) presents the cumulative distribution function of partners \(E(x,y)\). Darker points correspond to higher values.

5. Extensions and Discussion

5.1 Different Search Intensities

We now relax the assumption that on-the-match search efficiency is equal to search efficiency out of the match. Let \(\rho_0\) denote the search intensity of an unmatched agent and let \(\rho_1\) be the search efficiency of a matched one. The meeting rate is simply the product of the search intensities of those who meet.\(^{13}\) The following example illustrates how different values of \(\rho_1\) affect the equilibrium.

\(^{13}\)For example, there are \(\rho_0\rho_1 e(\ell,\emptyset) e(h,h)\) unmatched \(\ell\)-type agents who meet \(h\)-type agents matched to other \(h\)-type agents. Similarly, there are \((\rho_0)^2 e(\ell,\emptyset) e(h,\emptyset)\) unmatched \(\ell\)-type agents who meet unmatched \(h\)-type agents. Our approach here is similar to Bobbio [2009].
**Example 2.** Types are uniformly distributed in a 50-point grid between 0 and 1. Production is modular: \( f(x,y) = x + y \). \((\delta = 0.05, r = 0.1 \text{ and } \rho_0 = \sqrt{0.1})\). Consider three cases: (i) \( \rho_1 = 0 \), (ii) \( \rho_1 = \frac{1}{3}\rho_0 \), and (iii) \( \rho_1 = \frac{2}{3}\rho_0 \)

Figure 6: Different on-the-job and out-of-the-job Search Intensity in Example 2

<table>
<thead>
<tr>
<th>a: ( e(x,y) ) with ( \rho_1 = 0 )</th>
<th>c: ( e(x,y) ) with ( \rho_1 = \frac{1}{3}\rho_0 )</th>
<th>e: ( e(x,y) ) with ( \rho_1 = \frac{2}{3}\rho_0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>b: ( E(x,y) ) with ( \rho_1 = 0 )</td>
<td>d: ( E(x,y) ) with ( \rho_1 = \frac{1}{3}\rho_0 )</td>
<td>f: ( E(x,y) ) with ( \rho_1 = \frac{2}{3}\rho_0 )</td>
</tr>
</tbody>
</table>

Note: Panels a, c and e present the density of matches \( e(x,y) \) for different values of \( \rho_1 \). Panels b, d and f present the cumulative distribution function of partners for different values of \( \rho_1 \). Darker points correspond to higher values.

Figure 6 shows how in Example 2 positive sorting does not hold for low values of \( \rho_1 \), but it does as \( \rho_1 \) increases. Upper panels in Figure 6 show densities \( e(x,y) \) while lower panels show cumulative distributions \( E(x,y) \). Without on-the-match search \( (\rho_1 = 0) \) the equilibrium features hyperphily and negative assortative matching: the contour lines of \( E(x,y) \) are decreasing in \( x \) in panel b. When we allow agents to search on the match but with low search efficiency, there is assortative matching for agents of high type, but low agents still prefer to wait unmatched for more profitable partners. Therefore matching is not positively assortative for low type agents. With the parameter values used in these
simulations, for values of search efficiency on the match as low as two thirds of the search efficiency out of the match, there is no difference in acceptance sets between unmatched agents of different types. All unmatched agents accept all partners, and since agents search on the match and prefer better partners, there is positive assortative matching in the whole support of types.

Example 2 also illustrates how, if on-the-match search is not allowed, hyperphily does not imply positive sorting. This is true in general in a model like Shimer and Smith [2000]. Since only unmatched agents search, the steady state distribution of types is shaped by acceptance sets. In an equilibrium with hyperphily, the total surplus of the match is strictly increasing in types. In this way, if a partner of type \( x \) is accepted by an agent of type \( y \), then every \( x' \geq x \) is also accepted by every \( y' \geq y \). Therefore, the lower bound of the acceptance set is non increasing in own type (and there is no upper bound for hyperphily). As a result, when all agents accept everybody there is random sorting. Otherwise, there is negative sorting.

The disconnect between hyperphily and positive sorting when agents cannot search on the match explains why with a modular, or slightly supermodular production function there is no positive sorting in Shimer and Smith [2000]. It is easy to show from Equations (5) and (8) in Shimer and Smith [2000] that if complementarity in production is weak enough the equilibrium features hyperphily. Then, a slightly supermodular production function does not lead to positive assortative matching.

5.2 An Example with Renegotiation

The main result in this paper is that frictions can lead to positive sorting, even without productive complementarity. In our stylized model, agents are not allowed to renegotiate how to split production when one of them meets an alternative partner. However, the mechanism we highlight can also hold if agents are allowed to renegotiate. We present next an example with on-the-match search, renegotiation and no complementarity in production that features hyperphily and positive assortative matching.

Since both partners’ outside options change, modeling renegotiation with bilateral
on-the-match search is not straightforward. Kiyotaki and Lagos [2007] present a search model where both the firm and the worker search on the match. In their setting, contracts can be renegotiated if a partner has a credible threat to dissolve the match. When an agent finds an alternative partner, her current partner and the poaching one compete à la Bertrand. Kiyotaki and Lagos do not study sorting since agents are homogeneous in their model. Matches are heterogeneous only due to a fixed match-specific productivity shock.

For simplicity, consider infinitely impatient agents of one of two types who bargain à la Kiyotaki and Lagos. As discussed in Section 4.1, in the limit the value of the match depends only on the flow-payoff received, and outside options converge to zero. Whenever an unmatched agent meets a matched one, both agents competing (the poaching one, and the current partner) have being unmatched (with zero value) as outside option. The more productive one can make a better offer, so he always wins. However, when matched agents meet other matched agents, some of the transitions that occur without renegotiation no longer happen.

Renegotiation prevents inefficient separations: the sum of the surplus of the destroyed matches cannot exceed the surplus of the newly created one. Therefore, in this simple example, when an $h$-type agent matched to an $\ell$-type meets another $h$-type also matched to an $\ell$-type, both $h$-type agents renegotiate their contracts and no match is destroyed (see Proposition 1 in Kiyotaki and Lagos). However, both $h$ and $\ell$ still leave $\ell$ when they find an unmatched $h$. Therefore the steady-state distribution of partners of $h$ stochastically dominates the distribution of partners of $\ell$, as we show in Appendix A.5.1. In this way, although no inefficient separations take place when renegotiation is allowed, frictions can lead to positive assortative matching without productive complementarity.

6. Conclusion

In this paper we argue that frictions represent a driving force towards positive assortative matching. While in frictionless markets payoffs reflect individual contributions, when it takes time to find a partner the division of output becomes more even. Therefore
in markets with large frictions, the total production of the match, rather than individual contributions, becomes the main determinant of preferences over partners. Since production is increasing in partner’s type, an endogenous preference for better types arises. If individuals search while matched, more productive agents upgrade their partners more often, and therefore the equilibrium distribution becomes positively assortative.

A key element in our analysis is that agents are allowed to search while matched. Match-to-match transitions are pervasive in markets with two-sided heterogeneity. We present a partnership model that includes the key elements of Shimer and Smith [2000] but allows for bilateral on-the-match search.\(^{14}\) We first analyze the case where agents are of one of two types: either low or high productivity. We provide necessary and sufficient conditions for hyperphilic and show that this preference leads to positive assortative matching. We highlight conditions such that positive assortative matching arises even with a modular or submodular production function. Moreover, when frictions are sufficiently large, this is the unique equilibrium. Our result extends to the case of any (finite) number of types. We show that as agents become impatient or frictions large, an equilibrium where agents endogenously prefer higher types exists and it is the only possible one.

The conventional wisdom states that stronger frictions require stronger complementary in production for positively assortative matching to arise. In contrast, this paper highlights that frictions can lead to positive sorting. Our result refutes the interpretation of sorting in markets with frictions as evidence of complementary in production.

### A. Appendix

#### A.1 Bargaining Sets are Compact

**Lemma 2.** Take any state of the economy \((S^*, q^*)\). Then, bargaining sets \(S_{xy}\) under on-the-match search are compact.

\(^{14}\)Most recent studies on assortative matching in markets with frictions and transferable utility, such as Lopes de Melo [2013], Hagedorn, Law, and Manovskii [2012], and Lise, Meghir, and Robin [2013], take the canonical model of Shimer and Smith [2000] as a starting point.
Proof. Since \( r > 0 \) and \( f(x,y) \) is finite, \( S_{xy} \) is bounded. We show next that \( S_{xy} \) is also closed. Take a sequence \( \{(S^n_1, S^n_2)\}_{n=1}^{\infty} \in S_{xy} \) generated by a sequence of consistent agreements \( \{(\hat{d}^n, \hat{\pi}^n)\}_{n=1}^{\infty} \) and with \( \lim_{n \to \infty} (S^n_1, S^n_2) = (\overline{S}_1, \overline{S}_2) \). We show there is a consistent agreement that generates \((\overline{S}_1, \overline{S}_2)\), and so \((\overline{S}_1, \overline{S}_2) \in S_{xy}\). Since \( \{(S^n_1, S^n_2)\}_{n=1}^{\infty} \) converges, there exists \( N \) such that \( \forall n > N, \)

\[
\max_{y' \in X} \{ S^* (x, y') : S^* (x, y') < \overline{S}_1 \} < S^n_1 < \min_{y' \in X} \{ S^* (x, y') : S^* (x, y') > \overline{S}_1 \} \quad \text{and} \quad \max_{x' \in X} \{ S^* (y, x') : S^* (y, x') < \overline{S}_2 \} < S^n_2 < \min_{x' \in X} \{ S^* (y, x') : S^* (y, x') > \overline{S}_2 \}.
\]

Whenever \( S^n_i > \overline{S}_i \) or \( S^n_i < \overline{S}_i \), for \( n > N \), \( \hat{d}^n_i \) is unique. We use this fact repeatedly in this proof.

We consider first the case where no \( i \in \{1,2\} \) has \( S^n_i = \overline{S}_i \) infinitely often. Then, there is a subsequence \( \{(S^m_{1m}, S^m_{2m})\}_{m=1}^{\infty} \) with either 1) \( S^m_{1m} > \overline{S}_1 \) and \( S^m_{2m} > \overline{S}_2 \), or 2) \( S^m_{1m} > \overline{S}_1 \) and \( S^m_{2m} < \overline{S}_2 \), or 3) \( S^m_{1m} < \overline{S}_1 \) and \( S^m_{2m} > \overline{S}_2 \), or finally 4) \( S^m_{1m} < \overline{S}_1 \) and \( S^m_{2m} < \overline{S}_2 \). In any such subsequence, for \( m \) big enough \( \hat{d}^m_{1m} = \overline{d} \) is constant. So \( S^m_{1m} \) and \( S^m_{2m} \) are simply linear functions of \( \hat{\pi}^m_{1m} \) and \( \hat{\pi}^m_{2m} \). Since \( S^m_{1m} \) converges, so does \( \hat{\pi}^m_{1m} \to \overline{\pi} \). Moreover, since \( \hat{\pi}^m_{1m} + \hat{\pi}^m_{2m} \leq f(x, y) \forall m \), then also \( \overline{\pi}_1 + \overline{\pi}_2 \leq f(x, y) \). Thus, \( \overline{\pi} = (\overline{d}, \overline{\pi}) \) generates \((\overline{S}_1, \overline{S}_2)\) and is consistent.

Next, we consider the case with \( S^n_i = \overline{S}_i \) infinitely often for some \( i \in \{1,2\} \). If \( (S^n_1, S^n_2) = (\overline{S}_1, \overline{S}_2) \) for some \( n \), then of course \((\overline{S}_1, \overline{S}_2) \in S_{xy}\). Otherwise, without loss of generality, let \( i = 1 \). Then there is a subsequence \( \{(S^m_{1m}, S^m_{2m})\}_{m=1}^{\infty} \) with \( S^m_{1m} = \overline{S}_1 \) and either always \( S^m_{2m} > \overline{S}_2 \), or always \( S^m_{2m} < \overline{S}_2 \). In any such subsequence for \( m \) big enough \( \hat{d}^m_{2m} = \overline{d} \). Since \( S^m_{1m} = \overline{S}_1 \), \( \hat{\pi}_1 = \overline{\pi}_1 \) is also constant. Define \( \overline{\pi}_2 = f(x, y) - \overline{\pi}_1 \geq \hat{\pi}^m_{2m} \).

Let \( \overline{d}_1 \) be the most beneficial to 2 (so 1 does not leave if indifferent). Let \( \overline{S} = (\overline{S}_1, \overline{S}_2) \) be induced by \( (\overline{d}_1, \overline{d}_2) \) and \( (\overline{\pi}_1, \overline{\pi}_2) \). \( \overline{d}_1 \) is no worse than what 2 gets in the subsequence, and \( \overline{\pi}_2 \geq \hat{\pi}^m_{2m} \). Then \( \overline{S}^m_{2m} \leq \overline{S}_2 \). Thus, \( \overline{S}_2 = \lim_{m \to \infty} S^m_2 \leq \overline{S}_2 \). If \( \overline{S}_2 = \overline{S}_2 \) we are done. Otherwise, decrease \( \overline{\pi}_2 \) to make it so. ■

A.2 Details on Multiplicity of Equilibria

In the Online Appendix we provide a simple example of how on-the-match search can lead to some uninteresting multiplicity of equilibrium. The third condition in our definition of equilibrium guarantees that equilibria are stable in the following sense.

Assume that agents have to pay a small cost \( t > 0 \) each time they quit their current partner to form a new match. Surplus from matches are then given by the following
We show our result by contradiction. Assume resulting from that agreement. Then, we show first that if \(\hat{\text{s}}\) build a willing partner. Moreover, again for small \(\epsilon > 0\), the product of individual surpluses is larger. Then, the original pair \((d^*, \pi^*)\) does not solve the bargaining problem.

\[
S^*(x, y) = \left( r + \delta + \sum_{x' \in X} d^*(y, x, x') q^*(y, x') \right)^{-1} \left[ \pi^*(x, y) + \sum_{y' \in X} d^*(x, y, y') q^*(x, y') \left[ S^*(x, y') - S^*(x, y) - t \right] - \sum_{y' \in X} q^*(x, y') S^*(x, y') \right]
\]

Take a pair \((d^*, \pi^*)\) satisfying the first two conditions in our equilibrium definition. We show next that \(S^*(x, y) > S^*(y, x) \Rightarrow \exists y' : S^*(x, y) = S^*(x, y') - t\) must be satisfied. Assume it is not. Then, there exists an alternative consistent agreement between \(x\) and \(y\) which leads to a higher product of individual surpluses. To build it, keep the decision function unchanged but pick \(\tilde{\pi}(x, y) = \pi^*(x, y) - \epsilon\) and \(\tilde{\pi}(y, x) = \pi^*(y, x) + \epsilon\). For small \(\epsilon > 0\), agent \(x\) does not change his behavior. Thus, the new pair \((d^*, \tilde{\pi})\) is consistent. Moreover, again for small \(\epsilon > 0\), the product of individual surpluses is larger. Then, the original pair \((d^*, \pi^*)\) does not solve the bargaining problem.

A.3 Only Strict Heterophily with \(d^*(h, h, \ell) = 1\)

**Lemma 3.** \(S^*(h, h) > S^*(h, h) \Rightarrow S^*(\ell, h) > S^*(\ell, \ell)\).

**Proof.** First, since \(S^*(h, \ell) > S^*(h, h)\), the third condition in the equilibrium definition guarantees \(S^*(\ell, h) \geq S^*(h, \ell)\). Next, consider the following alternative agreement for \((h, h)\): they never leave each other and they split production. Let \(\tilde{S}\) denote the surplus resulting from that agreement. Then,

\[
S^*(h, \ell) \geq \tilde{S} = (r + \delta)^{-1} \left[ h - q^*(h, \ell) S^*(h, \ell) - q^*(h, h)S^*(h, h) \right]
\]

We show our result by contradiction. Assume \(S^*(\ell, \ell) \geq S^*(h, h)\). Note that \(q^*(\ell, h) \geq q^*(h, h)\) and \(q^*(\ell, \ell) \geq q^*(h, \ell)\), since both agents prefer low types (at least weakly). Then,

\[
S^*(\ell, \ell) = (r + \delta)^{-1} \left[ \ell - q^*(\ell, \ell) S^*(\ell, \ell) - q^*(\ell, h)S^*(\ell, h) \right] < \tilde{S}
\]

To sum up, \(\tilde{S} > S^*(\ell, \ell) \geq S^*(\ell, h) \geq S^*(h, \ell) \geq \tilde{S}\). That is our contradiction. ■

A.4 Details on Proposition 3

We show first that if \((d^*, \pi^*)\) is an equilibrium, then it features hyperphily. To do this, we build a simple agreement for match \((x, y)\): they share production evenly and leave for any willing partner. \(\tilde{S} = \tilde{S}(x, y) = \tilde{S}(y, x)\) denotes the surplus from such agreement:

\[
\tilde{S} = \left( r + \delta + \sum_{x' \in X} q^*(y, x') + \sum_{y' \in X} q^*(x, y') \right)^{-1} \frac{f(x, y)}{2}
\]
Moreover, surplus is bounded above – see (1): \((r + \delta)\, S^* (x, y) \leq \pi^* (x, y)\). Then,

\[
\pi^* (x, y) \pi^* (y, x) \geq (r + \delta)^2 S^* (x, y) S^* (y, x) \geq \left( \frac{r + \delta}{r + \delta + 2\rho} \frac{f(x, y)}{2} \right)^2
\]

which implies

\[
(1 - k) \frac{f(x, y)}{2} \leq \pi^* (x, y) \leq (1 + k) \frac{f(x, y)}{2} \quad \text{with } k = \sqrt{1 - \left( 1 + 2\frac{\rho}{r + \delta} \right)^{-2}}. \tag{4}
\]

Next, consider matches \((x, y)\) and \((x, y + 1)\). In the best possible case for \(x\) when matched to \(y\), \(y\) never leaves her, and pays her the maximum value in (4). Let \(\mathcal{X}(y) = \{ y' \in X : d^* (x, y, y') = 1 \}\) and \(\mathcal{X}(y) = \{ y' \in X : d^* (x, y, y') = 0 \}\). Note that \(X = \mathcal{X}(y) \cup \mathcal{X}(y)\) and \(\mathcal{X}(y) \cap \mathcal{X}(y) = \emptyset\). Then \(S^* (x, y)\) is bounded above as follows:

\[
(r + \delta)\, S^* (x, y) \leq \pi^* (x, y) - \left[ \sum_{y' \in \mathcal{X}(y)} q^* (x, y') \right] S^* (x, y) - \sum_{y' \in \mathcal{X}(y)} q^* (x, y') S^* (x, y')
\]

\[
\leq \frac{f(x, y)}{2} (1 + k) - \left[ \sum_{y' \in \mathcal{X}(y)} q^* (x, y') \right] S^* (x, y) - \sum_{y' \in \mathcal{X}(y)} q^* (x, y') S^* (x, y')
\]

In the worst possible case when matched to \(y + 1\), \(y + 1\) always leaves \(x\), and pays her the minimum value in (4). Then \(S^* (x, y + 1)\) is bounded below as follows:

\[
(r + \delta)\, S^* (x, y + 1) \geq \pi^* (x, y + 1) - \left[ \sum_{x' \in X} q^* (y + 1, x') + \sum_{y' \in \mathcal{X}(y)} q^* (x, y') \right] S^* (x, y + 1)
\]

\[
- \sum_{y' \in \mathcal{X}(y)} q^* (x, y') S^* (x, y')
\]

\[
\geq (1 - k) \frac{f(x, y + 1)}{2} - \left[ \rho + \sum_{y' \in \mathcal{X}(y)} q^* (x, y') \right] S^* (x, y + 1)
\]

\[
- \sum_{y' \in \mathcal{X}(y)} q^* (x, y') S^* (x, y')
\]

\[
\geq \left( 1 - k - \frac{\rho}{r + \delta} (1 + k) \right) \frac{f(x, y + 1)}{2}
\]

\(^{15}\)The argument is straightforward. Consider the simple agreement described. Calculate \(x\)'s surplus. See who would \(x\) actually optimally choose to leave for. Assume \(x\) behaves that way. Notice now \(y\)'s surplus is weakly larger. Calculate \(y\)'s best response now. At each step, neither \(x\) nor \(y\) can be worse off. So they leave each time for less people. Eventually, the process stops. That behavior is consistent.
Then, whenever the following condition holds, \( S^*(x, y + 1) > S^*(x, y) \):

\[
\left( 1 - k - \frac{\rho}{r + \delta} (1 + k) \right) \frac{f(x, y + 1)}{2} > \frac{f(x, y)}{2} (1 + k)
\]

\[
\frac{1 - k - \frac{\rho}{r + \delta}}{1 + k} > \frac{f(x, y)}{f(x, y + 1)}
\]

And it is easy to show that \( 1 + 3 \frac{\rho}{r + \delta} - 4 \sqrt{\frac{\rho}{r + \delta} (1 + \frac{\rho}{r + \delta})} = \frac{1 - k}{1 + k} - \frac{\rho}{r + \delta} \). Then, whenever the following condition holds, only hyperphily can be an equilibrium:

\[
\max_{x, y} \frac{f(x, y)}{f(x, y + 1)} \leq 1 - \frac{\rho}{r + \delta} \left[ 4 \sqrt{1 + \left( \frac{\rho}{r + \delta} \right)^{-1}} - 3 \right] \quad (5)
\]

We show next that hyperphily is an equilibrium. Hyperphily is characterized by \( d^*(x, y, y') = 1 \) if and only if \( y' > y \) and \( \pi^* \) such that \( S^*(x, y) = S^*(y, x) \) for all \((x, y)\). We need to show that for low enough \( \frac{\rho}{r + \delta} \), \( (d^*, \pi^*) \) is an equilibrium.

First, under hyperphily, surpluses are given by:

\[
\left( r + \delta + \sum_{x' > x} q^*(y, x') + \sum_{y' > y} q^*(x, y') \right) S^*(x, y) + \sum_{y' \leq y} q^*(x, y') S^*(x, y') = \pi^*(x, y)
\]

\[
\left( r + \delta + \sum_{y' > y} q^*(x, y') + \sum_{x' > x} q^*(y, x') \right) S^*(y, x) + \sum_{x' \leq x} q^*(y, x') S^*(y, x') = \pi^*(y, x)
\]

\( S^*(x, y) = S^*(y, x) \) and \( \pi^*(x, y) + \pi^*(y, x) = f(x, y) \). Then, individual surpluses are given by

\[
2 \left( r + \delta + \sum_{x' > x} q^*(y, x') + \sum_{y' > y} q^*(x, y') \right) S^*(x, y) = f(x, y)
\]

\[
- \sum_{y' \leq y} q^*(x, y') S^*(x, y') - \sum_{x' \leq x} q^*(y, x') S^*(y, x') .
\]

Let \( \overline{F} = \max_{x, y} f(x, y) \). It is easy to find the following lower bound from (6):

\[
2 \left( r + \delta + \sum_{x' > x} q^*(y, x') + \sum_{y' > y} q^*(x, y') \right) S^*(x, y) \geq f(x, y)
\]
\[- \left( \sum_{x' \leq x} q^*(x, y') + \sum_{y' \leq y} q^*(y, x') \right) \frac{F}{r + \delta} \]

Then,

\[ 2 (r + \delta + 2\rho) S^*(x, y) \geq f(x, y) - 2F \frac{\rho}{r + \delta} \]

From (6), we create lower and upper bounds for \( S^*(x, y) \),

\[ \frac{f(x, y)}{2} - F \frac{\rho}{r + \delta} \leq S^*(x, y) \leq \frac{f(x, y)}{2} \]

Consider matches \((x, y)\) and \((x, y + 1)\). Given (7),

\[ S^*(x, y) \leq \frac{f(x, y)}{2}, \quad \text{and} \quad \frac{f(x, y + 1)}{2} - F \frac{\rho}{r + \delta} \leq S^*(x, y + 1). \]

For \( \frac{\rho}{r + \delta} \) small, \( f(x, y) < \frac{f(x, y + 1)}{2 - \frac{\rho}{r + \delta}} \). Then \( S^*(x, y + 1) > S^*(x, y) \) and thus \((d^*, \pi^*)\) is consistent. We need to verify next that no consistent agreement \( c \) leads to a higher product of surpluses. Any such agreement must have \( \tilde{S}^c_1 \geq S^*(x, y + 1) \) or \( \tilde{S}^c_2 \geq S^*(y, x + 1) \) or both.\(^\text{16}\) Assume without loss of generality that \( \tilde{S}^c_1 \geq S^*(x, y + 1) \). Next, note that for any agreement \( \tilde{S}^c_1 + \tilde{S}^c_2 \leq \frac{f(x, y)}{r + \delta} \). Again, pick \( \frac{\rho}{r + \delta} \) small, so \( S^*(x, y + 1) \geq \frac{f(x, y)}{2 - \frac{\rho}{r + \delta}} \). Then the product of surpluses must be bounded:

\[
\tilde{S}^c_1 \tilde{S}^c_2 \leq \frac{f(x, y + 1)}{2} - F \frac{\rho}{r + \delta} \left[ \frac{f(x, y)}{r + \delta} - \frac{f(x, y + 1)}{2} - F \frac{\rho}{r + \delta} \right]^{\frac{1}{2}} \leq S^*(x, y)S^*(y, x)
\]

where again the last inequality holds for small \( \frac{\rho}{r + \delta} \).

**A.5 Hyperphil and Positive Assortative Matching**

Proof. \( h \)'s distribution of partners first order stochastically dominates \( \ell \)'s if and only if \( e(\ell, \emptyset) > e(h, \emptyset) \) and \( e(\ell, \emptyset) + e(\ell, \ell) > e(h, \emptyset) + e(h, \ell) \). Steady state conditions for \(^{16}\)To see this, note that if \( \tilde{S}^c_1 < S^*(x, y + 1) \) and \( \tilde{S}^c_2 < S^*(y, x + 1) \) then neither agent leaves the other less often. Then \( \tilde{S}^c_1 + \tilde{S}^c_2 \leq 2S^*(x, y) \).
\( e(\ell, \varnothing) \) and \( e(h, \varnothing) \) require, respectively, that:

\[
\delta[\frac{1}{2} - e(\ell, \varnothing)] + \rho e(\ell, \ell) e(h, \varnothing) + \rho e(\ell, h) [e(h, \varnothing) + e(\ell, h)] = \rho e(\ell, \varnothing) [e(\ell, \varnothing) + e(h, \varnothing)]
\]

(8)

\[
\delta[\frac{1}{2} - e(h, \varnothing)] = \rho e(h, \varnothing) [e(\ell, \varnothing) + e(\ell, \ell) + e(h, \varnothing) + e(h, \ell)]
\]

(9)

Then \( e(\ell, \varnothing) > e(h, \varnothing) \). Otherwise, \( e(\ell, \varnothing) \) and \( e(h, \varnothing) \) cannot jointly be in steady state.\(^{17}\) Next, consider steady state conditions for \( e(\ell, h) \) and \( e(h, h) \), respectively:

\[
\rho[\frac{1}{2} - e(\ell, h)] e(h, \varnothing) = e(h, \ell) [\delta + \rho e(h, \varnothing) + \rho e(h, \ell)]
\]

(10)

\[
\rho[\frac{1}{2} - e(h, h)]^2 = \delta e(h, h)
\]

(11)

So \( e(h, h) > e(h, \ell) \). Otherwise, \( e(\ell, \varnothing) \) and \( e(h, \varnothing) \) cannot jointly be in steady state.\(^{18}\)

A.5.1 The Two Type Case with Renegotiation

Renegotiation prevents inefficient separations. In the example presented in Section 5.2 individuals who meet unmatched agents switch partners as often as with hyperphil with on-the-match-search. However, if the production function is not supermodular, as \( 2F > 2h \), a type-\( h \) agent matched to a \( \ell \)-type one does not break the match when she finds a matched \( h \)-type agent. In contrast, in an equilibrium featuring hyperphil with on-the-match-search, \( h \) leaves \( \ell \) if she meets another \( h \) matched to \( \ell \). Therefore the steady state distribution of matches in a model with and without renegotiation may differ.\(^{19}\) However, it is still straightforward to show that \( h \)'s distribution of partners first order stochastically dominates \( \ell \)'s.

With renegotiation, steady state conditions for \( e(\ell, \varnothing), e(h, \varnothing), e(\ell, h), \) and \( e(h, h) \) now require, respectively, that:

\[
\delta[\frac{1}{2} - e(\ell, \varnothing)] + \rho e(\ell, \ell) e(h, \varnothing) + \rho e(\ell, h) e(h, \varnothing) = \rho e(\ell, \varnothing) [e(\ell, \varnothing) + e(h, \varnothing)]
\]

\[
\delta[\frac{1}{2} - e(h, \varnothing)] = \rho e(h, \varnothing) [e(\ell, \varnothing) + e(\ell, \ell) + e(h, \varnothing) + e(h, \ell)]
\]

\[
\rho[\frac{1}{2} - e(\ell, h)] e(h, \varnothing) = e(h, \ell) [\delta + \rho e(h, \varnothing)]
\]

A proof analogous to that without renegotiation shows that \( h \)'s distribution of partners first order stochastically dominates \( \ell \)'s.

---

\(^{17}\) Assume to the contrary that \( e(\ell, \varnothing) > e(h, \varnothing) \). Then, the right hand side is lower in (8) than in (9), but the left hand side is lower in (10) than in (8).

\(^{18}\) Assume to the contrary that \( e(h, h) > e(h, \ell) \). Then, the right hand side is lower in (11) than in (10), but the left hand side is lower in (10) than in (11).

\(^{19}\) If only one side of the market searches on the match, the distribution of matches in a model with or without renegotiation is the same (see Bartolucci [2013]). This is because a matched agent can never meet another matched agent.
A.6 Conditions for Hyperphily

Under hyperphily \(d(\ell, \ell, h) = d(h, \ell, h) = 1\) and \(d(\ell, h, \ell) = d(h, h, \ell) = 0\). Then, the steady state conditions become:

\[
e(\ell, \ell) [\delta + q(\ell, h)] + e(\ell, h) [\delta + q(h, h)] = e(\ell, \emptyset) [q(\ell, \ell) + q(\ell, h)]
\]

\[
e(\ell, \emptyset) q(\ell, \ell) = e(\ell, \ell) [\delta + 2q(\ell, h)]
\]

\[
[e(\ell, \emptyset) + e(\ell, \ell)] q(\ell, h) = e(\ell, h) [\delta + q(h, h)]
\]

\[
\delta [e(h, \ell) + e(h, h)] = e(h, \emptyset) [q(h, \ell) + q(h, h)]
\]

\[
[e(h, \emptyset) + e(h, \ell)] q(h, h) = \delta e(h, h)
\]

The successful meeting rates become, \(q(\ell, \ell) = \rho e(\ell, \emptyset), q(\ell, h) = \rho e(h, \emptyset), q(h, \ell) = \rho [e(\ell, \emptyset) + e(\ell, \ell)]\) and \(q(h, h) = \rho [e(h, \emptyset) + e(h, \ell)]\). Substituting these into the steady state conditions, dividing by \(\rho\), and setting \(\kappa = \frac{\ell}{2}\), we get

\[
e(\ell, \ell) \left[\kappa^{-1} + e(\ell, \emptyset)\right] + e(\ell, h) \left[\kappa^{-1} + \frac{1}{2} - e(h, h)\right] = e(\ell, \emptyset) \left[e(\ell, \emptyset) + e(h, \emptyset)\right] \tag{12}
\]

\[
e(\ell, \emptyset)^2 = e(\ell, \ell) \left[\kappa^{-1} + 2e(h, \emptyset)\right] \tag{13}
\]

\[
[e(\ell, \emptyset) + e(\ell, \ell)] e(h, \emptyset) = e(\ell, h) \left[\kappa^{-1} + e(h, \emptyset) + e(h, \ell)\right] \tag{14}
\]

\[
\kappa^{-1} [e(h, \ell) + e(h, h)] = e(h, \emptyset) \left[e(\ell, \emptyset) + e(\ell, \ell) + e(h, \emptyset) + e(h, \ell)\right] \tag{15}
\]

\[
[e(h, \emptyset) + e(h, \ell)]^2 = \kappa^{-1} e(h, h) \tag{16}
\]

Solving Equation (15) for \(e(h, \emptyset)\), gives a quadratic equation in the unknown \(e(h, \emptyset)\), \(\kappa^{-1} \left[\frac{1}{2} - e(h, \emptyset)\right] = e(h, \emptyset) \left[\frac{1}{2} + e(h, \emptyset)\right]\), the positive solution of which is

\[
e(h, \emptyset) = \frac{1}{2} \left(\sqrt{\kappa^{-2} + 3\kappa^{-1} + \frac{1}{4} - \kappa^{-1} - \frac{1}{2}}\right)
\]

A similar procedure on Equation (16), gives

\[
e(h, h) = \frac{1}{2} \left(1 + \kappa^{-1} - \sqrt{\kappa^{-2} + 2\kappa^{-1}}\right)
\]

Using these two results together with the normalization condition gives

\[
e(h, \ell) = e(\ell, h) = \frac{1}{2} \left(\frac{1}{2} + \sqrt{\kappa^{-2} + 2\kappa^{-1} - \sqrt{\kappa^{-2} + 3\kappa^{-1} + \frac{1}{4}}\right)
\]

Solving Equation (12) for \(e(\ell, \emptyset)\) yields the following quadratic equation (the \(e(h, \cdot)\)
are all known by now):

\[ e(\ell, \emptyset)^2 + \left[ \kappa^{-1} + 2e(h, \emptyset) \right] e(\ell, \emptyset) - \frac{1}{2} \kappa^{-1} - \frac{1}{2} e(h, \emptyset) - e(h, \ell)^2 = 0 \]

Its positive solution, after substituting in the values of \( e(h, \emptyset) \) and \( e(h, \ell) \), is

\[
e(\ell, \emptyset) = \frac{1}{4} - \frac{1}{2} \sqrt{\kappa^{-2} + 3\kappa^{-1} + \frac{1}{4}} + \frac{1}{2} \sqrt{3\kappa^{-2} + 9\kappa^{-1} + \frac{1}{2} - \frac{1}{2} \sqrt{\kappa^{-2} + 3\kappa^{-1} + \frac{1}{4}} (2\sqrt{\kappa^{-2} + 2\kappa^{-1} + 1}) + \sqrt{\kappa^{-2} + 2\kappa^{-1}}}
\]

Finally, since \( e(\ell, \ell) = \frac{1}{2} - e(\ell, \emptyset) - e(h, \ell) \) we get

\[
e(\ell, \ell) = \sqrt{\kappa^{-2} + 3\kappa^{-1} + \frac{1}{4}} - \frac{1}{2} \sqrt{\kappa^{-2} + 2\kappa^{-1}} - \frac{1}{2} \sqrt{3\kappa^{-2} + 9\kappa^{-1} + \frac{1}{2} - \frac{1}{2} \sqrt{\kappa^{-2} + 3\kappa^{-1} + \frac{1}{4}} (2\sqrt{\kappa^{-2} + 2\kappa^{-1} + 1}) + \sqrt{\kappa^{-2} + 2\kappa^{-1}}}
\]

Under hyperphily, surpluses are:

\[
S^*(h, \ell) = \left[r + \delta + q^*(h, \ell) + q^*(h, h)\right]^{-1} \left[F - \pi^*(\ell, h)\right]
\]

\[
S^*(h, h) = \left[r + \delta + q^*(h, h)\right]^{-1} \left[h - q^*(h, \ell)S^*(h, \ell)\right]
\]

\[
S^*(\ell, h) = \left[r + \delta + q^*(\ell, h) + q^*(h, h)\right]^{-1} \left[\pi^*(\ell, h) - q^*(\ell, \ell)S^*(\ell, \ell)\right]
\]

\[
S^*(\ell, \ell) = \left[r + \delta + 2q^*(\ell, h) + q^*(\ell, \ell)\right]^{-1} \ell
\]

Surplus equalization \( S^*(h, \ell) = S^*(\ell, h) \) requires:

\[
\pi^*(\ell, h) = \left[r + \delta + q^*(\ell, h) + q^*(h, h)\right] F + \frac{r + \delta + q^*(h, \ell) + q^*(h, h)}{r + \delta + 2q^*(h, h) + q^*(\ell, h) + q^*(h, \ell)} q^*(\ell, \ell) \ell
\]

As discussed in Section 3.1, a pair \((d^*, \pi^*)\) is consistent in an equilibrium with hyperphily if \( G_{HYP}^1 \equiv S^*(h, h) - S^*(h, \ell) > 0 \) and \( G_{HYP}^2 \equiv S^*(\ell, h) - S^*(\ell, \ell) > 0 \).

**Bargaining in match** \((\ell, h)\)

Condition 1 in the main text lists the three kinds of agreement which may prevent \((d^*, \pi^*)\) from solving the bargaining problem in the match \((\ell, h)\). We check next when these agreements are not feasible. First, in agreement \(c_1\), neither \(\ell\) nor \(h\) leave each other, and \(h\) is made indifferent. \(\ell\) obtains \(\tilde{S}^{c_1}_\ell\). We need then \( G_{HYP}^3 \equiv S^*(\ell, h) - \tilde{S}^{c_1}_\ell > 0 \). In agreement \(c_2\), \(\tilde{S}^{c_2}_h = S^*(h, h)\) (thus \(h\) never leaves) and \(\ell\) only leaves when she finds an \(h\), leading to surplus \(\tilde{S}^{c_2}_\ell\). We need then \( G_{HYP}^4 \equiv S^*(\ell, h)S^*(h, \ell) - \tilde{S}^{c_2}_\ell S^*(h, h) > 0 \). Fi-
nally, agreement \( c_3 \) also has \( \hat{S}_{c_3} \equiv \hat{S}_1(h,h) = S^*(h,h) \), but now \( \ell \) always leaves. We need \( G_{\text{HYP}}^5 \equiv S^*(\ell,h)S^*(h,\ell) - \hat{S}_{c_3} \ell S^*(h,h) \geq 0 \).

**Bargaining in match** \((\ell, \ell)\)

Condition 2 lists the three kinds of agreements which may prevent \((d^*, \pi^*)\) from solving the bargaining problem in match \((\ell, \ell)\). We check next when these agreements are not feasible. First, let \( \hat{S}_{c_4} \) be the surplus obtained by either agent in match \((\ell, \ell)\) when they do not leave each other. If \( c_4 \) were consistent, it would lead to a higher product of surpluses, as both agents would receive a higher surplus. Therefore \( G_{\text{HYP}}^6 \equiv S^*(\ell,h) - \hat{S}_{c_4} \ell > 0 \) must hold for hyperphily to be an equilibrium. Next, in agreement \( c_5 \) one agent \( \ell \) obtains \( S^*(\ell,h) \) and does not leave, whereas the other one leaves only when meeting agent \( h \). We need then \( G_{\text{HYP}}^7 \equiv S^*(\ell,h) - \hat{S}_{c_5} \ell > 0 \). Finally, in agreement \( c_6 \), \( \hat{S}_{c_6} \) is the surplus obtained by an \( \ell \) agent in match \((\ell, \ell)\) when she always leaves and her partner is indifferent between this match and one with \( h \). For hyperphily to solve the bargaining problem, it must be the case that \( G_{\text{HYP}}^8 \equiv S^*(\ell,\ell)^2 - \hat{S}_{c_6} \ell \hat{S}_{c_6} \geq 0 \).

**Bargaining in match** \((h, h)\)

In match \((h, h)\) there is no endogenous destruction and agents equalize surplus, therefore they are maximizing the product of surpluses.

**Details on equilibrium conditions**

We characterize each condition as a function of primitives. Let us start with match \((\ell, h)\). In alternative agreement \( c_1 \) for match \((\ell, h)\), we have \( \hat{d}_\ell = \hat{d}_h = 0 \) and \( \hat{\pi}_h = h \). Note that for any lower \( \hat{\pi}_h \), \( h \) leaves. Next, any higher \( \hat{\pi}_h \) leads to a lower product of surpluses. It suffices then to focus on this agreement. We need to show that \( G_{\text{HYP}}^3 \equiv S^*(\ell,h) - \hat{S}_{c_1} \ell > 0 \).

\[
\hat{S}_{c_1} = (r + \delta)^{-1} \left[ F - h - q^*(\ell,\ell)S^*(\ell,\ell) - q^*(\ell,h)S^*(\ell,h) \right] \quad \text{and} \quad S^*(\ell,h) = \frac{F - r\delta + 2q^*(\ell,h) + q^*(\ell,\ell)}{2(r + \delta + q^*(h,h)) + q^*(\ell,h) + q^*(h,\ell)}
\]

Therefore, we need

\[
F - h - \frac{q^*(\ell,\ell)\ell}{r + \delta + 2q^*(\ell,h) + q^*(\ell,\ell)} < \frac{[r + \delta + q^*(\ell,h)] \left[ F - \frac{q^*(\ell,h)\ell}{r + \delta + 2q^*(\ell,h) + q^*(\ell,\ell)} \right]}{2[r + \delta + q^*(h,h)] + q^*(\ell,h) + q^*(h,\ell)}
\]

Thus

\[
F < h \left( 1 + \frac{r + \delta + q^*(\ell,h)}{r + \delta + 2q^*(\ell,h) + q^*(\ell,\ell)} \right) + \frac{q^*(\ell,\ell)\ell}{r + \delta + 2q^*(\ell,h) + q^*(\ell,\ell)} \quad (\text{HYP 1})
\]
Next, in agreement $c_2$, $\widehat{a}_h = 0$, $\widehat{a}_e(\ell) = 0$, $\widehat{a}_e(h) = 1$ and $\widehat{\pi}_e$ is such that $\widehat{S}_e^{c_2} = S^*(h, h)$. Note that for any lower $\widehat{\pi}_h = F - \widehat{\pi}_e$, $h$ leaves. Next, any higher $\widehat{\pi}_h$ leads to a lower product of surpluses. It suffices then to focus on this agreement. We need to show that $G_{HYP}^4 = S^*(\ell, h)S^*(h, \ell) - \widehat{S}_e^{c_2} \geq 0$. Surpluses are:

\[
\begin{align*}
\widehat{S}_h^{c_2} &= (r + \delta + q^*(\ell, h))^{-1} (F - \widehat{\pi}_h - q^*(h, h)S^*(h, h) - q^*(h, \ell)S^*(h, \ell)) = S^*(h, h) \\
\widehat{S}_e^{c_2} &= (r + \delta + q^*(\ell, h))^{-1} (\widehat{\pi}_e - q^*(\ell, \ell)S^*(\ell, \ell))
\end{align*}
\]

From this we can recover $\widehat{\pi}_e$:

\[
\widehat{\pi}_e = F - [r + \delta + q^*(h, h)]^{-1} \left[ (r + \delta + q^*(\ell, h) + q^*(h, h)) h - \frac{q^*(h, h)q^*(\ell, h)(F - \pi^*(\ell, h))}{r + \delta + q^*(h, \ell) + q^*(h, h)} \right]
\]

From now on, we work with $\pi^*(\ell, h) = A_1 + B_1 F$ and $\widehat{\pi}_e = A_2 + B_2 F$, with:

\[
\begin{align*}
A_1 &= \frac{r + \delta + q^*(\ell, h) + q^*(h, h)}{r + \delta + 2q^*(\ell, h) + q^*(h, h)} \frac{q^*(\ell, h)\ell}{2 (r + \delta + q^*(\ell, h) + q^*(h, h)) + q^*(h, h)} \\
B_1 &= \frac{q^*(h, h)q^*(\ell, h)(1 - B_1)}{r + \delta + q^*(\ell, h) + q^*(h, h)} \\
A_2 &= -\frac{r + \delta + q^*(\ell, h) + q^*(h, h)}{r + \delta + q^*(h, h)} h - \frac{q^*(h, h)q^*(\ell, h)A_1}{(r + \delta + q^*(h, \ell) + q^*(h, h)) (r + \delta + q^*(h, h))} \\
B_2 &= 1 + \frac{q^*(h, h)q^*(\ell, h)(1 - B_1)}{(r + \delta + q^*(h, h)) (r + \delta + q^*(h, \ell) + q^*(h, h))}
\end{align*}
\]

We first check that $\widehat{S}_e^{c_2} \geq S^*(\ell, \ell)$. This occurs whenever

\[
F \geq B_2^{-1} \left( \frac{r + \delta + q^*(\ell, h) + q^*(h, h)}{r + \delta + 2q^*(\ell, h) + q^*(h, h)} \ell - A_2 \right)
\]

If agreement $c_2$ is consistent we still have to check that $F$ is large enough to make the product of surpluses larger, that is, $S^*(h, \ell)S^*(\ell, h) \geq \widehat{S}_h^{c_2} \widehat{S}_e^{c_2}$:

\[
[(1 - B_1)F - A_1]^2 \geq C_1 \left( B_2F + A_2 - \frac{q^*(\ell, h)\ell}{r + \delta + 2q^*(\ell, h) + q^*(h, h)} \right) \\
\times \left( h - q^*(h, h) \frac{1 - B_1} {r + \delta + q^*(h, \ell) + q^*(h, h)} \right)
\]

with

\[
C_1 = \frac{(r + \delta + q^*(h, h) + q^*(h, h))^2}{(r + \delta + q^*(h, h))(r + \delta + q^*(h, h))}.
\]

The previous expression holds with equality for $F$ given by:

\[
\left[ (1 - B_1)^2 + C_1 B_2 \frac{q^*(h, h)(1 - B_1)}{r + \delta + q^*(h, \ell) + q^*(h, h)} \right] F^2
\]

40
Since \((1 - B_1)\), \(C_1\) and \(B_2\) are positive, \(G_{HYP}^4\) is a convex function of \(F\). In order to have an equilibrium with hyperphily, \(F\) has to be smaller than the lower root or larger than the higher one. Only the first of these two conditions is relevant. To see this, note that there exists an \(\tilde{F}\) such that \(S^*(\ell, h) = S^*(h, h)\). For \(F = \tilde{F}\), \(S_{\ell}^c \leq S_{h}^c > S^*(\ell, h)S^*(h, \ell)\) holds. Therefore, \(F\) larger than the large root of \(G_{HYP}^4 = 0\) requires that \(F > \tilde{F}\). However, consistency condition \(G_{HYP}^1\) states than an equilibrium with hyperphily requires \(F < \tilde{F}\). Therefore if \(F_{HYP}^4\) is the small root of \(G_{HYP}^4 = 0\), an equilibrium with hyperphily requires:

\[
F \leq \max \left( F_{HYP}^4, B_2 \frac{r + \delta + q^*(\ell, h) + q^*(\ell, \ell)}{r + \delta + 2q^*(\ell, h) + q^*(\ell, \ell)} - A_2 \right) \quad \text{(HYP 2)}
\]

We move next to alternative agreement \(c_3\). Surpluses are:

\[
\hat{S}_{h}^c = \left[ r + \delta + q^*(\ell, h) + q^*(\ell, \ell) \right]^{-1} \left[ F - \tilde{\pi}_{\ell} - q^*(h, h)S^*(h, h) - q^*(\ell, h)S^*(h, \ell) \right] = S^*(h, h)
\]

\[
\hat{S}_{\ell}^c = \left[ r + \delta + q^*(\ell, h) + q^*(\ell, \ell) \right]^{-1} \tilde{\pi}_{\ell}
\]

Then \(\tilde{\pi}_{\ell} = A_3 + B_3F\), with

\[
A_3 = \frac{r + \delta + q^*(\ell, h) + q^*(\ell, \ell) + q^*(h, h)}{r + \delta + q^*(h, h)} - \frac{q^*(h, h)(q^*(\ell, h) + q^*(\ell, \ell))A_1}{r + \delta + q^*(h, h) + q^*(h, \ell)}
\]

\[
B_3 = 1 + \frac{q^*(h, h)(q^*(\ell, h) + q^*(\ell, \ell))B_1}{r + \delta + q^*(h, h) + q^*(h, \ell)}
\]

Condition \(G_{HYP}^5\) \(\equiv S^*(h, \ell)S^*(\ell, h) - \hat{S}_{h}^c \hat{S}_{\ell}^c \geq 0\) holds if:

\[
[(1 - B_1)F - A_1]^2 \geq C_2 \left( B_3F + A_3 \right) \left( h - q^*(h, \ell) \right) \frac{(1 - B_1)F - A_1}{r + \delta + q^*(\ell, h) + q^*(h, h)}
\]

\(\text{with}\)

\(\text{if } F = \tilde{F}, \text{ this is equivalent to } \hat{S}_{\ell}^c \leq S^*(\ell, h) \text{ because } \hat{S}_{h}^c = S^*(h, h) = S^*(h, \ell). \text{ Add } \hat{S}_{h}^c = S^*(h, h) \text{ on both sides of the inequality and rearange terms to get:}\)

\[
\frac{\tilde{F} - q^*(h, h)S^*(h, h) - q^*(\ell, h)S^*(h, \ell) - q^*(\ell, \ell)S^*(h, \ell)}{r + \delta + q^*(\ell, h)} > \frac{\tilde{F} + [q^*(\ell, h) - q^*(h, \ell) - q^*(h, h)]S^*(h, \ell) - q^*(\ell, \ell)S^*(\ell, \ell) - q^*(h, h)S^*(\ell, \ell)}{r + \delta + q^*(\ell, h)}
\]

Comparing numerators gives \(q^*(h, h) > q^*(\ell, h)\), which indeed holds.
\[ C_2 = \frac{(r + \delta + q^*(h, h) + q^*(h, \ell))^2}{(r + \delta + q^*(\ell, h) + q^*(\ell, \ell))(r + \delta + q^*(h, h))} \]

Therefore:
\[
G_{HYP}^5 = \left[ (1 - B_1)^2 + C_2 B_3 \frac{q^*(h, \ell)(1 - B_1)}{r + \delta + q^*(h, \ell) + q^*(h, h)} \right] F^2 \\
+ \left[ -2(1 - B_1) A_1 - C_2 B_3 \left( h + \frac{q^*(h, \ell) A_1}{r + \delta + q^*(h, \ell) + q^*(h, h)} \right) + \frac{q^*(h, \ell)(1 - B_1)}{r + \delta + q^*(h, \ell) + q^*(h, h)} C_2 A_3 \right] F \\
+ A_1^2 - \left( h + \frac{q^*(h, \ell) A_1}{r + \delta + q^*(h, \ell) + q^*(h, h)} \right) C_2 A_3 \geq 0
\]

Since \((1 - B_1), C_2, \text{ and } B_3\) are positive, \(G_{HYP}^6\) is a convex function of \(F\). There are two values \(F_{HYP}^5 < F_{HYP}^{5'}\) of \(F\) that equalize the product of the surplus. In order to have an equilibrium with hyperphily \(F\) has to be smaller than \(F_{HYP}^5\) or larger then \(F_{HYP}^{5'}\):

\[ F \notin \left( F_{HYP}^5, F_{HYP}^{5'} \right) \quad \text{(HYP 3)} \]

We move next to match \((\ell, \ell)\). Consider alternative agreement \(c_4\). We need to show that \(G_{HYP}^6 \equiv S^*(\ell, h) - \tilde{S}_1^{c_4} > 0\), with
\[
\tilde{S}_1^{c_4} = (r + \delta)^{-1} [\ell - q^*(\ell, h)S^*(\ell, h) - q^*(\ell, \ell)S^*(\ell, \ell)]
\]

This occurs whenever
\[
F > \frac{[2(r + \delta + q^*(h, h)) + q^*(\ell, h) + q^*(h, \ell)][r + \delta + 2q^*(\ell, h)] + [r + \delta + q^*(\ell, h)]q^*(\ell, \ell)}{r + \delta + q^*(\ell, h)[r + \delta + q^*(\ell, \ell) + 2q^*(\ell, h)]} \ell
\]

\[ (HYP 4) \]

Next, consider agreement \(c_5\). Agent \(\ell\) indexed by 2 does not leave so \(\tilde{S}_2^{c_5} = S^*(\ell, h)\), with
\[
\tilde{S}_2^{c_5} = (r + \delta + q^*(\ell, h))^{-1}[\hat{\tau}_2 - q^*(\ell, \ell)S^*(\ell, \ell) - q^*(\ell, h)S^*(\ell, h)]
\]

This requires
\[
\hat{\tau}_2 = \frac{r + \delta + 2q^*(\ell, h)}{2(r + \delta + q^*(h, h)) + q^*(\ell, h) + q^*(h, \ell)} F \\
+ \frac{r + \delta + 2q^*(h, h) - q^*(\ell, h) + q^*(h, \ell)}{[r + \delta + 2q^*(\ell, h) + q^*(\ell, \ell)](2(r + \delta + q^*(h, h)) + q^*(h, h) + q^*(h, \ell))} q^*(\ell, \ell) \ell
\]

We need to verify now that \(G_{HYP}^7 \equiv S^*(\ell, \ell) - \tilde{S}_1^{c_5} > 0\) with
\[
\tilde{S}_1^{c_5} = \frac{2\ell - \hat{\tau}_2 - q^*(\ell, \ell)S^*(\ell, \ell)}{r + \delta + q^*(\ell, h)} < S^*(\ell, \ell)
\]
If condition $G_{\text{HYP}}^6$ holds, it must be the case that $\hat{S}_1^c < S^*(\ell, h)$. We look for the maximum $F$ that makes the agreement $(\hat{S}_1^c, \hat{S}_2^c)$ consistent:

$$F > \frac{[r + \delta + 3q^*(\ell, h) + q^*(\ell, \ell)][2(r + \delta + q^*(h, h)) + q^*(\ell, h) + q^*(h, \ell)]}{[r + \delta + 2q^*(\ell, h)][r + \delta + 2q^*(\ell, h) + q^*(\ell, \ell)]} - \frac{[r + \delta + 2q^*(h, h) - q^*(\ell, h) + q^*(h, \ell)]}{[r + \delta + 2q^*(\ell, h)][r + \delta + 2q^*(\ell, h) + q^*(\ell, \ell)]}$$

(HYP 5)

Next, consider agreement $c_6$. We need to verify that $G_{\text{HYP}}^8 \equiv S^*(\ell, \ell)^2 - \hat{S}_1^c \hat{S}_2^c \geq 0$ with

$$\hat{S}_1^c = \frac{2\ell - \hat{\rho}_2}{r + \delta + q^*(\ell, h) + q^*(\ell, \ell)} \quad \text{and} \quad \hat{S}_2^c = \frac{\hat{\rho}_2 q^*(\ell, h) S^*(\ell, h) - q^*(\ell, \ell) S^*(\ell, \ell)}{r + \delta + q^*(\ell, \ell) + q^*(\ell, h)}$$

Note that

$$\hat{S}_1^c + \hat{S}_2^c = \frac{2\ell q^*(\ell, h) S^*(\ell, h) - q^*(\ell, \ell) S^*(\ell, \ell)}{r + \delta + q^*(\ell, \ell) + q^*(\ell, h)} = \frac{2\ell - 2q^*(\ell, h) S^*(\ell, \ell)}{r + \delta + q^*(\ell, \ell) + q^*(\ell, h)}$$

Since $S^*(\ell, \ell) < S^*(\ell, h)$ and $q^*(\ell, \ell) = \rho e(\ell, \emptyset) > \rho e(h, \emptyset) = q^*(\ell, h)$, then $2S^*(\ell, \ell) > \hat{S}_1^c + \hat{S}_2^c$. Both $\ell$ agents equalize surplus in $S^*(\ell, \ell)$, and no agreement in the same segment of the frontier or in an interior segment of the frontier can generate a larger product of surpluses. Therefore condition $G_{\text{HYP}}^8$ always holds.

Finally, we check consistency of the equilibrium with hyperphily. For condition $G_{\text{HYP}}^1$, note that $S^*(h, h) > S^*(h, \ell)$ if and only if $h > \pi^*(h, \ell)$. The following expression holds:

$$S^*(\ell, h) - \hat{S}_1^c = (r + \delta)^{-1} \left[ \pi^*(\ell, h) - (F - h) - q^*(h, h) S^*(\ell, h) \right].$$

Condition $G_{\text{HYP}}^3$ implies the previous expression is positive. Thus $\pi^*(\ell, h) - F + h > 0 \Rightarrow F - \pi^*(h, \ell) - F + h > 0 \Rightarrow h > \pi^*(h, \ell)$, so $G_{\text{HYP}}^1$ holds.

$G_{\text{HYP}}^2$ holds whenever $G_{\text{HYP}}^6$ holds. To see this, note that

$$\hat{S}_1^c = S^*(\ell, \ell) + \frac{q^*(\ell, h) [2S^*(\ell, \ell) - S^*(h, h)]}{r + \delta}.$$
References


