Multidimensional Uncertainty and Hypersensitive Asset Prices

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Abstract

We consider a dynamic asset pricing model with one asset, in which one informed trader trades against liquidity traders and competitive market makers. Informed trader has private information about the fundamental value of the asset as well as the exogenous demand shock on the market. We characterize the unique linear Markov equilibrium of the model. With just the private information about fundamentals the price converges to the fundamental value in a monotone way (Kyle ’85). We show that the model with arbitrarily small demand shocks exhibits a price bubble. The bubble is created for strategic reasons by the informed trader, who follows a so called pump-and-dump strategy. He initially exacerbates the demand shock, trading at a loss (contrarian behavior), and gains later on by trading at an inflated price. Finally, both payoff relevant and payoff irrelevant information is revealed to the market.

Preliminary and Incomplete

1 Introduction

Asset prices diverge, often significantly, from the underlying values. While the fact is well documented, it is not easily reconciled with the classical economic theory: What stands in the way is essentially an arbitrage argument. Most of the current theories
explaining diverging prices resort to exogenous shocks: either demand shocks, or the shocks to the structure of the market. In this paper we present a model, in which the price is destabilized endogenously, for strategic reasons. Moreover, we will argue that the model with no strategic motives to destabilize the price is nongeneric.

In the paper we consider a dynamic asset pricing model with one asset, in which one informed trader trades against liquidity traders and competitive market makers. The model is based on the well known model by Kyle ’85. However, we let the informed player have private information about the fundamental value of the asset as well as the exogenous demand shock on the market. The demand shock is the expected demand by the liquidity traders, at any point of time. This multidimensional private information offers a rich spectrum of strategies to the informed trader. Most importantly, we show that even when the extent of the demand shocks is completely negligible, it will trigger a price bubble that is created strategically by the informed trader.

In our model the informed trader has both fundamental (value of the asset) and nonfundamental (demand shock) private information. While the first dimension of private information is standard in the finance literature, the second dimension is meant to capture the empirically well documented “soft” private information about the market conditions.

On a technical side, the private signals have joint normal structure. This helps us establish an explicit analytical solution of the model, which is the first contribution of the paper. More precisely, we establish that the continuous time version of the model has a unique linear Markov equilibrium strategy of the informed trader, which is linear in the extents of market biases about the value and the demand shock. We would like to contrast our analytical solutions with the predominantly numerical results for the existing models of multidimensional private information, in which the second dimension is binary (is the trader informed or not, has an event that changes the fundamental asset value occurred or not).

Second, we establish that in the course of trading the informed trader reveals all the private information to the market. That he reveals the private information about the value is fairly standard. It follows from the basic arbitrage argument: instead of ending with private information on his hands, the informed trader could have traded heavier on it, making larger profits but also pushing the price closer to its value. That the informed trader also reveals his information about the payoff irrelevant demand shock
is less intuitive, since there is no direct arbitraging on the payoff irrelevant information. However, we show how a novel “indirect arbitrage argument” extends to this case as well. This leads to full information revelation, much as in the case of the value.

Third, our results on the hypersensitive asset prices hinge on the insider using a so-called *pump-and-dump* strategy. Such a strategy involves the investor cheaply purchasing a stock and artificially increasing the stock’s value before selling the overvalued stock for a profit. While empirically well documented, we believe ours is the first justification of pump-and-dump strategies in a rational, equilibrium setting. Indeed, informed trader’s “fooling” of the market by pumping the price seems inconsistent with the equilibrium solution concept, in which his strategy is assumed to be known. With 1 dimensional uncertainty a pump-and-dump strategy is unprofitable. In our case even a small demand shock allows the price to rise relatively quickly in the process of pumping, and fall slowly in the process of dumping. On net, the informed trader makes more sales at high prices, squeezing an additional profit. Let us also point out that the strategy exhibits a persistent type of contrarian behavior. In the initial process of pumping, the informed trader purchases the asset at a price exceeding the value.

The fourth contribution is the extreme sensitivity of the asset prices with respect to the demand shock, stressed above. When, say, the asset value is zero, in the model with no demand shocks by Kyle ’85 the price is expected to stay equal to zero throughout the trading process. In the paper we characterize the limit distribution of prices, as the variance of demand shocks shrinks to zero, given a realized value and a demand shock. We show that in the limit, a positive demand shock results in a price that is expected to rise well above the true value, before falling back to zero (implied by the information revelation). This discontinuity results from the fact that as the extent of the demand shocks shrinks, the optimal strategy of the informed trader involves exacerbating, or trading on the demand shock in a way that keeps the effect of the shock on the order flow relatively constant. In effect, with almost inconsequential demand shocks, the price is destabilized almost exclusively for strategic reasons.

Finally, we show that the model implies the hypersensitivity not only of the expected price levels, but also price volatility. We extend the model to include a public signal of the demand shock, in the middle of the trading process. We show that this signal brings about a discrete price jumps with variance bounded away from zero, even if the variance
of the demand shocks is arbitrarily small. In other words, discrete price swings can be caused by the arrival of almost irrelevant public information.

1.1 Related Literature

To be completed

2 The Model

There is a single risky asset with a value $v$, which will be publicly announced at time 1. At time 0 the single risk-neutral informed trader learns $v$ as well as the demand/supply shock $m$. At any time $t$ prior to 1 the asset is traded continuously among the noise (liquidity) traders, the informed trader and competitive risk-neutral market makers. The cumulative order from the liquidity traders $Z_t$ follows

$$dZ_t = md t + \sigma dB_t,$$

where $B_t$ is a one-dimensional Brownian Motion. The informed trader chooses his cumulative order $S_t$ strategically. At any time $t$ the cumulative total order $O_t = S_t + Z_t$ is publicly observed and the asset price $P_t$ is determined as the expected value of the asset given $O_t$ as well as the strategy of the informed trader.

Notice that in the model only the informed trader acts strategically. He has private information about both the payoff-relevant value of the asset ("fundamentals") as well as the payoff-irrelevant demand shock. Intuitively, private information about the fundamental value gives informed trader direct arbitrage opportunities: purchasing when the asset is undervalued or selling when it is overvalued. Private information about the demand shock allows him to service, or "trade against" it at favorable prices. This can alternatively be seen as a factor slowing down the release of information about the asset to the market: for example, selling given an unexpected demand shock is less likely to drive the price down.

We assume that $v$ and $m$ are independently normally distributed with zero means and variances $\sigma_v^2$ and $\sigma_m^2$ respectively.\footnote{We can easily extend the results to the case of correlated $m$ and $v$. Independence is made for the} A strategy of the insider $S = \{S_t\}_{t \in [0,1]}$ is a
continuous semimartingale with respect to the filtration generated by $Z_t$, $v$ and $m$. In order to exclude the “doubling strategies” (see Back ’92) we also require a strategy to satisfy

$$\mathbb{E} \left[ \int_{[0,1]} S_t^2 dt \right] < \infty.$$ 

In the paper we will focus on the equilibrium strategies in the following subclass. First, let $\overline{V}_t$ and $\overline{M}_t$ be the processes of market expectations of $v$ and $m$, which are measurable with respect to the filtration $\mathcal{F}_t^O$ generated by the total order process up to time $t$. A linear Markov strategy $S$ satisfies

$$dS_t = \beta_t (v - \overline{V}_t) dt + \delta_t (m - \overline{M}_t) dt,$$

for some deterministic functions $\beta_t$ and $\delta_t$ in $L_2 ([0,1])$. Thus, a linear Markov strategy depends on the private information and the history of trade only through two state variables, $v - \overline{V}_t$ and $m - \overline{M}_t$, which are the extents of market’s biases about $v$ and $m$. Moreover, at any point of time the order flow is linear in those biases.

We will assume that $P_t = \overline{V}_t$, which is motivated by the competitive fringe of the market makers. Finally, for a strategy $S$ and a price process $P$ the wealth of the informed trader at time $T \leq 1$ is defined as (see Back ’92)

$$W_T = \int_{[0,T]} (v - P_t) dS_t - [P, S]_T.$$

The term $[P, S]_T$ is the cross-variation process, whose differential is usually written as $dP_t dS_t$. It captures the fact that a large order - in the sense of having a positive quadratic variation - by the informed trader at time $t$ affects the execution price at time $t$, equal to $P_t + dP_t$, just like in discrete time models. On the other hand, if the process $S$ has

sake of tractability and because it makes the interpretation of some of our results easier.

$^2$ $S_t$ can be written as $S_t = S_{1,t} - S_{2,t} + M_t$, where $S_{i,t}$ are positive, increasing and continuous processes, and $M_t$ is a continuous local martingale independent of $B_t$ (see Karatzas Shreve). The fact that $M_t$ and $B_t$ are independent reflects the fact that at any given moment the order flow from the informed trader is independent of (“is submitted before”) the demand from the liquidity traders. We exclude discontinuous strategies for the sake of tractability only: we comment in the proof of Lemma 3 that the informed trader would not benefit from discrete orders (see also Back ’92).

$^3$ If instead we assumed that the current order $dX_t$ is executed at the “past” price $P_{t-}$, then the informed trader could freely destabilize the market estimates by adding a Brownian component to his strategy, and so freely create private information. Indeed, the value function in Theorem 2 is a semi-positive quadratic function in the state variables $v - \overline{V}_t$ and $m - \overline{M}_t$ (see also Kyle ’85). Therefore,
differentiable paths, as in the case of a linear Markov strategy, we have \([P,S]_T = 0\).

### 2.1 Discrete Time Model

In the paper we will occasionally refer to the following discrete time version of the model. The model is parametrized by the period length \(\Delta = 1/N, \; N \in \mathbb{N}\), together with the parameters of the continuous time model above, \(\sigma_v^2, \sigma_m^2\). In each period \(t = 0, \Delta, 1 - \Delta\) the timing of the game is as follows: First, the informed trader submits the order \(s_t ("= dS_t'\))\); then the liquidity traders submit their order \(z_t ("= dZ_t'\))\), which is drawn from \(N(m\Delta, \sigma\Delta)\); finally at the end of period \(t\) the price \(P_t\) is set competitively to be \(\overline{V}_t\), as in continuous time. A linear Markov strategy in period \(t\) thus takes the form

\[
    s_t = \beta_t (v - \overline{V}_{t-\Delta}) + \delta_t (m - \overline{M}_{t-\Delta}) .
\]

(1)

The wealth of informed trader at the end of period \(T\) is \(W_T = \sum_{t=0}^{T} (v - P_t) s_t\).

We cannot hope to obtain tractable results in the discrete time framework for any period length. However, let us solve the simple static version of the model (with period length \(\Delta = 1\)) in order to develop some basic intuitions about the strategic behavior of the informed trader. The linear strategy in this setting is simply a strategy that is linear in the realized value and the demand shock, since the ex-ante expectations are fixed and equal to zero. An equilibrium strategy \(s_0\) is a strategy that maximizes informed trader’s payoff given that the price is equal to the expected value conditional on the realized order flow and the strategy \(s_0\) being used.

**Lemma 1** In the static model there is a unique linear equilibrium strategy. The strategy, price and the wealth of the informed trader are given by

\[
    s_0 = \beta v + \delta m ,
\]

\[
    p = \lambda (s_0 + m + \varepsilon) ,
\]

\[
    \mathbb{E} [W_1 | S, v, m] = \frac{\beta}{2} (v - \lambda m)^2 ,
\]

without the cross-variation component the informed trader would strictly prefer to destabilize the market estimates in this way.
where
\[ \lambda = \left( \frac{\sigma_v^2}{\sigma_m^2 + 4\sigma^2} \right)^{0.5}, \quad \beta = \frac{1}{2\lambda}, \quad \delta = -0.5. \]

Few properties are worth pointing out. First, the informed trader benefits from the known demand shocks. This is because his payoffs are proportional to the squared difference between the value of the asset \( v \) and the price \( \lambda m \) that would result if he abstained from trading. The demand shocks destabilize this price, increasing his profits. It is also easy to verify the comparative statics result that informed trader’s expected equilibrium profits are increasing in the variance of the demand shock \( \sigma_m^2 \).

Second, the informed trader benefits if the shocks to the value \( v \) and the demand \( m \) have opposite signs: when the asset is undervalued, in which case he wants to buy it, and there is a supply shock (or vice versa). One way to interpret it is that in this case his demand for the asset will be undetected by the market maker, and so will affect the price less. Alternatively, with \( v \) positive a supply shock will increase the difference between the value and the price \( \lambda m \) that results if he abstains from trading.

Finally, \( \delta \) equal to negative half means that the informed trader trades against, or dampens the demand shocks. This again should be intuitive. If, say, \( v = 0 \), a demand shock will push the price above the value, and so in order to make profits the informed trader will submit a partially offsetting sell order. This offsetting by the informed trader means that fixing the magnitude of the demand shock \( m + \varepsilon \), the larger the share of the unexpected part of the shock \( \varepsilon \) the bigger effect on the price it will have. On the other hand, the demand shocks are not fully offset by the informed trader, and so a demand shock will push the price up, and the supply shock - down.

3 Equilibrium

Let us focus back on the continuous-time model. We are interested in equilibrium strategies of the informed trader defined as follows. Such a strategy must be optimal, starting from any time \( t \) onwards (and conditional on any history), given that market maker assumes that the strategy is indeed being followed. More precisely, fix a strategy \( S \) and let \( P_t = \overline{V}_t \) be the process of expected values if the market believes the informed trader uses \( S \). We will say that \( S \) is an equilibrium strategy if for any other strategy \( \tilde{S} \) and time
\[ t < 1^4 \]

\[ \mathbb{E} \left[ \int_{[t,1]} (v - P_t) \, dS_r - [P, S][t,1] | S, v, m, \mathcal{F}_t^Z \right] \geq \mathbb{E} \left[ \int_{[t,1]} (v - P_{\tilde{t}}) \, d\tilde{S}_r - [P, \tilde{S}][t,1] | \tilde{S}, v, m, \mathcal{F}_t^Z \right]. \quad (2) \]

In the rest of the section we characterize the unique linear Markov equilibrium strategy. The proof follows in five steps. In Lemma 2 we characterize the Bayesian learning by the market makers when the informed trader follows a linear Markov strategy. Lemma 3 shows that in equilibrium the informed trader must be indifferent between any strategies that reveal the same amount of information to the market. Lemma 5 establishes that in equilibrium market beliefs about \( m \) and \( v \) cannot be perfectly correlated before the end of trading. Theorem 1 shows that market makers must learn the exact values of \( m \) and \( v \) from the equilibrium trading process, and so all the private information is revealed to the market endogenously. Finally, Theorem 2 establishes that there is a unique linear Markov strategy that satisfies the conditions of Lemmas 2, 3 and the Theorem 1 - indifference, no colinearity and information revelation - and verifies that this is indeed an equilibrium strategy.

For the rest of the section fix the parameters of the model: the ex-ante variances of the fundamental value and the demand shock \( \sigma_v^2 \) and \( \sigma_m^2 \) as well as the volatility of the noise trader demand \( \sigma^2 \). For a fixed strategy \( S \) that market believes the informed trader is following and the corresponding processes of expectations \( \bar{V}_t \) and \( \bar{M}_t \) denote the biases of the market as

\[ X_t = v - \bar{V}_t = v - P_t, \]
\[ Y_t = m - \bar{M}_t. \]

Consequently, a linear Markov strategy \( \tilde{S} \) takes the form

\[ d\tilde{S}_t = (\beta_t X_t + \delta_t Y_t) \, dt, \quad (3) \]

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for \( \beta_t \) and \( \delta_t \) in \( L_2([0,1]) \). Finally, let \( \Sigma_t \) be the market makers’ posterior covariance matrix

\[
\Sigma_t = \mathbb{E} \left[ \begin{array}{ccc}
X_t^2 & X_tY_t & |S,F_t^0| \\
X_tY_t & Y_t^2 & 0
\end{array} \right]. \tag{4}
\]

**Lemma 2** Fix a linear Markov strategy \( S \) that market believes the informed trader is using. If the informed trader uses a strategy \( \tilde{S}_t \) then the market biases follow

\[
\begin{align*}
dX_t &= -\lambda_t(d\tilde{S}_t + Y_tdt + \sigma dB_t), \\
dY_t &= -\phi_t(d\tilde{S}_t + Y_tdt + \sigma dB_t).
\end{align*} \tag{5}
\]

The gain functions \( \lambda_t \) and \( \phi_t \) are defined as

\[
\begin{bmatrix}
\lambda_t \\
\phi_t
\end{bmatrix} = \Sigma_t \begin{bmatrix}
\beta_t \\
\delta_t + 1
\end{bmatrix} \sigma^{-2}, \tag{6}
\]

where \( \Sigma_t \) is the market makers posterior covariance matrix (4). Moreover, \( \Sigma_t \) is the unique solution to

\[
d\Sigma_t = -\begin{bmatrix}
\lambda_t^2 & \lambda_t\phi_t \\
\lambda_t\phi_t & \phi_t^2
\end{bmatrix} \sigma^2 dt, \tag{7}
\]

with the initial conditions \( \sigma_{11,0} = \sigma_v^2, \sigma_{22,0} = \sigma_m^2 \) and \( \sigma_{12,0} = \sigma_{21,0} = 0 \).

In the case when \( \tilde{S} \equiv S \) the result is an application of the Kalman - Bucy filter (see Theorem 10.2 in Lipster, Shiryaev). Intuitively, market makers learn about two variables, \( v \) and \( m \), from a single signal, total order flow. If the informed trader uses a linear strategy then the order flow, \( v \) and \( m \) are jointly Normally distributed at any instant, and so the posterior estimates change linearly with the unexpected order flow. The gain parameters \( \lambda_t \) and \( \phi_t \) are proportional to the covariance of the signal with the variables of interest.

The fact that the market cannot distinguish which of the two variables \( m \) or \( v \) drives the total order (through the informed trader’s demand) immediately suggests that the informed trader benefits from the demand shocks. Just as in the static case, no matter what strategy he uses - and in particular if he abstains from trading - a demand shock will add a deterministic drift to \( X_t \) as in (5), destabilizing the price. We believe this is a
generic feature of trading models with multidimensional private information. In Section 4 we analyze how the informed trader optimally responds to the demand shocks.

For any other strategy $\tilde{S} \neq S$ the result follows from the Girsanov’s Theorem (see Karatzas Shreve). Therefore, instead of thinking of a deviation strategy $\tilde{S}$ as changing the measure over the paths of total order $O_t$ and the biases $X_t$ and $Y_t$ (as in (2)), one can simply think of it as steering or changing the drift of those biases, thought of as state variables, which are additionally perturbed by a fixed Brownian motion process $B_t$. Put differently, the informed trader chooses his trading strategy to maximize the integral of his flow payoffs $X_t d\tilde{S}_t - dP_t d\tilde{S}_t$, given the law of motions in (5).

For any strategy $\tilde{S}$ let $[\tilde{S}]_t$ be its quadratic variation process. Note that for any linear Markov strategy $\tilde{S}$ we have $[\tilde{S}]_t \equiv 0$.

Lemma 3 Fix a linear Markov equilibrium strategy $S$ and let $T^* \leq 1$ be the first time when $X_t$ and $Y_t$ are perfectly correlated, i.e., $\det \Sigma_t = 0$. Then there are $a, b, c > 0$ such that with $b_t = b + t/2$

i) for any strategy $\tilde{S}$ and any $t < T < T^*$ we have

$$
\mathbb{E} \left[ \int_{[t,T]} \left( v - P_r \right) d\tilde{S}_r - [P, \tilde{S}]_{[t,T]} \right] = aX_t^2 + 2b_tX_tY_t + cY_t^2 + d_t - \mathbb{E} \left[ \int_{[t,T]} \lambda_r d[\tilde{S}]_r [\tilde{S}, v, m, F^Z_t] \right]
$$

ii) at any $t < T^*$ the gain functions satisfy

$$
\lambda_t = \frac{c}{2(ac - b_t^2)}, \quad \phi_t = \frac{-b_t}{2(ac - b_t^2)};
$$

where $d_t$ is defined by $d_T = 0$ and $dd_t = -\sigma^2 \left( a\lambda_t^2 + 2b_t \lambda_t \phi_t + c\phi_t^2 \right) dt$.

Proof. In the Appendix.

The main point of the Lemma is that in equilibrium the informed trader must be indifferent between any strategies with $[\tilde{S}] = 0$ that reveal the same amount of information to the market, in the sense that they result in the same covariance matrix of $X_T$ and $Y_T$ (in Lemma 5 we establish that $T = 1$). This is equivalent to the fact that the informed trader is indifferent between the size of his order flow at any point of time, just as in
the continuous-time model with a single dimension of uncertainty (see Kyle ’85, Back ’92). Intuitively, in continuous-time the market is infinitely deep: there is large amount of liquidity trading and so, as long as $S_t = 0$, essentially no effect of informed trader’s order flow on the current price. Consequently, the flow payoffs $X_t dS_t$ are linear in the “action” $dS_t$. Since $dS_t$ is unrestricted and the movement of the state variables $X_t$ and $Y_t$ is also linear in $dS_t$, the only way for the equilibrium to exist is to have the informed trader indifferent at any point of time.

The Lemma also suggests that using a strategy with $S_t \neq 0$ is suboptimal. Indeed, a strategy with such “infinite” order flows affects adversely the price at which the trade is executed at a given time, and is dominated by one that in the end reveals the same amount of information to the market but trades smoothly, with finite order flows.

The indifference also implies the conditions (9) on the gain parameters as well as conditions on the parameters $a, c$ and $b_t$ in the payoff function. For example, when $Y_t = 0$ and the informed trader abstains from trading the market biases have no drift. Thus, indifference between trading now and abstaining for an instant requires $a$ to be constant. The case of $c$ and $b_t$ is slightly more complicated and relies on the fact that the state variable $Y_t$ and $dS_t$ affect the biases in exactly the same way.

For the rest of the proof we will use the following technical Lemma. The implication is that in any linear Markov equilibrium, as long as the market does not believe $m$ and $v$ to be perfectly correlated (and so the gain parameters $\lambda_t$ and $\phi_t$ are not colinear - see (9)), the informed trader can drive the biases $X_t$ and $Y_t$ to any arbitrary vector. 

**Lemma 4** Fix two bounded functions $f_t$ and $g_t$ on $[0, 1]$ and consider two processes $X_t$ and $Y_t$ that solve

$$dX_t = f_t(u_t dt + dB_t),$$

$$dY_t = g_t(u_t dt + dB_t),$$

For example, in the case when only $X_t = 0$ and the informed trader abstains from trading the market biases will drift. However, the effect of this drift is the same as the effect of $dS_t$, and so equal to zero. Thus, to have the informed trader indifferent between trading now and abstaining from trade $c$ must be constant.

It would not be difficult to strengthen the result to avoid $\varepsilon$ in the statement and have convergence a.e. The weaker result below is sufficient and, in our view, more instructive. Also, the generalization to more dimensions is straightforward.
where \( u_t \) is a \( \mathcal{F}^{X,Y}_t \)-measurable control and \( B_t \) is a Brownian Motion. Suppose that for every \( T < 1 \) \( f_t \) and \( g_t \) are not colinear on \([T, 1]\). Then for every \( x, y \in \mathbb{R} \) and \( \varepsilon > 0 \) there exists a control \( u_t \) such that \( X_t \) and \( Y_t \) satisfy

\[
\mathbb{E}[X_1] = x, \quad \mathbb{E}[(X_1 - x)^2] < \varepsilon, \\
\mathbb{E}[Y_1] = y, \quad \mathbb{E}[(Y_1 - y)^2] < \varepsilon.
\]

**Proof.** In the Appendix. ■

The following Lemma shows that in linear Markov equilibrium the market cannot believe before the end of trading that \( m \) and \( v \) are perfectly correlated. Otherwise, the informed trader could make arbitrarily high profits by first fooling the market that, say, the demand shock is greater than the truly realized one (see Lemma 5). Then, once the market stops learning independently about the demand shock, the trader would drive the price arbitrarily low for a long period of time and purchase the asset undetected by the market, proxying in for the demand shock.

**Lemma 5** Fix a linear Markov equilibrium strategy \( S \) and let \( \Sigma_t \) be the market makers’ posterior covariance matrix (see (4)). Then, for every \( t < 1 \) \( \det \Sigma_t \neq 0 \), i.e., the equilibrium biases \( X_t \) and \( Y_t \) are not colinear.

**Proof.** In the Appendix. ■

The following Theorem is one of the first main result of this section.

**Theorem 1** In any linear Markov equilibrium all the information is revealed by the end of trading, \( \lim_{t \to 1} \Sigma_t = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \).

**Proof.** Fix an equilibrium strategy \( S \) and let \( a, b_t \) and \( c \) be the parameters in the expected payoff function (8). We claim that it is sufficient to show that the matrix \( \begin{bmatrix} a & b_t \\ b_t & c \end{bmatrix} \) is strictly positive definite for some \( t \in (0, 1) \). This is because since for any strategy the expected payoffs satisfy (8), if \( \Sigma_1 \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) a strategy \( \tilde{S} \) such that \( \mathbb{E}[X_t^2 | \tilde{S}], \mathbb{E}[Y_t^2 | \tilde{S}] < \varepsilon \) (see (9) and Lemma 4) would constitute a profitable deviation, for \( \varepsilon \) sufficiently small.
Recall from Lemma 3 that $db_t = 1/2 dt$, which is implied by $Y_t \neq 0$ adding the drift to the biases (see proof of Lemma 3). Thus if the matrix $\begin{bmatrix} a & b_t \\ b_t & c \end{bmatrix}$ was negative definite everywhere, it would be strictly negative definite at some $t \in (0, 1)$. This implies that one eigenvalue $e$ of the matrix is strictly negative. But then, if $(x_e, y_e)$ is the associated eigenvector, a strategy $\tilde{S}$ resulting in $(E[X_t|\tilde{S}], E[Y_t|\tilde{S}])$ sufficiently close to $(Mx_e, My_e)$ would yield arbitrarily high profits as $M \to \infty$, in view of (8). This establishes the proof of the Theorem.

That the informed trader reveals all the information about the value $(\sigma_{11, t} = 0)$ is fairly intuitive. Note that private information about $v$ is directly payoff relevant, and offers easy arbitrage: If, say, the informed trader knows that the asset is undervalued he can use this information and make money by buying the asset. In other words, ending the game with private information about the value implies some unrealized arbitrage opportunities.

The argument why the information about the demand shock is revealed as well $(\sigma_{22, 1} = 0)$ is more delicate and, to the best of our knowledge, new. Note that the demand shock is orthogonal to the fundamental value $v$ of the asset, and so private information about it does not offer direct arbitrage opportunities. There is no direct way to trade on the under/overestimated demand shock. However, the private information about the demand shock offers indirect arbitrage opportunity, in the following way. Even if price is equal to the value of the asset, if the market is wrong about the demand shock, the price will drift deterministically away from the value, as in (5). Once this happens, the informed trader can arbitrage away this difference, as before. In other words, ending the game with private information about the demand shock implies some unrealized indirect arbitrage opportunities. We believe that a similar indirect arbitrage argument applies to other models of nonfundamental private information, such as information about own payoff function, etc.

Preceding Lemmas and Theorem 1 establish the following, which is the second main result of this section.

**Theorem 2** There exists a unique linear Markov equilibrium strategy $S$. The equilibrium
is characterized by

\[ dS_t = \beta_t X_t \, dt + \delta_t Y_t \, dt, \]
\[ dX_t = -\lambda_t \left( dS_t + Y_t \, dt + \sigma dB_t \right), \]
\[ dY_t = -\phi_t \left( dS_t + Y_t \, dt + \sigma dB_t \right), \]
\[ \mathbb{E} \left[ W_1 | S, v, m, \mathcal{F}_t^Z \right] = a X_t^2 + 2 b_t X_t Y_t + c Y_t^2 + d_t, \]
\[ \Sigma_t = \mathbb{E} \left[ \begin{array}{ccc} X_t^2 & X_t Y_t & Y_t^2 \\ X_t Y_t & Y_t^2 & |S, \mathcal{F}_t^O \right], \]

where the deterministic parameters \( a, b_t, c, d_t, \beta_t, \delta_t, \lambda_t, \phi_t \) and \( \Sigma_t \) are such that:

**(Value)** \( b_t = (t/2 - 1/4) \), \( a \) and \( c \) are the unique solution with \( a, c > 0, ac > 1/16 \) to

\[
\frac{1}{\sqrt{ac} - \frac{1}{4}} - \frac{1}{\sqrt{ac} + \frac{1}{4}} + \frac{1}{\sqrt{ac}} \left[ \ln(\sqrt{ac} - \frac{1}{4}) - \ln(\sqrt{ac} + \frac{1}{4}) \right] = 4a^2 \sigma_m^{-2}, \tag{11}
\]
\[
\frac{1}{\sqrt{ac} - \frac{1}{4}} - \frac{1}{\sqrt{ac} + \frac{1}{4}} - \frac{1}{\sqrt{ac}} \left[ \ln(\sqrt{ac} - \frac{1}{4}) - \ln(\sqrt{ac} + \frac{1}{4}) \right] = 4a^2 s_v^2 \sigma_m^{-2},
\]

and \( d_t \) is the solution to \( d_1 = 0 \) and \( dd_t = -\sigma^2 (a \lambda_t^2 + 2b_t \lambda_t \phi_t + c \phi_t^2) \, dt; \)

**(Learning)** \( \lambda_t \) and \( \phi_t \) are defined in (9), while \( \Sigma_t \) is the unique solutions of the ordinary differential equation (7) with the initial conditions \( \sigma_{11,0} = \sigma_v^2, \sigma_{22,0} = \sigma_m^2 \) and \( \sigma_{12,0} = \sigma_{21,0} = 0; \)

**(Strategy)** \( \beta_t \) and \( \delta_t \) are the solutions to (6).

**Proof. Verification.** It is easy to verify that the equations (11) are equivalent to

\[
\sigma_v^2 \sigma_m^{-2} = \int_{[0,1]} \lambda_t^2 \, dt = \int_{[0,1]} \frac{c^2}{4(ac - b_t^2)} \, dt, \tag{12}
\]
\[
\sigma_m^2 \sigma_v^{-2} = \int_{[0,1]} \phi_t^2 \, dt = \int_{[0,1]} \frac{b_t^2}{4(ac - b_t^2)} \, dt.
\]

In other words, given (7) in Lemma 2, the strategy \( S \) reveals all the information to the market (if it is believed to be followed). Also, \( [S] \equiv 0 \). Therefore it follows from the formula for the expected profits in part i) of Lemma 3 and the positive definiteness of the matrix \( \begin{bmatrix} a & b_t \\ b_t & c \end{bmatrix} \) that the strategy \( S \) is optimal.
**Uniqueness.** Lemma 2 establishes that $\beta_t$ and $\delta_t$ is uniquely pinned down by $\lambda_t$, $\phi_t$ and $\Sigma_t$ by (6), and that $\Sigma_t$ in turn is pinned down by $\lambda_t$ and $\phi_t$. Lemmas 3 and 5 establish the form of the value function as well as $\lambda_t$ and $\phi_t$ for some $a, c, b_t$ with $db_t = 0.5dt$ and $d_t$ as in the Theorem. Consequently, the three “free” parameters, through which all the other parameters in the equilibrium are defined, are $a, b_0$ and $c$. (7) in Lemma 2 and Theorem 1 imply that those parameters must solve (12) together with

$$0 = \int_{[0,1]} \lambda_t \phi_t dt = \int_{[0,1]} \frac{-cb_t}{4(ac - b_t^2)^2} dt.$$ 

It is easy to see that for the above equation to be satisfied it must be that $b_0 = -1/4$, and so $b_t = (t/2 - 1/4)$. Given that, equations (12) are equivalent to equations (11). Lemma 8 in the Appendix shows that for the given $\sigma^2$, $\sigma^2_v$ and $\sigma^2_m$ equations (11) have a unique solution $(a, c)$ with $a, c > 0, ac > 1/16$. This concludes the proof of the uniqueness and the theorem.

**4 Properties of the Equilibrium**

Recall that in the static model the informed trader benefits from the large demand shocks, since they destabilize the price. Also, the informed trader benefits if the shocks to $v$ and $m$ have opposite signs. Let us first verify that the same effects are present in the dynamic environment. Later we will investigate in more detail the informed trader’s response to the demand shock.

That the informed trader benefits from the demand shocks is best seen if we focus on the case when he chooses not to engage in trade until (just before) time 1. As discussed above Theorem 1, even if $v = 0$ and so $X_0 = 0$, say, a positive demand shock $Y_0 > 0$ will add a drift to the price, which can be eventually exploited by the informed trader. More precisely, it is easy to verify that conditional on no trading by the informed trader, we have $E[X_t^2], E[X_t Y_t]$ and $E[Y_t^2]$ linear in $Y_0^2$, with all of the coefficients strictly positive.

In order to better illustrate how the informed trader can benefit from additional private information about demand shock, we introduce the following change of variables. Fix the equilibrium strategy $S$ from Theorem 2 together with the covariance matrix
function $\Sigma_t$. Let $H_t = \frac{\sigma_{12,t}}{\sigma_{11,t}^2}$ and define

$$\hat{Y}_t = Y_t - \frac{\sigma_{12,t}}{\sigma_{11,t}^2}X_t$$

to be the orthogonalized part of $Y$. In other words, we have $\mathbb{E}[\hat{Y}_tX_t|\mathcal{F}_t^O] = 0$, for any $t < 1$.

**Lemma 6** For the equilibrium strategy $S$ as in Theorem 2 we have

$$\mathbb{E}[W_t|S,v,m,\mathcal{F}_t^Z] = \tilde{a}_tX_t^2 + 2\tilde{b}_tX_t\hat{Y}_t + c\hat{Y}_t^2 + d_t, \quad (13)$$

where

$$\tilde{a}_t = a + 2b_tH_t + cH_t^2,$$

$$\tilde{b}_t = b_t + cH_t,$$

with $a, b_t, c$ as in Theorem 2. Moreover, $\tilde{b}_t < 0$ for every $t < 1$.

The logic behind $\tilde{b}_t < 0$ is the same as in the static setting. When the asset is, say, undervalued and the demand from the liquidity traders overestimated, the informed trader can satisfy his demand with relatively little response of the price. An easy corollary to the Lemma is that the informed trader does not fully absorb the demand shock, just as in the static model.

**Corollary 1** In the equilibrium strategy as in Theorem 2 we have $\delta_t > -1$, for all $t$.

**Proof.** It is easy to verify that $d\hat{Y}_t = \frac{(\delta_t+1)}{\det \Sigma_t}(d\tilde{S}_t + Y_t + \sigma dB_t)$. Consequently, if $\delta_t \leq -1$ a strategy such that $d\tilde{S}_t + Y_t < 0$ both weakly increases the drift of $\hat{Y}_t$ and decreases that of $X_t$. Given the form of the expected value function in (13) and in particular $\tilde{b}_t < 0$ this contradicts the indifference of the informed trader. ■

It is clear why the informed trader does not fully absorb the demand shock. This is because the trader benefits precisely from the demand shocks destabilizing the price. However, the dynamic setting offers him many more ways to respond to the demand shock. In particular, in the static case the magnitudes and the signs of the market biases
are a matter of luck. In the dynamic setting, however, the informed trader can choose a strategy to “drive” the biases of the market in the desired regions. For example, instead of “trading against” the demand shock, as in the static setting, the trader can choose to first exacerbate the shock and thus destabilize the price even further. In fact, we have the following result.

**Proposition 1** In the equilibrium strategy as in Theorem 2 we have \( \delta_t > 0 \), for all \( t \).

**Proof.** To be completed. ■

To provide the intuition for the proof let us consider a discrete time model with two periods, \( t = 0, 1/2 \). It will be useful to write down the linear Markov strategy \( s_t \), \( t = 0, 1/2 \), as

\[
s_t = \tilde{\beta}_t(v - \tilde{P}_t) + \tilde{\delta}_t(m - \tilde{M}_t),
\]

(14)

where \( \tilde{P}_t \) and \( \tilde{M}_t \) are the expected price and the demand shock estimate at the end of period \( t \) if the informed trader abstains from trading,

\[
\tilde{P}_t = P_{t-\Delta} + \lambda_t(m - \tilde{M}_{t-\Delta}),
\]

\[
\tilde{M}_t = M_{t-\Delta} + \phi_t(m - \tilde{M}_{t-\Delta}).
\]

The reparametrization is motivated by the continuous time model, in which “\( \tilde{P}_t = P_{t-\Delta} \)” and “\( \tilde{M}_t = M_{t-\Delta} \)” (as long as the strategy has no quadratic variation). Thus, say, the continuous time state variable \( X_t \) can be alternatively interpreted as \( v - P_t \) or \( v - \tilde{P}_t \).

First, it is easy to see that \( \tilde{\delta}_{1/2} = 0 \). This is because with this reparametrization the static benefits from trading against the demand shock \( m - \tilde{M}_t \) disappear. The same is true in the continuous time model.

Second, just as in the static model analyzed in section 2.1, the expected payoff at the beginning of the second period is proportional to the squared difference between the value of the asset \( v \) and the price \( \tilde{P}_{1/2} \)

\[
\mathbb{E} \left[ W_1 | v, m, \varepsilon_0, s_0, s_{1/2} \right] = a_{1/2}(v - \tilde{P}_{1/2})^2,
\]

where \( a_{1/2} > 0 \). In equilibrium both the demand shock \( m \) and the order flow \( s_0 \) in the
first period move the price $\tilde{P}_{1/2}$ in the same direction.\footnote{The price is given by
\[
\tilde{P}_{1/2} = \lambda_0 (s_0 + m + \varepsilon_0) + \lambda_{1/2} (m - M_{1/2}) = \\
= (\lambda_0 - \lambda_{1/2}\phi_0) (s_0 + \varepsilon_0) + (\lambda_0 - \phi_0\lambda_{1/2} + \lambda_{1/2})m.
\] It can be verified that $(\lambda_0 - \lambda_{1/2}\phi_0) > 0$ for the linear Markov equilibrium strategy.}
Thus, in the dynamic model the informed trader has an additional benefit from accentuating the demand shock in the first period, since it helps further destabilize the price $\tilde{P}_{1/2}$ in the second period. This effect is absent in the static model. Here it results in $\tilde{\delta}_0 > 0$.

An immediate implication of Proposition 1 is that for a fixed level of a demand shock, its effect on the price is increasing in the extent to which it was expected by the informed trader. More precisely, for the linear Markov equilibrium strategy $S$, any time $t < 1$ and any magnitude of the positive order flow from the liquidity traders $dZ_t > 0$

\[
dP_t|S, dZ_t, m > dP_t|S, dZ_t, m',
\]
for any two levels of expectations by the informed trader $m > m' > 0$. In other words, in our model with private information monopolized by a single informed trader, the arbitraging behavior exacerbates instead of stabilizing the demand shocks’ effect on price.

The informed trader’s equilibrium strategy of accentuating the demand shocks is reminiscent of the familiar pump-and-dump strategy (see e.g. WSJ). Consider the event when $v = 0$ but there is a positive demand shock, $m > 0$, and for simplicity let’s focus on the deterministic dynamics of the means. Figures XX illustrate the paths of the expected market biases as well as expected cumulative position of the informed trader. The expected price starts at 0 and climbs up, in response to the exogenous demand shock. Initially the trader purchases the asset, pumping the price even more. Once the price climbs sufficiently high, the informed trader changes his strategy and starts dumping the asset. The selling pressure starts driving the price down, eventually back to the true realized value $v = 0$ (see Theorem 1).

**Figures**

Thus, given a positive demand shock $m > 0$ (and $v = 0$) the price is inflated for two reasons. One is a direct effect of the exogenous demand shock. The other effect is
strategic. The informed trader, knowing about the relatively price insensitive demand shock pushes the price even higher. Doing so allows him to later sell against the demand shock at inflated prices.

Let us point out that without the second dimension of uncertainty (see Kyle ‘85) such a pump-and-dump strategy of inflating the price cannot be profitable. Roughly speaking, with private information only about the value the losses incurred in the initial period of driving the price up are exactly offset by the additional profits of selling the asset in the later period of trade. In our model, the initial exogenous demand shock is expected to persist throughout the trading process. Thus the price will be rising quicker, driven by the additional demand shock, and falling slower. On the net, the trader makes more sales than purchases at the inflated prices, squeezing out an additional profit.

Finally, the equilibrium pump-and-dump strategy exhibits a persistent form of contrarian behavior: if the demand shock is big enough, initially the informed trader expects to incur losses in the process of pumping (which will be compensated in the later periods of dumping). Formally, we have the following.

**Proposition 2** Let $S$ be the linear Markov equilibrium strategy and consider the event $v = 0$ and $m \neq 0$. If $|m|$ is sufficiently high then there is $T > 0$ such that

$$
\mathbb{E} \left[ (v - P_t) \frac{dS_t}{dt} \middle| S, v, m \right] < 0, \text{ for } t \in (0, T).
$$

**Proof.** In the Appendix. ■

Proposition 1 shows that the price is destabilized by the exogenous shock as well as for strategic reasons. The following result pushes this argument further.

**Lemma 7** In the equilibrium strategy as in Theorem 2 we have

$$
\delta_t = \frac{\sigma}{\sigma_m} \times \frac{\sqrt{3}}{(1 - t)^2} + O(1), \quad \beta_t = \frac{\sigma}{\sigma_v} \times \frac{1 + 2t}{(1 - t)^2} + O(\sigma_m),
$$

$$
\lambda_t = \frac{\sigma_v}{\sigma} + O(\sigma_m), \quad \phi_t = \frac{\sqrt{3}\sigma_m}{\sigma} (1 - 2t) + o(\sigma_m),
$$

$$
a = \frac{\sigma}{2\sigma_v} + O(\sigma_m), \quad c = \frac{1}{4\sqrt{3}\sigma_m} + O(1).
$$

**Proof.** In the Appendix. ■
The most important point is that even as the exogenous shock becomes negligible ($\sigma_m$ small), the informed trader compensates for it by trading on it more heavily ($\delta_t \sim \frac{1}{\sigma_m}$). Thus the overall effect of the shock on the order flow, which is proportional to $\delta_t \times \sigma_m$, does not vanish even when the extent of the exogenous shock becomes negligible. Put otherwise, the trade is destabilized for purely strategic reasons. This translates into prices destabilized for purely strategic reasons, as we argue below.

Consider a model with a demand shock parametrized by $m = \alpha \sigma_m$. The $\alpha$ effectively measures how surprising the shock is, irrespectively of its magnitude. Note that for any $\alpha$, as $\sigma_m \rightarrow 0$ the “real” effect of the shock vanishes and the process of cumulative order from the liquidity traders $Z_t$ converges in distribution to a Brownian Motion. Thus, no matter how surprising the demand shock, for small $\sigma_m$ the models converge in some weak sense to the one with one dimensional uncertainty (Kyle ’85, Back ’92). The players are almost sure that the order from liquidity trader follows almost a Brownian Motion (while there is never common knowledge that it is exactly so).

The following Proposition and the Corrolaries analyze the corresponding limits of equilibrium distributions of biases, and so prices.

**Proposition 3** Conditional on a realized value $v$ of the asset as well as a demand shock $m = \alpha \sigma_m$, the processes of the equilibrium market biases $X_t$ and $Y_t/\sigma_m$ converge in distribution to the solutions $X_t^*$ and $Y_t^*$ of

\[
dX_t^* = -\frac{\sigma_v}{\sigma} \left( \frac{\sigma (1 + 2t)}{\sigma_v (1 - t)^2} X_t^* + \left( \frac{\sigma \sqrt{3}}{(1 - t)^2} + 1 \right) Y_t^* \right) dt + \sigma dB_t, \quad X_0^* = v,
\]

\[
dY_t^* = -\frac{\sqrt{3} (1 - 2t)}{\sigma} \left( \frac{\sigma (1 + 2t)}{\sigma_v (1 - t)^2} X_t^* + \left( \frac{\sigma \sqrt{3}}{(1 - t)^2} + 1 \right) Y_t^* \right) dt + \sigma dB_t, \quad Y_0^* = \alpha,
\]

as $\sigma_m \rightarrow 0$.

**Proof.** Follows from Lemma 7 and the continuous dependence of the solutions of stochastic ODE on the parameters (see Karatzas Shreve). ■

Recall that in the model with one dimension of uncertainty, for the realized value $v$ the process of the equilibrium market bias $X_t^1 = v - V_t = v - P_t$ follows (Kyle ’85, Theorem 3)

\[
dX_t^1 = -\frac{\sigma_v}{\sigma} \left( \frac{\sigma}{\sigma_v (1 - t)} X_t^1 dt + \sigma dB_t \right), \quad X_0^1 = v.
\]
One implication of the weak convergence of models as $\sigma_m \to \infty$ is this. Fix the equilibrium strategies of the insider and the learning by the market as in this model with one dimensional uncertainty, as well as $v$ and $\alpha$. If the demand from liquidity traders follows $dZ_t = \alpha \sigma_m dt + \sigma dB_t$, the process of market biases $X_t^1$ (and so prices) has an additional drift of $-\frac{\alpha v \sigma_m}{\sigma} dt$, and so converges to the process in the equilibrium of the Kyle model, as $\sigma_m \to 0$. However, the strategic response by the players upsets this continuity result.

**Corollary 2** Given a realized value $v$, the process $X_t^1$ of the market biases in the model with one dimensional uncertainty has a different distribution than the limit process $X_t^*$ of the equilibrium market biases, for any $\alpha$.

**Proof.** Lack of convergence for $\alpha \neq 0$ follows for example from comparing the drifts at $t = 0$ (see also Corollary 3). In the case $\alpha = 0$, the drift of $dX_t^1$ depends only on the value of $X_t^1$, while the drift of $dX_t^*$ depends both on $X_t^*$ and $Y_t^*$. The result thus follows form the fact that $X_t^*$ and $Y_t^*$ are not perfectly correlated. ■

The following illustrates the qualitative features of this strategic effect on the paths of market biases.

**Corollary 3** Fix $v$ and $\alpha$ as in Proposition (3). The expectations $x_t$ and $y_t$ of the limits of the equilibrium market biases $X_t^*$ and $Y_t^*$ are given by

$$
\begin{align*}
x_t &= v - (v + \sqrt{3}\alpha)t + \sqrt{3}\alpha t^2, \\
y_t &= \alpha - \sqrt{3}(v + \sqrt{3}\alpha)t + (\sqrt{3}v + 6\alpha)t^2 - 4\alpha t^2.
\end{align*}
$$

In particular, for $v = 0$ and $\alpha > 0$ the expected limit price path $p_t = -x_t$ is a strictly concave function with $p_0 = p_1 = 0$.

The intuition for the result is as follows. That given a positive demand shock $\alpha > 0$ and $v = 0$ the price starts climbing is true for any magnitude of the shock $\sigma_m$ (see the
discussion of the pump-and-dump strategies below Proposition 1). That the price climbs for “long” even for small $\sigma_m$ is a consequence of informed trader weighting in heavily to exacerbate the demand shock ($\delta_t \sim \frac{1}{\sigma_m}$, see Lemma (7)). If instead in equilibrium the informed trader was not trading so heavily on the demand shock when $\sigma_m$ is very small, and so $\delta_t$ was of order one, then the market would not be able to learn the demand shock. This is because the order flow would be dominated by the trade based on the mispricing of the asset (formally, the learning parameters $\lambda_t$ and $\phi_t$ would be determined by $\beta_t$ - see (6)). But then this would open the door to a profitable deviation to a pump-and-dump strategy: first inflate the price and then trade against the (undetected) demand shock. Put otherwise, in equilibrium the market must always entertain the possibility that high prices are due to an exacerbated demand shock.

Let us note that with small magnitude of the demand shocks ($\sigma^2_m \sim 0$), while the shocks affect the trend of the prices, they do not affect their volatility. The volatility of the prices at any point of time $t$ is approximately constant and equal to

$$\mathbb{E}[(dP_t)^2|S, v, m, \mathcal{F}_t^L] = \lambda_t^2 \sigma^2 \sim \sigma^2_v.$$

This is the same volatility as in the model with just private information about the value (Kyle ’85, Theorem 3). The reason essentially carries through to our model, due to upper-hemicontinuity, in the following way. In both models the volatility of prices is proportional to the square of the market depth $\lambda_t$. In the model by Kyle if the market depth $\lambda_t$ was not constant over time, the informed trader could make infinite profit by destabilizing the price: say, inflate the price when the market is thin, and sell when the market gets deeper. In our model the informed trader could profit from varying depth in exactly the same way, since a negligible demand shock would hardly affect the movement of prices and could be ignored.

Proposition 4 in the following section shows that the introduction of a public signal about the demand shock drastically affects the volatility of prices, even as the demand shock vanishes.
4.1 Public Information

Consider now the following extension of the model. Suppose that it is commonly known that at time $t = 1/2$ there will be a publicly announced signal $\gamma$ of the demand shock, where $\gamma = m + \varepsilon$, $\varepsilon \sim N(0, \sigma_\varepsilon^2)$. The proof of the following is similar to that of Theorem 2.

**Theorem 3** The model with publicly announced signal has a unique linear Markov equilibrium strategy. In the equilibrium the price changes discretely in response to the public signal with probability 1.

The sketch of the proof, together with the characterization of all the parameters is in the Appendix (to be completed).

The exogenous public signal is independent of the asset’s value. However, the market understands that in the equilibrium the informed trader trades both on the value as well as the demand shock, and so by the time of the announcement the beliefs of the market about both are endogenously correlated. Consequently, in equilibrium the announcement brings about a discrete update both in the estimate of the shock as well as the price. This is a simple example of how the seemingly irrelevant, or nonfundamental news can affect asset’s price.

The following analyzes the distribution of market beliefs (and so prices) as the size of the demand shock vanishes, analogously to Proposition 3.

**Proposition 4** Suppose that $\sigma_\varepsilon^2 = \kappa \sigma_m^2$. Conditional on a realized value $v$ of the asset as well as a demand shock $m = \alpha \sigma_m$, the processes of the equilibrium market biases $X_t$ and $Y_t/\sigma_m$ converge in distribution to the solutions $X^*_t$ and $Y^*_t$ of

\[
\begin{align*}
    dX^*_t &= -\lambda_t ((\beta_t X^*_t + (\delta_t + 1) Y^*_t)) dt + \sigma dB_t, \\
    dY^*_t &= -\phi_t ((\beta_t X^*_t + (\delta_t + 1) Y^*_t)) dt + \sigma dB_t, \\
    X^*_0 &= v, \\
    Y^*_0 &= \alpha, \\
    X^*_{1/2+} - X^*_{1/2} &= \alpha_1 (Y^*_{1/2} + \varepsilon), \\
    Y^*_{1/2+} - Y^*_{1/2} &= \alpha_2 Y^*_{1/2} + (1 - \alpha_2) \varepsilon,
\end{align*}
\]

where $\varepsilon \sim N(0, \kappa)$, $\alpha_1, \alpha_2 > 0$ and the function $\lambda_t$ is constant on the intervals $[0, 1/2]$ and $(1/2, 1)$. 

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Proof. To be completed.

The proposition shows that in the unique (linear Markov) equilibrium the prices must “jump” in response to the public news that are essentially irrelevant. The result is a natural counterpart and follows the similar argument to Proposition 3. Even a negligible demand shock leads to a price trending away from the fundamental value. News about the demand shock bring a sharp, discrete correction. Thus, while in either the model by Kyle or in the version with no public news and $\sigma_m^2 \sim 0$ the prices are continuous and have a constant volatility, here the public news brings about a discrete change in price, with variance $\kappa^2 \alpha_1$. Also, the volatility changes after the public announcement from $\lambda_{1/2}^2 \sigma^2$ to $\lambda_{1/2+}^2 \sigma^2$.

5 Conclusions

To be completed.

6 Appendix

Proof. (Lemma 3) Fix a linear equilibrium strategy $S$ and thus the gain functions $\lambda_t$ and $\phi_t$ as in Lemma 2. For any differentiable $a_t, b_t, c_t, d_t$, when the informed trader uses a linear strategy $\tilde{S}$ with drift $\theta_t$ then we have from Lemma 2 and Ito’s formula

$$d \left( a_t X_t^2 + 2b_t X_t Y_t + c_t Y_t^2 + d_t \right) + X d\tilde{S}_t =$$

$$= X_t \theta_t - 2X_t ( \theta_t + Y)(a_t \lambda_t + b_t \phi_t) - 2Y_t (\theta_t + Y)(b_t \lambda_t + c_t \phi_t) +$$

$$+ da_t X_t^2 + db_t X_t Y_t + dc_t Y_t^2 + dt \sigma dB_t + dd_t + \left( a_t \lambda_t^2 + 2b_t \lambda_t \phi_t + c_t \phi_t^2 \right) (\sigma^2 dt + d[\tilde{S}]_t),$$

for $h_t = -2(a_t \lambda_t X_t + b_t (\phi_t X_t + \lambda_t Y_t) + c_t \phi_t Y_t)$.

First, if $\tilde{S} = S$ and so $\theta_t = \beta_t X_t + \delta_t Y_t$ and $[\tilde{S}] \equiv 0$, as long as the functions $a_t, b_t, c_t, d_t$
satisfy the equations

\[ da_t + \beta_t(1 - 2a_t\lambda_t - 2b_t\phi_t)dt = 0, \quad (16) \]
\[ dc_t - 2(\delta_t + 1)(b_t\lambda_t + c_t\phi_t)dt = 0, \]
\[ 2db_t + \delta_t(1 - 2a_t\lambda_t - 2b_t\phi_t) - 2\beta_t(b_t\lambda_t + c_t\phi_t)|dt - 2(a_t\lambda_t + b_t\phi_t) = 0, \]
\[ dd_t - \sigma^2(a_t^2\lambda_t^2 + 2b_t\lambda_t\phi_t + c_t\phi_t^2) dt = 0, \]

then the drift of the term in (15) is zero. Also, it follows from the “no doubling” restriction that for any strategy \( \tilde{S}^8 \)

\[ \mathbb{E} \left[ \int_{[t,T]} h_t \sigma dB_t | \tilde{S}, v, m, \mathcal{F}_t \right] = 0. \quad (17) \]

and so (8) holds.

Second, (15) implies that

\[ (1 - 2a_t\lambda_t - 2b_t\phi_t) = (b_t\lambda_t + c_t\phi_t) = 0. \quad (18) \]

Otherwise the informed trader could profitably deviate from the strategy \( S \): for example, if \( (1 - 2a_t\lambda_t - 2b_t\phi_t) > 0 \) then choosing the drift \( \theta = (\beta_t + \gamma) X_t dt \), with \( \gamma > 0 \), in any event when \(|X_t|\) is large and \( Y_t \) is equal or sufficiently close to zero would increase the expected wealth at time \( T \) relative to strategy \( S \) (note that we are using here that \( t < T \), and so \( Y_t \) and \( X_t \) are not perfectly correlated). The necessary condition (18) implies (9) as well as, together with (16), that \( da_t = dc_t = 0 \) and \( db_t = 0.5dt \).

---

8Given that \( dX_t \) and \( dY_t \) are linear in \( d\tilde{S}_t \), \( Y_t \) and \( dB_t \) and the linear coefficients \( \lambda_t \) and \( \phi_t \) are uniformly bounded, it follows from the Ito formula that the “no doubling” restriction \( \mathbb{E} \left[ \int_{[0,1]} \tilde{S}_t^2 dt \right] < \infty \) implies that \( \mathbb{E} \left[ \int_{[0,1]} X_t^2 dt | \tilde{S} \right] < \infty. \) The last inequality is sufficient for the expectation in the text to be equal to zero (see KS). We leave the details to the reader.
Finally, for an arbitrary strategy \( \tilde{S} \) we have

\[
\begin{align*}
&d \left( a_t X_t^2 + 2b_t X_t Y_t + c_t Y_t^2 + d_t \right) + X d\tilde{S}_t - dP_t d\tilde{S}_t = \\
&= h_t \sigma dB_t + \left( a_t \lambda_t^2 + 2b_t \lambda_t \phi_t + c_t \phi_t^2 \right) d[\tilde{S}]_t - dP_t d\tilde{S}_t = \\
&= h_t \sigma dB_t + \left( a_t \lambda_t^2 + 2b_t \lambda_t \phi_t + c_t \phi_t^2 - \lambda_t \right) d[\tilde{S}]_t = \\
&= h_t \sigma dB_t - \frac{\lambda_t}{2} d[\tilde{S}]_t.
\end{align*}
\]

The first equality follows from (15), (16) and (18), the second from \( dP_t d\tilde{S}_t = -dX_t d\tilde{S}_t = \lambda_t d[\tilde{S}]_t \) and the last equality from \( a_t \lambda_t^2 + 2b_t \lambda_t \phi_t + c_t \phi_t^2 = \lambda_t^2 / 2 \). The formula, together with (17) establishes (8) and concludes the proof of the Lemma.

**Proof. (Lemma 4)** We construct the control \( u_t \) as follows. Fix \( T < 1 \) such that \( \int_{[T,1]} f_t^2 dt, \int_{[T,1]} g_t^2 dt < \varepsilon \). Consider functions \( f_t^T \) and \( g_t^T \) defined on \([T,1]\) such that

\[
\begin{align*}
&\int_{[T,1]} f_t f_t^T = 1, \quad \int_{[T,1]} g_t f_t^T = 0, \\
&\int_{[T,1]} g_t g_t^T = 1, \quad \int_{[T,1]} f_t g_t^T = 0.
\end{align*}
\]

The existence of such functions follows from the fact that \( f_t \) and \( g_t \) are not colinear: For example, \( f_t^T \) can be defined as the appropriately rescaled difference between \( f_t \) and the projection of \( g_t \) on \( f_t \)

\[
f_t^T = C \times \left( f_t - \frac{\int_{[T,1]} f_t g_t dt}{\int_{[T,1]} g_t^2 dt} g_t \right).
\]

Now, define the control \( u_t \) as follows

\[
\begin{align*}
u_t &= 0, \quad t \leq T, \\
u_t &= (x - X_T)^T f_t^T + (y - Y_T) g_t^T, \quad t > T.
\end{align*}
\]

\footnote{Observe also that the analogous equality would hold if we allowed discontinuous strategies: the only change would be an additional term \(-\frac{\lambda_t}{2} \Delta[\tilde{S}]_t\), where \( \Delta[\tilde{S}]_t \) are the squared differences of the cumulative order at the points of discontinuity (compare Back 92).}
Given the control $u_t$ we have
\[ X_1 = X_T + \int_{[T,1]} f_t((x - X_T)^T f_t^T + (y - Y_T) g_t^T)dt + f_t dB_t = x + \int_{[T,1]} f_t dB_t, \]
\[ Y_1 = Y_T + \int_{[T,1]} g_t((x - X_T)^T f_t^T + (y - Y_T) g_t^T)dt + g_t dB_t = y + \int_{[T,1]} g_t dB_t, \]
\[ \mathbb{E}[X_1] = x, \quad \mathbb{E}[(X_1 - x)^2] = \int_{[T,1]} f_t^2 dt < \varepsilon; \]
\[ \mathbb{E}[Y_1] = y, \quad \mathbb{E}[(Y_1 - y)^2] = \int_{[T,1]} g_t^2 dt < \varepsilon, \]
where the inequalities in the last two lines follow from the choice of $T$. This concludes the proof of the Lemma. □

**Proof. (Lemma 5)** We will establish that if the market makers were convinced that $X_t$ and $Y_t$ are perfectly correlated from some time $T < 1$ onwards, this would allow the informed trader to obtain arbitrarily high profits. Suppose by contradiction that on an nonempty interval $[T,1]$ we have $Y_t = \alpha X_t$ for some constant $\alpha$, and so the equilibrium strategy takes the form $dS_t = \tilde{\beta}_t X_t dt$. As in the proof of Lemma 3 on this interval we have, for any differentiable $\tilde{a}_t$ and $\tilde{d}_t$
\[ d(\tilde{a}_t X_t^2 + \tilde{d}_t) + X_t dS_t = \]
\[ = -2\tilde{\alpha}_t \lambda_t X_t \beta_t + X_t^2 d\tilde{a}_t + d\tilde{d}_t - \sigma^2 \tilde{a}_t \lambda_t^2 + \beta_t X_t^2 dt - 2\tilde{\alpha}_t \lambda_t X_t \sigma dB_t = \]
\[ = X_t^{2} (\beta_t (1 - 2\tilde{\alpha}_t \lambda_t) + d\tilde{a}_t - 2\alpha \tilde{a}_t \lambda_t) + d\tilde{d}_t - \sigma^2 \tilde{a}_t \lambda_t^2 - 2\tilde{\alpha}_t \lambda_t X_t \sigma dB_t, \]
and so
\[ \mathbb{E} \left[ \int_{[t,1]} (v - P_r) dS_v | S, v, m, \mathcal{F}_t^Z \right] = \tilde{a}_t X_t^2 + \tilde{d}_t - \mathbb{E} \left[ \tilde{a}_t X_t^2 | S, v, m, \mathcal{F}_t^Z \right], \]
for any $\tilde{a}_t$ and $\tilde{d}_t$ that solve $\beta_t (1 - 2\tilde{\alpha}_t \lambda_t) + d\tilde{a}_t - 2\alpha \tilde{a}_t \lambda_t = 0$, and $d\tilde{d}_t - \sigma^2 \tilde{a}_t \lambda_t^2$, $\tilde{d}_t = 0$.

On the other hand, if at time $t$ the informed trader deviates to a linear Markov strategy $\tilde{S}$ with drift $\theta dt$ then
\[ d(\tilde{a}_t X_t^2 + \tilde{d}_t) + X_t d\tilde{S}_t = X_t \theta_t (1 - 2\tilde{\alpha}_t \lambda_t) + d\tilde{d}_t - \sigma^2 \tilde{a}_t \lambda_t^2 - 2\tilde{\alpha}_t \lambda_t X_t \sigma dB_t, \]
and so to make sure that the informed trader has no profitable deviation it must be that 

\[ 1 - 2\tilde{a}_t \lambda_t = 0. \]

This implies, just as in Lemma 3, that for any strategy \( \tilde{S} \) with \( \tilde{S} = 0 \) 

\[
E \left[ \int_{[0,1]} (v - P_r) d\tilde{S}_r | \tilde{S}, v, m, F^Z_t \right] = \tilde{a}_t X^2_t + \tilde{d}_t - E \left[ \tilde{a}_1 X^2_1 | \tilde{S}, v, m, F^Z_t \right].
\] (19)

This establishes that conditional on the equilibrium play on the interval \([0, T]\), for any linear Markov strategy \( \tilde{S} \) the expected profits from trading on the interval \([T, 1]\) depend only on 

\[ E \left[ X^2_1 | \tilde{S}, v, m, F^Z_t \right]. \]

Let’s construct a deviation strategy \( \tilde{S} \) that yields arbitrarily high expected payoffs as follows. Intuitively, the strategy first "fools" the market to believe that there is a larger demand shock than the truly realized one. Then, on the interval \([T, 1]\) the strategy drives the price arbitrarily low (\( X_t \) arbitrarily high) for a long period of time, at which price the informed trader purchases the asset undetected by the market. More precisely, equations (9) imply that the conditions of Lemma 4 are satisfied, and so there is a strategy such that (with high probability) \( X_T \) is very close to 0, and \( Y_T \) is very close to \(-N\), for some \( N > 0 \). Notice that for any \( t \) in \([0, T]\), any public history \( \eta \) on \([T, t]\) and any two strategies \( \tilde{S} \) and \( \hat{S} \) we have 

\[
E \left[ dX_t | \tilde{S}, X_T = 0, Y_T = -N, \eta \right] = E \left[ dX_t | \hat{S}, X_T = 0, Y_T = 0, \eta \right]
\]

exactly when \( d\tilde{S}_t = d\hat{S}_t + N \). From this and (19) follows that conditional on \( X_T = 0, Y_T = -N \) the expected payoff on the interval \([T, 1]\) of a strategy \( \tilde{S} \) that results in \( X_1 = 0 \) is equal to 

\[
\tilde{d}_t + N \times E \left[ \int_{[T,1]} X_t dt | \tilde{S}, X_T = 0, Y_T = -N \right].
\]

Therefore, arbitrarily high expected profits can be made by a strategy \( \tilde{S} \) that results in \( X_1 = 0 \) and at the same time in the arbitrarily high expectation of \( \int_{[T,1]} X_t dt \). This establishes the desired contradiction and so the proof of the Lemma.

**Lemma 8** For given parameters \( \sigma^2, \sigma^2_v \) and \( \sigma^2_m \) equations (11) have a unique solution \((a, c)\) with \( a, c > 0, ac > 1/16 \).
Proof. If \( x = \sqrt{ac} - \frac{1}{4} \) then the LHS of the first equation in (??) becomes

\[
G(x) = \frac{1}{x} - \frac{1}{x + 0.5} + \frac{1}{x + 0.25} [\ln(x) - \ln(x + 0.5)]
\]

Clearly \( G(x) \rightarrow \infty \) as \( x \downarrow 0 \) and \( G(x) \rightarrow 0 \) as \( x \rightarrow \infty \): The first part follows from \( \frac{1}{x} \) heading to infinity faster than \(-\ln(x)\) when \( x \rightarrow 0 \). The second part follows as all three terms converge to zero as \( x \rightarrow \infty \).

The next step is to show that \( G(x) \) is strictly decreasing. Taking the first derivative, \( G'(x) \) for \( x > 0 \):

\[
G'(x) = -\frac{1}{x^2} + \frac{1}{(x + 0.5)^2} + \frac{\ln(1 + 0.5x^{-1})}{(x + 0.25)^2} + \frac{1}{(x + 0.25)} \left[ \frac{1}{x} - \frac{1}{x + 0.5} \right]
\]

\[
= -0.25 \left( \frac{1}{x^2} + \frac{1}{(x + 0.5)^2} - \frac{4\ln(1 + 0.5x^{-1})}{(x + 0.25)} \right)
\]

\[
\leq -0.25 \left( \frac{1}{x^2} + \frac{1}{(x + 0.5)^2} - \frac{4(0.5x^{-1})}{(x + 0.25)} \right)
\]

\[
\leq -0.25 \left( \frac{1}{x^2} + \frac{1}{(x + 0.5)^2} - \frac{2}{x(x + 0.5)} \right) < 0
\]

Thus, \( G(x) \) monotonically declines from \( \infty \) to \( 0 \) and so there is a unique solution to \( 4s^2_m \sigma^{-2} = G(x) \), which thus pins down unique \( ac > 0 \).

The LHS of the second equation in (11) is strictly greater than that of the first equation and thus strictly positive. Thus the second equation pins down unique \( a, c > 0 \) for a fixed value of \( ac > 0 \). ■

Proof. (Lemma 6) The form of the coefficients in the expected wealth function follows easily from comparing with the expected wealth function under the original variables \( X_t \) and \( Y_t \) in Theorem 2

Let us establish here that \( \hat{b}_t < 0 \) for every \( t < 1 \). Since \( H_0 = 0 \) we have \( \hat{b}_0 = -1/4 \). On the other hand, since \( \sigma_{12,t}, \sigma_{11,t}^2 \rightarrow 0 \) as \( t \rightarrow 1 \) we have from L’Hopital’s rule that as \( t \rightarrow 1 \)

\[
\hat{b}_t \rightarrow b_1 + c \cdot \frac{\sigma_{12}'}{\sigma_{11}^2} = b_1 + c \cdot \frac{-\lambda_1 \phi_1}{-\lambda_1^2} = b_1 + c \cdot \frac{-b_1}{c} = 0,
\]

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Also, we have

\[ \widehat{b}_t = b_t + cH_t = \frac{2(ac - b_t^2)}{\sigma_{H,t}^2} [\lambda_t \sigma_{12,t} - \phi_t \sigma_{11,t}], \]

\[ \widehat{b}'_t = \frac{1}{2} + cH'_t = \frac{1}{2} + c\frac{\lambda_t}{\sigma_{H,t}^4} [\lambda_t \sigma_{12,t} - \phi_t \sigma_{11,t}] = \frac{1}{2} + c\frac{\lambda_t}{\sigma_{H,t}^4} [\lambda_t \sigma_{12,t} - \phi_t \sigma_{11,t}] = \frac{1}{2} + c\lambda_t \sigma_{H,t}^2 \frac{\sigma_{11,t}}{\sigma_{11,t}^2} \widehat{b}_t. \]

This establishes that \( \widehat{b}_t > 0 \) implies \( \widehat{b}'_t > 0 \). Together with \( \widehat{b}_0 < 0 \) and \( \widehat{b}_1 = 0 \) this establishes the proof of the lemma.

**Proof. (Proposition 2)** Since \( dS_t = (\beta_t X_t + \delta_t Y_t)dt \), from Ito’s formula we have

\[
\begin{align*}
\frac{d}{dt} \mathbb{E} \left[ (v - P_t) \frac{dS_t}{dt} \right | S, v, m] &= \\
&= d\mathbb{E}[\beta_t X_t^2 + \delta_t X_t Y_t | S, v, m] = \\
&= \mathbb{E}[\beta_t X_t^2 + \delta_t X_t Y_t + 2\beta_t X_t dX_t + \delta_t (X_t dY_t + Y_t dX_t) + \beta_t (dX_t)^2 + \delta_t dX_t dY_t | S, v, m] = \\
&= \mathbb{E}[A_t X_t^2 + B_t X_t Y_t + C_t Y_t^2 + D_t | S, v, m] dt,
\end{align*}
\]

with appropriate continuous functions \( A_t, B_t, C_t \) and \( D_t \), and in particular

\[ C_t = -\delta_t (\delta_t + 1) \phi_t. \]

Since \( \mathbb{E}[X_t^2 | S, v, m], \mathbb{E}[X_t Y_t | S, v, m] \) and \( \mathbb{E}[Y_t^2 | S, v, m] \) are continuous, the function \( \mathbb{E} \left[ (v - P_t) \frac{dS_t}{dt} \right | S, v, m] \) is continuously differentiable. On the other hand, it follows from Proposition 1 that \( C_0 < 0 \), and so the derivative of \( \mathbb{E} \left[ (v - P_t) \frac{dS_t}{dt} \right | S, v, m] \) at \( t = 0 \) is strictly negative if \( v = 0 \) and \( m \) is sufficiently large. This establishes the proof.

**Proof. (Lemma 7)** To simplify the notation, for the parameters \( a, b_t, c \) of the value function defined in Theorem 2 (line (11)), denote

\[ \xi_t = \frac{ac}{ac - b_t^2}. \]

\( \xi_t \) depends on the parameters of the model \( \sigma_v^2, \sigma_m^2 \) and \( \sigma_v^2 \) - the dependence that we do not make explicit, for the sake of tractability. The function \( \widehat{\delta}_t \) in the proposition is based on
the approximation \( \xi_t \approx 1 \), which is justified since, as we will see, \( \sigma_m \downarrow 0 \) implies \( ac \uparrow \infty \). The approximation makes the explicit definitions of all the endogenous parameters in the model possible.

Since \( \Sigma_t \) satisfies the differential equation (7) and \( \Sigma_1 \) is the null matrix, we have

\[
\lambda_t = \frac{1}{2a} \xi_t = \frac{1}{2a} (1 + \varepsilon_1),
\]

\[
\phi_t = -\frac{b}{2ac} \xi_t = \frac{1}{4ac} \left( \frac{1}{2} - t \right) (1 + \varepsilon_1),
\]

\[
\sigma_{11,t}^{-2} = \int_{[t,1]} \lambda_s^2 ds = \int_{[t,1]} \frac{c^2}{4(ac - b_t^2)^2} ds = \frac{1}{4a^2} \int_{[t,1]} \xi_s^2 ds = \frac{1 - t}{4a^2} (1 + \varepsilon_2),
\]

\[
\sigma_{12,t}^{-2} = \int_{[t,1]} \lambda_s \phi_s ds = \frac{1}{4a^2c} \frac{t(t - 1)}{4} (1 + \varepsilon_3),
\]

\[
\sigma_{22,t}^{-2} = \int_{[t,1]} \phi_s^2 ds = \frac{1}{4a^2c^2} \int_{[t,1]} b_s^2 \xi_s^2 ds = \frac{1}{16a^2c^2} \frac{(1 - t)(1 - 2t + 4t^2)}{12} (1 + \varepsilon_4),
\]

where \( \varepsilon_1 \in [0, \max_t \xi_t - 1] \) and \( \varepsilon_i \in [0, (\max_t \xi_t)^2 - 1] \), \( i = 2, 3, 4 \) (we are not making explicit the dependence of the error terms on time). In particular, the last equation evaluated at \( t = 0 \) yields

\[
ac = \frac{1}{\sigma_m} \frac{\sigma}{8\sqrt{3}} + \frac{\sigma}{\sigma_m} \frac{\sqrt{1 + \varepsilon_4} - 1}{8\sqrt{3}}.
\]

Consequently

\[
ac = \frac{1}{\sigma_m} \frac{\sigma}{8\sqrt{3}} + O(1), \quad a = \frac{\sigma}{2\sigma_v} + O(\sigma_m),
\]

\[
\varepsilon_i = O(\sigma_m), \quad i = 1, ..., 4.
\]
By substituting those values we obtain the formulas for $\lambda_t$ and $\phi_t$. From the Bayes formula (6) in Lemma 2 we have

$$\delta_t + 1 = \frac{(\sigma_{11,t} \phi_t - \sigma_{12,t} \lambda_t)}{\det \Sigma_t} = \frac{1-t}{4a^2} (1 + \varepsilon_2) \frac{1}{3ac} \left( \frac{1}{2} - t \right) \left( 1 + \varepsilon_1 \right) - \frac{1}{4a^2 c} \frac{t(t-1)}{4} \left( 1 + \varepsilon_3 \right) \frac{1}{2a} \left( 1 + \varepsilon_1 \right) =$$

$$= ac \times \frac{24}{(1-t)^2} \times (1 + O(\sigma_m)) = \frac{\sigma}{\sigma_m} \times \frac{\sqrt{3}}{(1-t)^2} + O(1);$$

$$\beta_t = \frac{(\sigma_{22,t} \lambda_t - \sigma_{12,t} \phi_t)}{\det \Sigma_t} =$$

$$= \frac{1}{16a^2 c^2} \frac{(1-t)(1-2t+4t^2)}{12} \left( 1 + \varepsilon_4 \right) \frac{1}{2a} \left( 1 + \varepsilon_1 \right) - \frac{1}{4a^2 c} \frac{t(t-1)}{4} \left( 1 + \varepsilon_3 \right) \frac{1}{2a} \left( 1 - t \right) \left( 1 + \varepsilon_1 \right) =$$

$$= a \times \frac{2(1+2t)}{(1-t)^2} \times (1 + O(\sigma_m)) = \frac{\sigma}{\sigma_v} \times \frac{1+2t}{(1-t)^2} + O(\sigma_m).$$

$\blacksquare$