Implicit Markets in Economics via Pairwise Matching

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Abstract

I explore a class of probabilistic pairwise matching social settings without formal prices. I nevertheless find that they may be interpreted as implicit markets, equilibrated by probabilities. The seminar focuses on two primary examples: I first explore a perfect foresight equilibrium of a directed search model of journal rejection rates when papers have uncertain quality. I then turn to applications of implicit markets instead based on anonymous random matching. Here, I discuss passing games, such as an equilibrium model of prevalence rates with contagious diseases, and “cat and mouse” resource theft games as a foundation for a model of crime rates. I conclude with a model of counterfeit money, which merges these two games.
The Economics of Counterfeiting*

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Abstract

This paper develops a new tractable strategic theory of counterfeiting as a multi-market large game played by good and bad guys. There is free entry of bad guys, who choose whether to counterfeit, and what quality to produce. Opposing them is a continuum of good guys who select a costly verification effort. A counterfeiting equilibrium consists of a “cat and mouse” game between effort and quality, and a collateral “hot-potato” passing game among good guys. With log-concave verification costs, counterfeiters pro-duce better quality at higher notes, but verifiers try sufficiently harder that the verification rate still rises. We prove that the passed and counterfeiting rates vanish for low and high notes. We develop and use a graphical framework for deducing comparative statics.

Our theory applies to fixed-value counterfeits, like checks, money orders, or money. Focusing on counterfeit money, we assemble a unique data set from the U.S. Secret Service. We identify some new time series and cross-sectional patterns, and explain them: (1) the ratio of all counterfeit money (seized or passed) to passed money rises in the note, but less than proportionately; (2) the passed-circulation ratio rises in the note, and is very small at $1 notes; (3) the vast majority of counterfeit money used to be seized before circulation, but this is no longer true; and (4) the ratio of the internal Federal Reserve Banks passed rate to the economy-wide average falls in the note until the $100 note. Our theory explains how to estimate from data the counterfeiting rate, the street price of counterfeit notes, and the incredibly small costs expended verifying each note.

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*This is a radical revision of a 2009 submission to this journal. The paper began in 2005 as “Counterfeit $$$”, as a model just of the hot potato game; the cat and mouse game was developed while Lones visited the Cowles Foundation in 2006. We have profited from the insights and/or data of Charles Bruce (Director, National Check Fraud Center), Pierre Duguay (Deputy Governor, Bank of Canada), Antti Heinonen (European Central Bank, Counterfeit Deterrence Chairman), Ruth Judson (Federal Reserve), John Mackenzie (counterfeit specialist, Bank of Canada), Stephen Morris, and Lorelei Pagano (former Special Agent, Secret Service) — as well as comments at I.G.I.E.R. (Bocconi), the 2006 Bonn Matching Conference, the 2006 SED in Vancouver, the Workshop on Money at the Federal Reserve Bank of Cleveland, Tulane, Michigan, the Bank of Canada, the 2007 NBER-NSF GE conference at Northwestern, the 2008 Midwest Theory Conference in Columbus, the 2011 Yale Summer Theory Conference, Maryland, Pittsburgh, Stanford, and Georgetown.
## Contents

1 Introduction .................................................. 1

2 The Model .................................................. 6

3 Equilibrium Derivation .................................... 9

4 Equilibrium Comparative Statics .......................... 12

5 Predictions about Passed, Seized, and Missed Money .... 15

6 Conclusion and Relationship to the Literature ............ 17

A Appendix: Omitted Proofs ................................ 18

   A.1 Optimal Quality and Zero Profit Curves: Proof of Lemma \( \Pi(a) \) . . . . . . . . 18
   A.2 Constant Counterfeiting Rate Curve Slope: Proof of Lemma \( \Pi(b) \) . . . . . . 19
   A.3 Existence and Uniqueness: Proof of Theorem \( \Pi \) . . . . . . . . . . . . . . . . . 19
1 Introduction

Counterfeiting is a major economic problem, called “the world’s fastest growing crime wave” (Phillips, 2005). This paper explores the counterfeiting of stated-value financial documents like money, checks, or money orders. The domestic losses from check fraud may have exceeded $20 billion in 2003. Counterfeit money is much less common but still costly: The counterfeiting rate of the U.S. dollar is about one per 10,000 notes, with the domestic public losing $80 million in 2011, more than doubling since 2003. The indirect counterfeiting costs for money are much greater, forcing a U.S. currency re-design every 7–10 years. As well, many costs are borne by the public checking the authenticity of their money.

When we write counterfeit money (or checks), we have in mind two manifestations of it. Seized money is confiscated before it enters circulation. Passed money is found at a later stage, and leads to losses by the public. Whereas counterfeit money in toto is a stock variable, these manifestations are flows. We have gathered an original data set mostly from the Secret Service on seized and passed money over time and across denominations. All seized and passed counterfeit currency in the USA must be handed over to the Secret Service, and so very good data is available. Our USSS data includes all 5,594,062 seized and 8,541,972 passed counterfeit notes in the USA, 1995–2004, supplemented by aggregate USSS data 2005–7 and older published data. We have organized our data using two measures — the passed rate, or passed over circulation, and the seized-passed ratio (our creation). Seized is a more volatile series, as seen in Figure 1, as it owes to random, maybe large, counterfeiting discoveries, and is also contemporaneous counterfeit money. By contrast, passed money is twice averaged: It has been found by thousands of individuals, and may have long been circulating. We unearth some key counterfeiting facts and develop a model motivated by them:

#1. Since 1970, the seized-passed ratio has fallen 90%, and the passed rate has risen 75%.

There has been a sea change in the seized and passed money since 1980. Historically, seized vastly exceeded passed counterfeit money (Figure 1). But starting in 1986, and then accelerating in 1995, the seized-passed ratio began to tumble. Nowadays, most counterfeit money is passed, as the fraction passing into circulation has skyrocketed roughly from 10% to 80%. One brief time window 1995–98 witnessed a stunning 80% drop in the seized-passed rates for $5, $10, and $20 notes (Figure 2). Concurrently, the passed rates have proven much more stable, but have nonetheless drifted up by approximately 75% since 1964.

#2. One plus the seized-passed ratio (a) rises in the note, but (b) less than proportionately.

This clear trend holds in the U.S. denominations $1, $5, ..., $100 over the samples of

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1 Data here is sketchy. This estimate owes to a widely-cited Nilson Report (www.nilsonreport.com).
2 Arguably, the $500M budget of the Bureau of Printing and Engraving, and maybe $1B of the Secret Service and Treasury budgets owe to anti-counterfeiting. Also, there is a large private sector industry.
millions of passed and seized notes, as well as in Canada’s six paper notes. Slopes in the log-log diagram of Figure 2, i.e. elasticities, are positive but well below 1 (0.18, on average).

#3. The passed rate rises 40-fold from $1 to $20, and levels off, dropping at higher notes.

The rise is predictive of the U.S. dollar and Euro data. Figure 3 plots at the left the average fractions of passed notes over a long time horizon. The possibility of a falling passed rate at sufficiently high notes is not realized in the US data. Yet the Euro offers two higher value notes, and the passed rate of the 500 Euro note is less than one twelfth that of the 200.


Digitally-produced counterfeits, using scanners and color printers, are cheaper to make. As Table 1 depicts, the fraction of such notes falls in the denomination. Judson and Porter (2003) find that 73.6% of passed $100 bills were high quality circulars, but only 19.2% of $50 bills, and less than 3% of all others. The "Supernote" (circular 14342) is the best quality counterfeit ever. North Korea made this $100 note from bleached $1 notes, with the intaglio printing process used by the Bureau of Engraving and Printing. Banks cannot detect it.

The introduction of digital means of counterfeiting was very rapid, focused in 1994–98.

#5. Federal Reserve Banks find proportionately fewer counterfeits as the note rises until $100.

For Canada, from 1980-2005, the counterfeit-passed ratios are 0.095, 0.145, 0.161, 0.184, 0.202, and 3.054 for (respectively) $5, $10, $20, $50, $100, and $1000. The $1000 note was ended in 2000.

These ratios per million have averaged 1.96, 19.46, 71.21, 72.03, 49.94, 81.43, respectively.

The common claim that the most counterfeited note domestically on an annualized basis is the $20 is false over our time span. Accounting for the higher velocity of the $20, on a per-transaction basis (the relevant measure for decision-making), the $100 note is unambiguously the most counterfeited note.
Figure 2: Seized-Passed Ratios, Across Denominations. At left is one plus the seized-passed ratios, averaged over 1995–2008, for non-Colombian counterfeits in the USA. We do not have data for this time span for the $1 note; it averages 1.23 for the years 1998 and 2005–8. For this log-log graph, slopes are elasticities — positive and below one. At right, we instead plot seized-passed ratios for the $5, $10, and $20 notes from 1995–2004. The sample includes almost ten million passed notes, and about half as many seized notes.

Commercial banks transfer damaged or unneeded notes to the Federal Reserve Banks (FRB), who find $5–10 million of fake money yearly. The banking sector offers a reverse test of our model, since passed money at FRB missed earlier detection. The FRB computes its internal passed money rates; for the only years 1998, 2002, and 2005 with available data, the ratio of the internal FRB and passed rates monotonically falls from $1 through $50 (Figure 4). This reverse monotonicity might be surprising, as the lowest notes are the poorest quality counterfeits, and so easiest for innocent verifiers to catch before depositing into a bank.

Canada’s introduction of color notes temporarily nearly stopped counterfeiting. Canada almost eliminated passed money as it colorized each note in the 1970s (Table 2).

Counterfeiting is a crime that induces two linked conflicts: first, counterfeiters against verifiers and law enforcement, and then verifiers against each other. The extant literature focuses on the police-counterfeiter conflict yielding seized money. Thus overlooks the role of

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**Table 1:** Fraction of non-Columbian Passed Notes Digitally Produced, 1995–2004. Cheaper digital methods of production skyrocketed in 1995–98, focused on lower notes.

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<td>.250</td>
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<td>.75</td>
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<td>.460</td>
<td>.608</td>
<td>.564</td>
<td>.457</td>
<td>.472</td>
<td>.552</td>
<td>.43</td>
</tr>
</tbody>
</table>

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6See Table 6.1, 6.3 and 6.3 (resp.) in [Treasury](2000,2003,2006), and Table 1 in [Judson and Porter](2003).
the second conflict, which alone can explain passed money — which harms the public.

Counterfeiting is inherently a deception exercise, and a good theory ideally captures the rival efforts of bad guys to successfully fool victims, and of good guys to avoid being fooled. We develop a behavioral strategic theory with a continuum of players, integrating the passed and seized conflicts. We assume notes of a single denomination change hands every period. Some fake notes pass into circulation and a larger collateral game emerges: Good guys unwittingly pass on the fakes they acquire in an anonymous random matching exchange setting.

Inspired by Fact #6, we center our model on a new assumption in the money literature, namely, endogenous verification. Good guys expend efforts screening out passed counterfeit money handed them; more effort yields stochastically better scrutiny. This scrutiny firstly reduces the passing fraction, namely the share of counterfeit entering circulation, and thereby inflates the level of seized money. This allows us to speak to the time series of seized money.

But since good and bad guys cannot be distinguished, verification scrutiny also raises the level of passed counterfeit money. The interaction of good guys is a game of strategic complements: the more others verify, the more one should do likewise to protect oneself, *ceteris paribus*. Meanwhile, higher quality fakes cost bad guys more, but better deceive good guys, and so pass more often. Costly effort and counterfeit quality jointly determine the chance bad notes are caught: the *verification rate*. Our verification function confers a cardinal meaning to both effort and quality, with each subject to diminishing returns.

<table>
<thead>
<tr>
<th>Note</th>
<th>$5</th>
<th>$10</th>
<th>$20</th>
<th>$50</th>
<th>$100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Colored Notes Introduction</td>
<td>12/72</td>
<td>11/71</td>
<td>6/70</td>
<td>5/75</td>
<td>5/76</td>
</tr>
<tr>
<td>% Fall in Passed Rate</td>
<td>99</td>
<td>99</td>
<td>99</td>
<td>98</td>
<td>93</td>
</tr>
</tbody>
</table>

Table 2: “Scenes of Canada” / Multicoloured Series, 1971–76.
Figure 4: **Internal FRB / Average Passed Rate.** At left are the ratios of internal FRB and average passed rates in 1998 (dashed), 2002 (dotted), and 2005 (solid).

Our model confronts currency transactors a risky choice, trading off uncertain losses from counterfeiting against sure losses from more vigilant verification. The counterfeiting rate therefore emerges as the market-clearing quantity in our game. This rate must then vanish whenever marginal verification costs vanish. If the counterfeit quality was constant, then a standard vanishing marginal cost near zero forces a vanishing counterfeiting rate. In fact, our assumed endogenous and flexible counterfeit quality reinforces the logic: For criminals optimally face the weakest incentives to invest in the quality of the least counterfeit notes.

Our model is parsimonious: For with fixed verification effort, counterfeit optimization would ensure that quality rises in the note; thus, the verification rate would fall in the note, as would the counterfeit-passed ratio — contrary to data. If we instead tied the counterfeiters’ hands and fixed the quality, then to ensure zero profits, the passing fraction would move inversely to the note, and the counterfeit-passed ratio would rise in proportion to the note. In other words, with fixed quality, zero profits would require that the passing fraction scale by half moving from $5 to $10 to $20. In fact, the passing fraction rises much slower, since rising quality adds to costs. Loosely, endogenous verification effort explains why counterfeiting rate rises at low notes, while endogenous counterfeit quality justifies its eventual decline.

Ours is a stock-flow model — the amount of counterfeits produced equals the amount returned and the amounts passing into circulation balance those being discovered. Passed money reflects changes in the counterfeiting and discovery rates, and most cost changes push these in opposite directions.

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7This is reminiscent of Knowles, Persico, and Todd (2001), where a police search chance incentivizes a decision to carry drugs. But for us, good guys are pitted against each other, and the effort choice is not binary, and only co-determines losses with the counterfeiting rate.

8We estimate the unobserved counterfeiting rate from our data, and approximate the street price of counterfeit notes — agreeing with anecdotal evidence. Most curiously, we back out marginal verification costs from the passed rate. They equal the passed rate times the denomination, peaking around 1/4 cent for the $100 bill!
2 The Model

A. Players and Pairwise Matching. The story unfolds in periods 1, 2, 3, . . . . There are two types of risk neutral maximizing agents: a unit mass of good guys, and an infinitely elastic supply of bad guys, who are potential counterfeitors. Good guys either have or desire a note of value $\Delta > 0$; genuine notes are in fixed supply $M > 0$.

Money changes hands exogenously, from counterfeiters to good guys, among good guys, and between banks and good guys. Banks are a pass-through, returning notes to good guys. They swap a fraction $\phi > 0$ of notes with the Federal Reserve.

Each period, good guys with a note are randomly matched to those without a note, or to banks. And good guys without a note are randomly matched to good guys with a note, or to banks, or to bad guys. We explore a steady-state, where measures of all transactions remain unchanged each period. But since counterfeiters — although rare — spend and never acquire notes, and we assume that bank withdrawals balance bank deposits, good guys with a note are on the long side of the market; they meet a trading partner with chance slightly less than 1.

B. Actions and Payoffs. When good guys are matched, they cannot distinguish good guys from counterfeiters. They are aware that they may be handed a fake note. They reject a note if they notice that it is fake, whereupon it becomes worthless passed money — it is withdrawn from circulation, the passer losing its face value $\Delta$. The counterfeiting rate $\kappa \in [0, 1)$ is the fraction of transacted notes that are fake, from all transactors.

Good guys expend effort $e \geq 0$ scrutinizing any note before accepting it. Real notes are never mistaken for counterfeit. The verification rate $v \in [0, 1]$ is the chance that a fake note is noticed. This chance intuitively should rise in effort $e$ and fall in quality $q > 0$. Specifically, an increasing and convex function $\chi$ translates $e$ and $q$ into a endogenous verification rate $v$ via the implicit relation $e = q\chi(v)$, where $\chi$ is the verification cost function. Doubling quality requires twice the effort to secure the same verification rate. As quality is unobserved, the good guy does not know the actual verification rate $v$, but infers it in equilibrium.

Good guys without notes next period become good guys with notes upon accepting a note. If matched, they go to a bank with fixed chance $\beta \in (0, 1)$. Banks detect and confiscate fake money with a fixed chance $\alpha > 0$. With chance $1 - \beta$, the good guy with a note meets another randomly drawn transactor. So fake money is found in transactions at the discovery

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9We don’t model the rationing; we assume that good guys acquiring a note expects to soon spend it.
10Knowingly passing on fake currency is illegal by Title 18, Section 472 of the U.S. Criminal Code. We assume that no one engages in this crime of “uttering”, seeking a “greater fool” to accept bad money.
11To wit, this is an average of a 100% counterfeiting rate from counterfeiters and a smaller counterfeiting rate from good guys, because at least one good guy has already verified circulating money.
12Bank tellers told us that they used protocols, and were not incentivized to look at higher notes more carefully. As evidence of $\alpha < 1$, ATMs dispense fake money (personal communication, John Mackenzie, Bank of Canada).
rate $\delta(v) = \beta \alpha + (1 - \beta)v$. The passed rate $p = \delta \kappa$ is the ratio of passed money to circulation.

Each period, criminals choose whether to counterfeit a fixed finite expected quantity $x > 0$ and if so, what quality $q$ of notes to produce at an increasing and convex cost $c(q)$. Since verification efforts help the police, only an endogenous passing fraction $f(v) \leq 1 - v$ of production passes into circulation. Intuitively, the first verifier catches a fraction $v$ of notes, and police seize a share $1 - v - f(v)$. Criminals earn zero profits every period, net of legal penalty. As counterfeiters are invariably eventually caught, and the stated penalty is constant across notes, we assume a legal penalty $\ell > 0$, namely, the expected present loss from punishment.

C. Currency Verification. The verification function $v = V(e, q)$ is the derived screening rate, rising in effort and falling in quality: $V(e, q) \equiv \chi^{-1}(e/q)$ for all $e < q\chi(1)$, and $V(e, q) = 1$ for $e \geq q\chi(1)$. For any $e > 0$, verification is perfect for low enough quality $q \geq 0$.

The verification cost function is increasing and convex in $v$, and smooth, with $\chi'(v) > 0$ for $v > 0$, and initially $\chi(0) = \chi'(0) = 0$. When $v = V(e, q) < 1$, the verification function $V$ is smooth: the identity $q\chi(V(e, q)) \equiv e$ yields derivatives $q\chi'V_q + \chi \equiv 0$ and $q\chi'V_e \equiv 1$. Hence:

$$V_e(e, q) = 1/q\chi'(v) > 0 \quad \text{and} \quad V_q(e, q) = -\chi(v)/q\chi'(v) < 0 \quad (1)$$

Differentiating $q\chi(V(e, q)) \equiv e$ yields $q\chi'V_q + q\chi''V^2_e \equiv 0$, and so diminishing returns to effort: $q^2V_{ee}(e, q) = -\chi''(v)/(\chi'(v))^3 < 0$. To ensure that quality suffers diminishing returns in reducing verification, we assume $\chi$ is strictly log-concave, so that $\chi''/\chi' < \chi'/\chi$. For then:

$$q^2V_{qq} = \frac{\chi'}{\chi'} + \left(\frac{\chi'}{\chi'}\right)^2 \left(\frac{\chi''}{\chi'} - \frac{\chi''}{\chi'}\right) > 0 \geq \frac{\chi''}{(\chi')^2} \left(\frac{\chi''}{\chi'} - \frac{\chi''}{\chi'}\right) = q^2V_{eq} \quad (2)$$

Observe that we have also shown that verification is submodular in effort and quality. Effort raises the marginal efficacy of quality, but quality blunts the marginal fruits of effort.

We assume $v\chi''(v)/\chi'(v) \geq 1$, whence the limit elasticity $\lim_{v \to q} v\chi'(v)/\chi(v) \geq 2$ exists, by l’Hopital’s rule. Geometric cost functions $\chi(v) = v^B$ with $B \geq 2$ obey all assumptions.

D. The Verifier’s Problem. A verifier suffers transactions losses when three independent events simultaneously happen: (i) he is handed a fake note, and (ii) given such a fake note, his verifying efforts miss it, and (iii) given that he passes a fake note, the next agent catches

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13 Quantity is finite because the marginal distribution costs rise in output, as each passing attempt risks discovery: “If a counterfeiter goes out there and, you know, prints a million dollars, he’s going to get caught right away because when you flood the market with that much fake currency, the Secret Service is going to be all over you very quickly.” — Kersten (2005) [All Things Considered, July 23, 2005]

14 The Secret Service also advises anyone receiving suspected counterfeit money: “Do not return it to the passer. Delay the passer if possible. Observe the passer’s description.”

15 The Secret Service estimates that the conviction rate for counterfeiting arrests is close to 99%.

16 Weak convexity is clear: One can secure a verification chance $v$ at cost $(\chi(v - \varepsilon) + \chi(v + \varepsilon))/2$ by flipping a coin, and verifying at rates $v - \varepsilon$ or $v + \varepsilon$. In other words, $\chi(v) \leq (\chi(v - \varepsilon) + \chi(v + \varepsilon))/2$. 

7
Given an average verification rate \( v \), good guys choose their effort \( e \) to minimize their verification costs plus expected counterfeit losses next period:

\[
q \chi(V(e, q)) + \kappa(1 - V(e, q))\delta(v) \Delta
\]  

(3)

In this optimization, good guys act competitively, as they cannot affect the counterfeiting rate. Moreover, the model is in steady-state, and thus no time subscripts are needed.

E. The Counterfeiter’s Problem. We assume the passing fraction is a smooth, falling function obeying \( f(v) \leq 1 - v \) and \( f(0) > 0 \). This reflects how perfect verification chokes off passing (\( f(1) = 0 \)), and passing occurs if no one verifies. We also assume diminishing returns: \( f \) is weakly convex. The function is convex when verification increasingly helps police. We limit this effect in two ways. First, we assume \( f \) is strictly log-concave:

\[
\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} \geq 1
\]

(4)

For instance, any quadratic passing fraction \( f(v) = (1 - v)(1 - \gamma v) \) is monotone decreasing, convex and log-concave when \( 0 \leq \gamma < 1 \), and obeys \( f(0) > 0 = f(1) \). With geometric costs \( \chi(v) = v^B \), inequality (4) is slightly more restrictive, now requiring \( \gamma \leq (2B - 1)/(2B + 1) \).

The human and physical capital cost \( c(q) \) of the counterfeit quality \( q \) is smooth, with \( c', c'' > 0 \) for \( q > 0 \), \( c(0) = 0 \), and \( c'(q) \to \infty \) as \( q \uparrow \infty \). We assume a monotone cost of quality elasticity, and so a well-defined limit \( \eta = \lim_{q \to 0} qc'(q)/c(q) \geq 2 \):

\[
\left( \frac{qc'(q)}{c(q)} \right)' \geq 0
\]

(5)

A counterfeiter cares about his quality because it lessens the verification rate. Counterfeiters maximize profits equal to expected revenues \( f(v)x\Delta \) less costs \( c(q) + \ell \).

\[
\Pi(e, q, \Delta) = f(V(e, q))x\Delta - c(q) - \ell
\]

(6)

F. Equilibrium. Reflecting the decision margins of good and bad guys and the “rational expectations” verification rate, an equilibrium is a 4-tuple \((e^*, q^*, v^*, \kappa^*)\) such that: verification effort \( e^* \) minimizes costs (3) of good guys at the counterfeiting rate \( \kappa^* \); quality \( q^* \) maximizes

\footnotetext[17]{As is the norm, we ignore technicalities of randomness and independence for a continuum of events, and assume simply that probabilities of individual events correspond to measures of aggregate events.}

\footnotetext[18]{While good guys without notes face a two-period optimization, we can simply reduce it to a static one. We assume that \( \chi \) absorbs any discounting between periods in this simple optimization.}
profits \(6\) given \(e^*\); counterfeiting profits \(6\) vanish; and the verification rate is \(v^* = V(e^*, q^*)\).

Solving four equilibrium equations in four unknowns is in general hard. But since bad guys only supply notes, and never accept them, they don’t care about \(\kappa\) (bad guys don’t affect other bad guys). So our equilibrium admits a block recursive structure, parsing into two anonymous pairwise-matching games: The *cat and mouse game* pits good guys against bad guys; here, the profit maximization \(6\) and free entry condition link effort \(e\) and quality \(q\), irrespective of \(\kappa\). Next, in the *hot potato game*, good guys oppose both good and bad guys. Here, a cost minimization \(3\) relates effort \(e\), the counterfeiting rate \(\kappa\), and quality \(q\). Altogether, we solve equilibrium in stages: first, \(e^*, q^*\) solve the cat and mouse game; second, the verification rate is \(v^* = V(e^*, q^*)\); and finally, the counterfeiting rate \(\kappa^*\) solves the hot potato game.

### 3 Equilibrium Derivation

#### A. The Hot Potato Game.
Since \(V_e(e, q) > 0\), verification effort is a strategic complement in \(5\) to the average rate \(v\). One should examine a note more closely the more intensely it will be checked. So the minimizer \(\hat{\varepsilon}\) in \(5\) rises in \(v\). Since benefits are linear in \(v\), and \(\chi\) is strictly convex with \(\chi'(0) = 0\), any FOC solution with imperfect verification is a global minimum:

\[
1 = \kappa V_e(\hat{\varepsilon}, q)\delta(v) \Delta
\]

But identical good guys choose the same best response, i.e. \(\hat{\varepsilon} = V(\hat{\varepsilon}, q) = v\). As a product of weakly and strictly increasing functions, \(\chi'(v)/\delta(v) = [\chi'(v)/v][v/\delta(v)]\) is increasing. The derived counterfeiting demand curve \(v \mapsto \kappa\) slopes up, as fake notes are a bad, namely, the function:

\[
\kappa(v, q) = \frac{q\chi'(v)}{\delta(v)\Delta} = \frac{\text{marginal verification cost}}{\text{discovery rate} \times \text{denomination}}
\]

So verification \(v\) is an equilibrium at quality \(q\) for the counterfeiting rate \(\kappa(v, q)\).

#### B. The Cat and Mouse Game.
Given free entry, expected profits \(6\) vanish. In \((q, v)\)-space:

\[
\Delta x f(v) - c(q) - \ell = 0
\]

We assume \(\Delta > \Delta \equiv \ell/(xf(0)) > 0\). For if not, non-positive profits requires no verification and no quality costs; but then perfect verification arises with negligible effort. So \(e, q > 0\) in equilibrium, and obeys \(q < e/\chi(1)\). Since \(V(e, q)\) is smooth, the FOC \(10\) holds. It captures the tradeoff that higher quality notes pass better but cost more:

\[
\Pi_q(e, q, \Delta) \equiv \Delta x f'(V(e, q))V_q(e, q) - c'(q) = 0
\]
Theorem 1 (Existence and Uniqueness) If \( \sqrt{3}xf(0)\chi'(1) < (1 - \beta)c(1)^{1/\eta}\ell^{1 - 1/\eta} \), then a counterfeiting equilibrium uniquely exists for each note \( \Delta > \Delta \), and fails to exist for \( \Delta \leq \Delta \). Verification is positive and imperfect \( 0 < v < 1 \), and counterfeiting positive but finite, with:

\[
\kappa^* \leq \frac{\sqrt{3}xf(0)\chi'(1)}{(1 - \beta)c(1)^{1/\eta}\ell^{1 - 1/\eta}}
\]
Figure 5: The optimal quality locus $Q^*$ in (10) is steeper than the zero profit locus $\overline{\Pi}$ in (9), if it slopes down. The constant counterfeiting rate locus $K$ (constant (8)) lies between given (4).

The bound in (15) rises if counterfeiting is easier — specifically, lower unit quality costs $c(1)$, or legal costs $\ell$, or higher production $x$ or passing rate $f(0)$. The bound falls with better verification — namely, a higher bank chance $\beta$, or lower marginal verification costs $\chi'(1)$.

**D. Example.** Assume geometric costs $c(q) = q^A$ and $\chi(v) = v^B$ with $A, B \geq 2$, and $f(v) = 1 - v$ (i.e. no police). The zero profit and optimal quality equations (9) and (11) are:

$$\Delta x(1 - v) - q^A - \ell = 0 \quad \text{and} \quad Aq^A - \Delta xv/B = 0$$  \hspace{1cm} (16)

Solving the zero profit condition in (16), verification vanishes for notes $\Delta$ approaching $\Delta = \ell/x$. And as $\Delta \uparrow \infty$, the verification rate tends to $\bar{v} = AB/(1 + AB) < 1$, since:

$$q^A = (1 - \bar{v})(\Delta - \Delta) \quad \text{and} \quad v = \bar{v}(1 - \Delta/\Delta)$$  \hspace{1cm} (17)

So verification rises in the note $\Delta$, but is forever imperfect. While effort $e = qv^B$ rises in $\Delta$, quality rises much faster, and infinitely so initially as $B > 0$, as seen in Figure 6:

$$e = (1 - \bar{v})^{1/A} \bar{v}^B \Delta^{-B}(\Delta - \Delta)^{B+1/A}$$  \hspace{1cm} (18)

Substituting $q$ and $v$ from (17) into (8) yields the equilibrium $\kappa = Bqv^{B-1}/(\delta(v)\Delta)$, given the increasing discovery rate $\delta(v) = \beta\alpha + (1 - \beta)v$. Counterfeiting occurs for all $\Delta > \Delta$, and the counterfeiting rate $\kappa$ is unimodal in $\Delta$, vanishing for both $\Delta \downarrow \Delta$ and $\Delta \uparrow \infty$, since $A > 1$:

$$\kappa = \frac{B(1 - \bar{v})^{1/A} \bar{v}^{B-1} \Delta^{-B+1/A}(\Delta - \Delta)^{B-1+1/A}}{\beta\alpha\Delta + (1 - \beta)\bar{v}(\Delta - \Delta)}$$  \hspace{1cm} (19)

Figure 6 also depicts the like-shaped plot of the passed rate $p = \delta(v)\kappa$: It understates the counterfeiting rate, but the ratio $p/\kappa$ rises in $\Delta$, tending to $\bar{v} < 1$. Passed and counterfeiting rates vanish as $\Delta \downarrow \Delta$ (since $B > 1 + 1/A$), and both rates vanish like $\Delta^{1/A-1}$ as $\Delta \uparrow \infty$.

---

19 Given a strictly convex passing function $f(v) = (1 - v)/(1 - \gamma v)$ (i.e. with police), a quadratic equation fixes the verification rate $v$. More police presence (higher $\gamma$) depresses (“crowds out”) $v$, and elevates quality $q$.  

11
Figure 6: Verification, Effort, Quality, and Counterfeiting/Passed Rates in Example. Plots assume $A = 5, B = 3, x = 2, \ell = 10, \alpha = 4/5$ and $\beta = 1/4$. (Left) The verification rate rises from $\Delta = 5$ to $\bar{v} = 0.8$. (Middle) Effort and quality (solid/dashed) and $e/q$ rise from 0. (Right) The counterfeiting and passed rate curves (solid/dashed) vanish at small and large notes.

4 Equilibrium Comparative Statics

Equilibrium comparative statics analysis is possible largely by shifting curves in Figure 5.

**Theorem 2**

(a) Assume legal costs rise. Then the verification effort and rate fall; counterfeit quality falls at low and high notes $\Delta$, and always falls given (13); the counterfeiting rate falls.

(b) Assume that marginal verification costs fall. Then the verification effort and counterfeit quality fall, the verification rate rises, and the counterfeiting rate falls.

(c) If counterfeiting costs and marginal costs fall, then verification effort and counterfeit quality rise, the counterfeiting rate rises, and the verification rate falls if the cost elasticity $c'(q)/c(q)$ also falls.

(d) The verification effort and rate and counterfeit quality rise in the note if $\Delta > \Delta$. The verification effort and rate, and counterfeit quality all vanish as $\Delta \downarrow \Delta$. The verification effort and quality explode as the note $\Delta \uparrow \infty$. The counterfeiting rate vanishes as $\Delta \downarrow \Delta$ or $\Delta \uparrow \infty$.

Part (a) attests that a greater legal penalty displaces (“crowds out”) verification effort, but still deters counterfeiting. In parts (b)–(c), we see that while lower verification costs is not equivalent to greater counterfeiting costs, the sign of their effects on all variables is the same. Notably, more easily verified currency displaces verification effort so much that the verification rate rises and the counterfeiting rate falls. Part (d) rationalizes the hump-shaped plot in Figure 6(right panel) for the example in §3-D. We derive the effort comparative statics from our zero profit identity (9), and comparative statics of $(v, q, \kappa)$ via the graphical apparatus.

**Proof of (a):** Differentiate (9) in legal costs $\ell$ to get

$$\Pi_q \dot{q} + \Pi_e \dot{e} + \Pi_e = 0.$$ 

Since quality is optimal, $\Pi_q \dot{q} = 0$ — for $\dot{q} = 0$ if $q = 0$ in an open ball around $\ell$, and otherwise $\Pi_q = 0$. 

As a competitive game, the model inevitably admits one functional degree of freedom: Precisely scaling verification costs $\tilde{\chi} \equiv \chi/\nu$ is equivalent to inversely scaling the quality argument of counterfeiting costs to $c(q) \equiv c(\nu q)$, since $q \chi(v) \equiv (\nu q) \chi(q)$. Hence, $V(e/\nu, q) \equiv \chi^{-1}((e/\nu)/q) \equiv \chi^{-1}(e/(\nu q)) \equiv V(e, \nu q)$.

The least counterfeit note $\Delta$ rises in $\ell$ since it can no longer be profitably counterfeited, but this is inessential. The notation $\dot{x}$ denotes the derivative of $x$ in $\ell$. Later, it denotes derivatives in whatever parameter is changing.
Figure 7: Rising Legal Costs or Falling Verification Costs: Theorem 2 (a) and (b). Top: When legal costs rise, the zero profit curve $\Pi$ shifts down ($\Pi_L$ to $\Pi_H$), while $Q^*$ is unchanged. Verification worsens and quality falls if $Q^*$ slopes up — i.e. surely for low and high notes. The counterfeiting rate falls, with the lower $\bar{K}'$ locus. Bottom: When verification costs fall, the $Q^*$ locus shifts left ($Q^*_H$ to $Q^*_L$), and $\Pi$ is unchanged. Verification improves and quality falls. The counterfeiting rate falls: the $\bar{K}''$ locus and marginal verification cost function $\chi'$ are lower.

Since $\Pi_e = \Delta f' V_e < 0$ and $\Pi_\ell = -1 < 0$, we have $\dot{e} < 0$: Effort falls when legal costs rise.

Legal costs shift the zero profit curve down at each quality, because counterfeitters require less verification effort to avoid losses. The optimal quality locus $Q^*$ in (11) is unaffected by $\ell$, and so its shape alone governs changes in $(q, v)$. Verification falls, for either $Q^*$ slopes up, or slopes down and is steeper than $\bar{\Pi}$. Finally, if $Q^*$ is monotone, higher legal costs depress both quality and the verification rate, thus lowering the counterfeiting rate — as in Figure 7.

Proof of (b): Smoothly transform the old technology (say $t = 1$) into the new technology ($t > 1$) that renders money more readily verified. Assume a new verification cost function $\chi(v, t)$, with $\chi(v, 1) \equiv \chi(v)$ and a smaller slope $\chi_{vt}(v, t) < 0$ for $t \geq 1$; thus, by integration on $[1, v]$, we conclude $\chi_t(v, t) < 0$ for $t \geq 1$. Define $\mathcal{V}(e, q, t) = v$ iff $e = q\chi(v, t)$. First, the zero profit curve $\bar{\Pi}$ fixes how effort evolves. Differentiate its identity (9) in $t$:

$$\Delta x f(\mathcal{V}(e, q, t)) - c(q) - \ell = 0$$

Its $q$ derivatives cancel by (10). Then $\mathcal{V}_e \dot{e} + \mathcal{V}_\ell = 0$. Since $\mathcal{V}_e > 0$, effort $e$ falls in $t$.

For every $t \geq 1$, implicitly define $\nu(v, t) \leq v$ by $\chi(v, t) \equiv \chi(\nu(v, t))$, so that $\nu_t(v, t) < 0 < \nu_v(v, t)$. By log-concavity of $\chi$, $\chi'(\nu(v, t))/\chi(\nu(v, t))$ rises in $t$, for each $v$. So the $Q^*$ locus in (11) shifts left in $t$ for every $v$. As seen in Figure 7, quality falls, and verification rises. The counterfeiting rate falls as (i) the locus $\bar{K}$ shifts down to $\bar{K}''$, and (ii) each locus has a lower counterfeiting rate, since the marginal verification cost function $\chi_v$ in (8) falls in $t$. 

13
Figure 8: **Falling Counterfeit Costs or a Falling Note: Theorem (c) and (d) Depicted.** Top: The counterfeiting costs fall from $H$ to $L$, pushing the zero profit locus $\bar{\Pi}$ right more than the optimal quality locus $Q^*$, raising quality but lowering the verification rate. The counterfeiting rate falls, since the locus shifts left to $\bar{K}'$. Bottom: When the note $\Delta$ rises from $L$ to $H$, the locus $\bar{\Pi}$ shifts right more than $Q^*$, raising both quality and verification. The counterfeiting locus shifts right to $\bar{K}'$, but the rate might not rise, since $\Delta$ is higher.

**Proof of (c):** Write $c(q, \tau)$ as a smooth function of $\tau$, with $c_\tau, c_{q\tau} < 0$. Differentiate (9), $\Pi_q = 0$ implies $\Pi_\tau + \Pi_e \dot{e} = 0$. Since costs fall in $\tau$, we conclude $\Pi_e < 0 < \Pi_\tau$, and so $\dot{e} > 0$.

Given $T', U' > 0$, the $\bar{\Pi}$ and $Q^*$ loci shift right when $\tau$ rises, as $q$ rise to maintain equality in (12). Since costs smoothly fall in $\tau$, so too does $c(q, \tau) / (c(q, \tau) + \ell)$. If $c'/c$ also falls:

$$U_\tau - T_\tau = \frac{d}{d\tau} \log \left( \frac{qc'(q, \tau)}{c(q, \tau) + \ell} \right) = \frac{d}{d\tau} \log \left( \frac{qc'(q, \tau)}{c(q, \tau)} \right) + \frac{d}{d\tau} \log \left( \frac{c(q, \tau)}{c(q, \tau) + \ell} \right) < 0 \quad (20)$$

Thus, as $\tau$ rises, $Q^*$ shifts right more than $\bar{\Pi}$ does, lowering the verification rate (Figure 8).

Next, when $G' > 0$, the optimal quality locus $Q^*$ slopes up, and quality rises when $Q^*$ and $\bar{\Pi}$ shift right, as seen in Figure 8. But if $G' < 0$, then $Q^*$ slopes down, and both $\bar{\Pi}$ and $Q^*$ shift up: in this case, since $Q^*$ is steeper than $\bar{\Pi}$ at an equilibrium by Lemma 1, quality rises in $\tau$ exactly when $Q^*$ shifts up more than $\bar{\Pi}$ at each $q$. This occurs due to two re-enforcing effects. First, given (20), $U$ shifts up more than $T$, for fixed $q$; thus, $G$ must rise more than $F$. Next, since $F'(v) < G'(v) < 0$ by (14), this translates into a greater vertical shift in $Q^*$ than $\bar{\Pi}$. In both cases, the counterfeiting loci shifts right to $\bar{K}'$, and the counterfeiting rate rises. □

**Proof of (d):** That effort rises in the note ($\dot{e} > 0$) follows by differentiating the identity (9) in $\Delta$, and using $\Pi_q = 0$:

$$\Pi_e \dot{e} + \Pi_\Delta = 0 \quad (21)$$
and so \( \dot{e} > 0 \) since \( \Pi_e < 0 < \Pi_\Delta \) by (6). Next, when \( \Delta \) rises, \( \bar{\Pi} \) and \( Q^* \) shift right in (12). If \( Q^* \) slopes up, then quality clearly rises: But when \( Q^* \) slopes down, then as in Figure 8 (bottom), equilibrium quality rises if \( \bar{\Pi} \) vertically rises more than \( Q^* \); this holds since \( |F'(v)| < |G'(v)| \) by (14). Next, \( Q^* \) shifts right more than \( \bar{\Pi} \) by (20), and thus the verification rate rises.

As the note \( \Delta \downarrow \Delta = \ell/(xf(0)) \), the \( \bar{\Pi} \) locus (9) tends to the origin. So \( v \) and \( q \) vanish in equilibrium, where \( \bar{\Pi} \) and \( Q^* \) cross, as does \( e = q\chi(v) \). Next, as \( \Delta \uparrow \infty \), the left side of (11) explodes as \( \chi/\chi' \) weakly rises in \( v \), \( v \) weakly rises in \( \Delta \), and \( -f'(v) \geq -f'(0) > 0 \). So \( \kappa \) explodes. So \( e = q\chi(v) \) explodes as \( \Delta \uparrow \infty \), for \( v \) is monotone in \( \Delta \).

Finally, we analyze the counterfeiting rate. While the \( \bar{K}' \) locus at the higher denomination in Figure 8 (bottom) is right of the \( K \) locus, the counterfeiting rate (8) is also exogenously depressed by the higher note \( \Delta \). Thus, we must proceed analytically. First, the counterfeiting rate vanishes for low notes \( \Delta \downarrow \Delta \), since \( q \) and \( v \) vanish in (8), while the discovery rate obeys \( \delta(v) \geq \beta \alpha > 0 \). Next, assume \( \Delta \uparrow \infty \). Substitute the optimal quality condition (10) into (8):

\[
\kappa = \frac{q\chi'(v)}{\delta(v)\Delta} = \frac{q\chi'(v)xf'(v)\nu_e(q,e,q)}{-xf'(v)\chi(v)\delta(v)c'(q)} = \frac{-xf'(v)\chi(v)}{\delta(v)c'(q)}
\]

Since quality explodes, so too does marginal cost \( c'(q) \). Now, \( \chi(v) \leq \chi(1) < \infty \), and \( -f'(1) \leq -f'(0) < \infty \) since \( f \) is convex. Hence, the counterfeiting rate vanishes: \( \kappa \rightarrow 0 \). □

By Theorem 2 (d), counterfeit quality falls as the note rises, consistent that the switch to lower quality digital counterfeiting is more pronounced at the lower notes (Table 1). The highest quality “Supernote” was only for the $100 note.

## 5 Predictions about Passed, Seized, and Missed Money

We now return to the motivating data. Fix \( \Delta > \Delta \). Denote annual passed and seized money by \( P[\Delta] \) and \( S[\Delta] \), and equilibrium quality and verification as \( q[\Delta] \) and \( v[\Delta] \). By two steady-state approximations, new counterfeit production passing into circulation balances passed money outflows, and new counterfeit production replenishes the outflow of seized and passed money. Together, these imply \( P[\Delta] = f(v[\Delta])(S[\Delta] + P[\Delta]) \). Our zero profit equation (9) more specifically implies that seized-passed ratio \( S[\Delta]/P[\Delta] \) obeys:

\[
1 + S[\Delta]/P[\Delta] = 1/f(v[\Delta]) = \frac{x\Delta}{c(q[\Delta]) + \ell}
\]

Since \( v \) rises in \( \Delta \) by Theorem 2 (d), \( S[\Delta]/P[\Delta] \) rises in \( \Delta \). With fixed rather than endogenous quality, the inverse passing fraction rises proportionately to \( \Delta \). But since costly quality rises in \( \Delta \) by Theorem 2 (d), \( 1 + S/P \) rises less than proportionately to \( \Delta \).
By the formula (8) for the counterfeit rate $\kappa[\Delta]$, the passed rate satisfies:

$$p[\Delta] \approx \delta[\Delta]\kappa[\Delta] = \frac{q[\Delta]\chi'(v[\Delta])}{\Delta} = \text{marginal verification cost} \over \text{denomination} \tag{24}$$

Since the discovery rate $\delta(v[\Delta])$ increases in the note $\Delta$ by Theorem 2(d), so too is the ratio $p[\Delta]/\kappa[\Delta]$. So if the counterfeiting rate levels off, the passed rate continues to rise — for instance, it peaks at a higher note than the counterfeiting rate does in the example in §3.D.

Theorem 2(a)–(d) has respective implications for seized and passed money:

**Corollary 1**

(a) If legal costs rise, then the seized-passed ratio and the passed rate both fall.

(b) If verification costs fall, then the seized-passed ratio rises.

(c) Assume that counterfeiting costs and marginal costs fall. Then the seized-passed ratio falls — and the passed rate rises if $c'(q)/[c(q) + \ell]$ also does not fall.

(d) One plus the seized-passed ratio monotonically rises in $\Delta > \Delta_{\text{min}}$, but rises less than proportionately to the note. The passed rate vanishes as $\Delta \downarrow \Delta_{\text{min}}$ or $\Delta \uparrow \infty$.

**Proofs:** Parts (a)–(c) owe to Theorem 2(a)–(c) and the proven facts that the discovery rate $\delta$ and the seized-passed ratio $S/P$ rise in the verification rate. To wit, the passed rate $\delta\kappa$ falls in part (a); it moves ambiguously when verification or counterfeiting costs change, since the counterfeiting rate moves opposite to the discovery rate in (b) and (c) — except for (c), when $c'(q)/[c(q) + \ell]$ does not fall. We have shown the first half of (d); for the second half, the passed rate vanishes when the counterfeiting rate does, as the discovery rate is bounded by 1. \[\square\]

By part (a), raising the counterfeiting penalty effectively deters seized and passed money.

Part (c) speaks to the passed rate crash after Canada’s 1970–76 introduction of color notes. Part (c) also sheds light on the opposite phenomenon in the USA, in 1994–98, when digital technology lowered the non-Columbian costs of counterfeiting notes (Table 1). As seen in Figure 2(right), these years produced precipitous 80% drops in the seized-passed ratios of the non-Columbian $5, $10, and $20 notes, and less string drops in other notes.

The initial claim in part (d) simultaneously makes sense of two aspects of Figure 2(left) — first, that the slope is positive, and second that it is much less than one (i.e., direct proportionality). For instance, $1 + S/P$ does not even double moving from $5 to the $100 note.

Define the annualized passed rate $p_a[\Delta] = P[\Delta]/M[\Delta]$, namely, the ratio of yearly passed money to circulation. This equals the passed rate $p[\Delta]$ times the velocity of a note, which falls in the denomination. The claim that the annualized passed rate vanishes near the smallest notes is fully consistent with the passed rates of U.S. denominations seen in Figure 3. The

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22While circulation includes the counterfeit money, ignoring it has a negligible effect, as the approximation $p_a[\Delta] \approx P[\Delta]/M[\Delta] = \delta(v[\Delta])\kappa[\Delta]$ is accurate within 1% of 1%.

23The velocity is intuitively falling in the note, and thus as $\Delta$ rises, the annualized passed rate $p_a[\Delta]$ increasingly overstates the passed rate that our theory speaks to. Lower denomination notes wear out faster, surely due to
plot for the Euro also illustrates our additional claim that the passed rate vanishes for very large notes — the passed rate of the 500 Euro is less than 8% of the 200 Euro.

We now return to one final motivational data puzzle: passed money at the FRB. A fake note lands at an FRB if the following sequence of independent events transpires: it is fake, is deposited into a bank, it is not found, and then it is transferred to an FRB. With its perfect counterfeit detection, the \( \text{internal FRB passed rate} \) is the counterfeit fraction of transferred notes:

\[
\zeta = \frac{\text{fake notes hitting FRB}}{\text{total notes hitting FRB}} = \frac{\kappa \beta (1 - \alpha) \phi}{\beta (1 - \kappa) \phi + \kappa \beta (1 - \alpha) \phi} \approx \kappa (1 - \alpha)
\]

The approximation is accurate within \( \kappa \approx 0.0001 \), or 0.01%. While this depends on the unobserved counterfeiting rate, the \( \text{FRB ratio} \ \frac{\zeta}{p} \approx \frac{(1 - \alpha)}{\delta} \) does not. The discovery rate \( \delta \) rises in the note \( \Delta \), since \( v \) does by Theorem 2\( (d) \). So our theory predicts a monotonically falling FRB ratio, matching Figure 4 except at the $100 bill. Our simplifying assumption of constant \( \alpha \) is most strained here: If the bank detection chance of the high quality fake $100 note is sufficiently lower, so that \( \alpha[100] < \alpha[50] \), then the FRB ratio rises at $100.

6 Conclusion and Relationship to the Literature

Existing work on counterfeiting is predicated on a general equilibrium value of money. Our point of departure is thus to replace a priced asset with a new decision margin — verification effort. For a useful point of comparison, Williamson and Wright (1994) assumes that transactors observe fixed signals of the authenticity of money, albeit after acquiring it. We instead build an entire theory on costly verification efforts that individuals expend before accepting money. Their work could not explain any counterfeiting data, since the signals in no way respond to the payoff stakes. Simply put, we argue that \textit{exogenous attention cannot rationalize the facts of counterfeiting} — the assumption common to almost all existing work.

If we added general equilibrium effects to our model, they would be second order and add almost nothing to our explanations of passed and seized money at the current 1 in 10,000 counterfeiting rates, for they would only discount prices infinitesimally.

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24 In Green and Weber (1996), only government agents can discern fake notes, whose stock is assumed exogenous, unlike here. Williamson (2002) admits counterfeits of private bank notes that are found with fixed chance; counterfeiting does not occur in most of his equilibria. Verification is also random and exogenous in Nosal and Wallace (2007), who find no counterfeiting in equilibrium with a high enough counterfeiting cost. Li and Rocheteau (2011) subsequently questioned this.

25 An outlier in this literature is Banerjee and Maskin (1996). In our language, their verification is either perfect or worthless for each good: Agents either can or cannot distinguish good and bad qualities.
A Appendix: Omitted Proofs

A.1 Optimal Quality and Zero Profit Curves: Proof of Lemma \( \Pi \) (a)

The \( Q^* \) locus starts at \( q = v = 0 \) and is initially flat — since \(-\infty < f'(0) < 0 \) and the limit of \( v\chi'(v)/\chi(v) \) as \( v \to 0 \) finitely exists, whereas \( v/q = -[v\chi'(v)/\chi(v)][c'(q)/\Delta x f'(v)] \to 0 \) as \( q, v \to 0 \). Also, \( Q^* \) hits \( v = 0 \) at quality \( q^\Delta < \infty \), where \( q^\Delta c'(q^\Delta) = -\Delta f'(1)\chi(1)/\chi'(1) > 0 \), for \( 1 - v \geq f(v) > 0 \) and the convex passing fraction \( f \) implies a slope \( f'(v) \geq -1 \) as \( v \uparrow 1 \).

**Claim 1 (Strict SOC)** The second order condition at an optimum is strict: \( \Pi_{qq} < 0 \).

**Proof of Claim:** The SOC for maximizing \( \Pi(e, q, \Delta) \) is locally necessary:

\[
\Pi_{qq} = \Delta x f' V_{qq} + \Delta x f'' V_q^2 - c'' \leq 0 \tag{25}
\]

The derivative of the quality first order condition (10) in the note \( \Delta \) yields:

\[
0 = \Pi_{qq}\dot{q} + \Pi_{q\dot{e}} \dot{e} + \Pi_{q\Delta} \tag{26}
\]

For a contradiction, assume \( \Pi_{qq} = 0 \). Then (21) and (26) must be linearly dependent. Since \( \Pi_{q\dot{e}} = \Delta (f'V_{qe} + f''V_eV_q) \) and \( \Pi_{q\Delta} = f'V_q \), then exploiting (11) and (2):

\[
\frac{f'V_{qe} + f''V_eV_q}{f'V_q} = \frac{f'V_e}{f} \quad \Rightarrow \quad 0 < \frac{V_{qe}}{V_q} = \left( \frac{f''}{f'} - \frac{f''}{f'} \right) V_e
\]

This is a contradiction, for \( V_e > 0 \) and \( f'/f < f''/f' \) by strict log-concavity of \( f \). \( \Box \)

**Claim 2** If \( Q^* \) slopes down at an equilibrium, then it is steeper than \( \bar{\Pi} \).

**Proof:** We now argue that the SOC reduces to \( G'(v)T'(q) > F'(v)U'(q) \). Since the respective slopes of the \( \bar{\Pi} \) and \( Q^* \) curves are \( T'(q)/F'(v) \) and \( U'(q)/G'(v) \), this says that if \( Q^* \) is negatively sloped, then it is absolutely steeper than \( \bar{\Pi} \) — in other words, \( G'(v) < 0 \) implies \( T'(q)/F'(v) > U'(q)/G'(v) \). Reformulating the SOC (25), we find:

\[
0 > \Pi_{qq}(v, q, \Delta) = c' \left[ \frac{V_{qq}}{V_q} + \frac{f''}{f'} V_q \right] - c''(q)
\]

\[
= c' \left[ \frac{-1}{q} \left( 1 + \frac{\chi}{\chi'} \left( \frac{\chi'}{\chi} - \frac{\chi''}{\chi'} \right) \right) - \frac{f''}{f'} \left( \frac{\chi'}{q\chi'} \right) \right] - c''(q) \tag{27}
\]

by (9) and (11) and (2). Taking the quotient of (10) and (9), using \( V_q = -\chi/(q\chi') \), we find:

\[
\frac{f'}{f} = -\frac{qc'(q)}{c(q) + \ell \chi} \Rightarrow -\frac{q\chi'}{\chi} = -\frac{F'(v)/T'(q)}{q\chi'} \tag{28}
\]
That \( G'(v)T'(q) > F'(v)U'(q) \) follows from (27) and (28), for they yield

\[
\frac{\chi''}{\chi'} - \frac{\chi'}{\chi} + \frac{f''}{f'} - \frac{f'''}{f''} < \frac{\chi'q\epsilon'(q)}{\chi c'(q)} + \frac{\chi'}{\chi} = \frac{q\epsilon'(q) + c'(q)}{\epsilon'(q)} - \frac{c'(q)}{c'(q) + \ell}
\]

and thus, (14) yields \( F'(v) - G'(v) = -[F'(v)/T'(q)](U'(q) - T'(q)) \), as desired. □

A.2 Constant Counterfeiting Rate Curve Slope: Proof of Lemma 1

Differentiating the log of (8), the proportionate change in the counterfeiting rate is

\[
\frac{dk}{k} = \frac{dq}{q} + \left( \frac{v\chi''(v)}{\chi'(v)} - \frac{v\delta'(v)}{\delta(v)} \right) \frac{dv}{v} - \frac{d\Delta}{\Delta}
\]

Holding \( k \) and \( \Delta \) fixed, the change in quality along the \( \bar{K} \) locus obeys

\[
\frac{dq}{q} \bigg|_{\bar{K}} = \left( \frac{v\delta'(v)}{\delta} - \frac{v\chi''(v)}{\chi'(v)} \right) \frac{dv}{v}
\]

(29)

Along the \( \bar{P} \) locus, the change in quality obeys

\[
\frac{dq}{q} \bigg|_{\bar{P}} = \frac{\Delta xv f'(v)}{qc'(q)} \frac{dv}{v} = -\frac{v\chi'(v)}{\chi(v)} \frac{dv}{v}
\]

(30)

after substituting (11). By log-concavity of \( \chi \), we see that (29) strictly exceeds (30). Thus, the slope of \( \bar{K} \) exceeds that of \( \bar{P} \), but we now show that it is less than the slope of \( Q^* \). This is clear when \( Q^* \) has positive slope. Indeed, log-differentiating (11):

\[
\left( 1 + \frac{qc''(q)}{c'(q)} \right) \frac{dq}{q} \bigg|_{Q^*} = \left( \frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} - \frac{v\chi''(v)}{\chi'(v)} \right) \frac{dv}{v}
\]

When \( Q^* \) has negative slope, it is steeper than \( \bar{K} \) since \( c''(q)/c'(q) \geq 0 \) and by (4):

\[
\frac{vf''(v)}{f'(v)} + \frac{v\chi'(v)}{\chi(v)} \geq 1 > \frac{v\delta'(v)}{\delta(v)}
\]

A.3 Existence and Uniqueness: Proof of Theorem 1

Since by free entry, the counterfeiting rate \( k \) is derived from (8), an equilibrium is formally established in \( (v, q, e) \) space. The existence proof proceeds in \( (e, q, v) \).

A. Existence. Assume \( \Delta > \Delta_0 \). In this case, we exhibit a solution to the zero profit and optimal quality equations (9) and (11), at the left of Figure 5. Since \( f' < 0 < c' \), the zero
profit equation (9) implicitly defines a continuous and decreasing function \( q = Q_0(v) \). We must have \( Q_0(0) > 0 \), because \( c(Q_0(0)) = \Delta x f(0) - \ell > 0 \) when \( \Delta > \Delta_0 \). Since \( \Delta x f(0) > \ell \) and \( f(1) = 0 \), we may choose \( \hat{v} < 1 \) so that \( \Delta x f(\hat{v}) = \ell \). Then \( Q_0(v) \to 0 \) as \( v \to \hat{v} \). By the Implicit Function Theorem (IFT), because \( q c'(q) \) is strictly increasing, the quality FOC (11) implicitly defines a differentiable function \( q = Q_1(v) \). Since the limit \( v \chi'(v)/\chi(v) \) exists and is positive as \( v \to 0 \), both sides of (11) vanish, and so \( Q_1(0) = 0 \). Easily, (11) is positive at \( v = \hat{v} \), and thus \( Q_1(\hat{v}) > 0 \). Given \( Q_1(0) = 0 < Q_0(0) \) and \( Q_1(\hat{v}) > 0 = Q_0(\hat{v}) \), the Intermediate Value Theorem yields \( v \in (0, \hat{v}) \) with \( Q_0(v) = Q_1(v) \). But then \( 0 < v < 1 \) and \( 0 < q = Q_1(v) < Q_0(v) < \infty \). So \( \kappa > 0 \) by (8). Finally, since \( Q_0(v), Q_1(v) \) are differentiable in \( \Delta \), so is \( q[\Delta] \) and \( v[\Delta] \). (This also follows by applying the IFT to the system (9) and (11).)

**B. Uniqueness.** Assume two equilibria \((e_1, q_1)\) and \((e_2, q_2)\) for a note \( \Delta \). If \( q_1 = q_2 \) then \( e_1 = e_2 \), as profits fall in effort. Assume \( q_1 < q_2 \). By a line integral of \( \Pi \) along the smooth optimal quality curve from \((e_1, q_1)\) to \((e_2, q_2)\), i.e. \( Q^* = \{(e, q) : \Pi_q(e, q) = 0, q_1 \leq q \leq q_2\} \):

\[
0 - 0 = \Pi(e_2, q_2) - \Pi(e_1, q_1) = \int_{Q^*} (\Pi_e, \Pi_q) \cdot (de, dq) = \int_{e_1}^{e_2} \Pi_e de
\]

Since \( \Pi_e < 0 \), \( e_1 = e_2 \). Then \( v_1 > v_2 \), and so profits are higher at \((e_2, q_2)\) than \((e_1, q_1)\), a contradiction. (Also, \( 0 < v_i < 1 \), since \( \Pi \) has positive intercepts and \( Q^* \) rises from the origin.)

**C. The Peak Counterfeiting Rate.** We proceed in three steps.

**STEP 1.** Modifying the counterfeiting rate formula (22) for zero profits (9), we find:

\[
\kappa(v) = -\frac{xf'(v)\chi(v)}{\delta(v)c'(q)} = \frac{xf(v)\chi'(v)}{\delta(v)} \frac{q}{c(q) + \ell}
\]

Since \((c(q) + \ell)/q\) is minimized when \(qc'(q) - c(q) = \ell\), where it equals the marginal cost \(c'(q)\), and since \(\delta(v) \geq (1 - \beta)v\) and \(\chi'(v)/v\) is weakly increasing, we have:

\[
\kappa(v) \leq \frac{xf(0)\chi'(1)}{(1 - \beta)c'(q)}
\]  

(31)

**STEP 2: LOWER BOUND ON COST & MARGINAL COST OF QUALITY.** Since \(qc'(q)/c(q)\) weakly increases by (5), \(c'(q)/c(q) \geq \eta/q\) if \( q > 0 \). Integrating on \([1, q]\) yields \(\log c(q) - \log c(1) \geq \log q^n\) if \( q \geq 1 \). So \(c(q) \geq c(1)q^n\). Then \(c'(q)/c(q) \geq \eta/q\) implies \(c'(q) \geq c(1)\eta q^{n-1}\).

**STEP 3: A FIXED UPPER BOUND FOR THE COUNTERFEITING RATE.** Define producer surplus \(\pi(q) \equiv qc'(q) - c(q)\). Let \(Q(\ell)\) be the quality that yields producer surplus \(\pi(Q(\ell)) \equiv \ell\). Then by the cost bounds in Step 2, we deduce

\[
\ell = \pi(Q(\ell)) = Q(\ell)c'(Q(\ell)) - c(Q(\ell)) \geq c(1)\eta Q(\ell)^n - c(1)Q(\ell)^n
\]
This implies the following lower bound that allows us to simplify (31):

\[
c'(Q(\ell)) > \frac{\pi(Q(\ell))}{Q(\ell)} \geq \frac{\ell}{Q(\ell)} \geq \frac{\ell}{(\ell/c\eta(\eta + 1))^{1/\eta}} \geq c(1)^{1/\eta}\ell^{1-1/\eta}/\sqrt{3}
\]

since \((1 + \eta)^{1/\eta}\) is monotone decreasing in \(\eta > 1\), and we assumed \(\eta \geq 2\).

References


