Collateral Risk, Repo Rollover and Shadow Banking

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Abstract

I build a dynamic model of the shadow banking system that intermediates funds through the interbank repo market to understand its efficiency and stability. The model emphasizes a key friction: the maturity mismatch between the short-term repo and the long-term investment that banks need to finance. The haircut, interest rate, default rate of the repo contract and liquidity hoarding of banks are all determined endogenously in the equilibrium with repo rollover. And default is contagious. When collateral risk increases unexpectedly, the haircut and interest rate overshoot, triggering massive initial default and persistently hiking default rate and depressed investment.

Keywords: interbank repo market; shadow banking; haircut; liquidity hoarding; solvency and liquidity; counter-party risk

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1 Introduction

The shadow banking system is a fundamental process of credit intermediation in modern banking. The process relies heavily on such debt contracts as the repurchase contract, a short-term collateralized debt contract with safe harbor provisions. Right before the great recession, the gross outstanding volume reached $10 trillion in both the US and Euro-zone repo market, both about 70% of the GDP in the respective area in 2007.

In 2007, the risk on collateral asset increased unexpectedly due to the sharp decline in housing prices. Concerned about the quality of collateral asset, financial intermediaries reduced their repo exposure to each other and began hoarding liquidity. Haircut and interest rate shot up. This eventually led to the downfall of Lehman Brothers, which ran out of resources to finance its long-term investments.

The crisis in the repo market and the shadow banking system in the great recession exposed the instability of the system and left us the following questions. What is the systemic risk in the shadow banking system? Are the efficiency and stability of the system affected by frictions in the repo market? What triggered the crisis? Why has the system not fully recovered even 5 years after the breakout of the crisis? To answer these questions, I build a dynamic model of the shadow banking system that intermediates funds through the interbank repo market.

In the model, banks wait for their investment opportunities. The shadow banking system is where they lend their funding while they wait and where they borrow to finance their investment when they find their opportunities. The model emphasizes a key friction: the maturity mismatch between the short-term repo and the long-term investment that banks need to finance. The maturity

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1See Pozsar et al. [2013] for the credit intermediation process in the shadow banking system. And I will explain in more detail its institutional features in the next section.

2In legal terms, repo contract is a combination of two outright transactions, sales at the moment the contract is signed and purchase at a future date at a price according to the contract. Since it can be interpreted either as a combination of two spot trade or as a secured loan. It helps some financial institutions circumvent legal restrictions to lending other institutions or carrying out spot trade. Another difference between the repo contract and a secured loan is that when a borrower defaults, the collateral asset is not subject to automatic stay. The safe harbor provision makes financing through repo contract popular. See Garbade [2006] for more details.

3See Hördahl and King [2008].

4ABX indices, price index for CDS over a collection of mortgage backed securities, dropped.

mismatch manifests itself in borrowers who have to roll over their debt until they receive return from their investment and are able to repay their loan. Since the funds may not arrive before borrowers reach their endogenous debt limit, default happens in equilibrium.

The haircut, interest rate and default rate of the repo contract are all endogenously determined in the equilibrium with repo rollover. As we will see, the endogenous haircut and default rate allow me to study how changes in the environment affect borrowing constraint and the externality of equilibrium default.

I show that systemic risk arises from the externality of counterparty default in the environment of interbank repo lending: default triggers more default. Counterparty default is a shock to the lender’s portfolio, upon which her lending is turned into collateral asset. And an investor is more likely to default if she has to rely more on repo borrowing to finance her long-term investment because she has less funding of her own and more collateral asset. In combination, this means that default is contagious: counterparty default makes a lender more likely to default in the future when she finances her investment using the repo contract.

To understand the failing mechanism of the shadow banking system and to test the theory, I extend the model to allow for collateral risk and find that collateral risk affects adversely repo rollover and increases counterparty default risk. When collateral risk increases, the repo market dries up through two channels: liquidity hoarding, and counterparty default. When collateral risk increases, banks hoard more liquidity to secure funding for their own investment in the future. With less funding in the repo market, the equilibrium haircut and interest rate of the repo contract increases, repo rollover is harder and default is more likely. Counterparty default transforms lenders’ funding to collateral asset. As a result, supply of funding to the repo market further decreases. Therefore, as collateral risk increases, the efficiency of the financial system drops.

With the dynamic model, I can study not only the efficiency of the shadow banking system in a steady state but also the stability of the system in response to a shock. The stability of the system can be measured by two metrics: the magnitude of the initial response and the persistence of efficiency loss on the transition path to the new steady state. To understand the stability of

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6At the time, 50% of primary broker-dealers’ repo contracts are backed by such less liquid securities as corporate securities, mortgage backed securities and other asset backed securities and 65% of the contracts are overnight. See Adrian et al. [2009].
the financial system in a situation similar to the onset of the great recession, I characterize the transition dynamics triggered by an unexpected increase in the collateral risk. On impact, the hoarding motive of lenders causes the supply of funds to the repo market to fall abruptly and the haircut and interest rate of the repo contract increases sharply. This leads to a sudden tightening of borrowers’ borrowing capacity and, as a result, massive default the moment the shock hits the economy. Additionally, the debt overhang problem troubling the rest of borrowers who started borrowing before the crisis also increases default rate at the beginning of the crisis. As counterparty default is contagious, the massive initial default and the increase in default rate because of the debt overhang problem have a long lasting effect on the equilibrium allocation: repo lending and investment remain lows and default rate remains high for an extended period of time. The systemic risk from contagious counterparty default increases both the magnitude and persistence of efficiency loss on the dynamic path triggered by the shock.

**Literature review**

The paper takes a different stand on the source of the financial crisis from the literature on asymmetric information, market failure and the financial crisis (Chiu and Koepppl [2011], Camargo and Lester [2011]). Dang et al. [2009, 2012], Gorton and Ordonez [2012], Farhi and Tirole [2012], Hellwig and Zhang [2012] take one more step to study the effect of endogenous information structure and market liquidity. In this paper, I explore the possibility of a simpler explanation for the crisis, risk on the collateral asset, and study the mechanism through the risk could lead to a crisis.

The collateral risk I study is closely related to Kocherlakota [2001], where the risk on collateral makes it harder to enforce the borrower to share social return of a project to lender. As in Kocherlakota [2001] it is too costly to collect resources from borrowers other than their collateral and collateral asset is risky. The difference is that when the value of collateral asset drops, lenders are not able to withdraw funding from borrowers, even though only a tiny fraction of lenders observe the shock. This is repayment of repo contracts is mostly supported by debt rollover. But equilibrium rollover will collapse when a small fraction of lenders want to withdraw on observing the shock.

The term structure of repo borrowing in my model is exogenous. Brunnermeier and Oehmke [2013] shows that the exemption from automatic stay of short-term repo contracts triggers a ma-
turity rat race so banks borrow inefficiently short-term in equilibrium. But it would interesting to endogenize the term structure in the future and study its effect, as in Williamson [2013].

The financial system in my model is a place banks or investors share profitable opportunities, as in Kiyotaki and Moore [2002] and Berentsen et al. [2007]. While a lot are known about the traditional banking crisis with the workhorse model by Diamond and Dybvig [1983], the supply of models about the repo market is small. Martin et al. [2011] focuses on the repo market between cash providers and financial intermediaries. I focus on repo lending between financial intermediaries to give a complementary angle to understand the systemic risk in the secured banking system. Other models of the shadow banking system in the spirit of Diamond and Dybvig [1983] include Gennaioli et al. [2013]. In the empirical work by Gorton and Metrick [2012a], the author take the view that the recent crisis is a system-wide self-fulfilling run. While fragility of the financial system is important, it is unclear how it can happen in the bilateral repo market, as collateral asset plays the role of deposit insurance so a self-fulfilling bank run in Diamond-Dybvig model would have been prevented. I focus instead on studying the systemic risk and equilibrium dynamics without runs on the repo market.

My paper follows the view of Sargent [2013] on liquidity of the financial market, that liquidity problem is a result of market incompleteness and the answer to a liquidity problem is model dependent. I model the shadow banking sector with two questions raised in Moore [2011] in mind, that why financial intermediaries hold mutual gross positions and whether the gross positions create systemic risk.

The paper is related to the study of banks' risk taking behavior started by Allen and Gale [2001]. In my paper, investment in the profitable long-term project is risky as it is uncertain when the project will mature.

In my paper, a lender whose counterparty defaults is more likely to default when she invests in her long-term project. The financial contagion effect in the paper is in the spirit of Allen and Gale [2000]. Here the effect takes place on the dynamic equilibrium path and has explicit implication on such objectives as haircut.

Liquidity hoarding that arise from collateral risk reminds us of precautionary demand for funding as in Frenkel and Jovanovic [1980]. Another channel for liquidity hoarding is the speculative motive to buy asset under fire sale in the future, as in Gale and Yorulmazer [2013]. Gale and Yorul-
mazer [2013] listed two possible explanations for the phenomenon liquidity hoarding: counter-party default risk and the fear participation in lending may comprise a lender's future access to liquidity. Both ingredients contribute to the freeze of repo market and liquidity hoarding of bankers in my model.

Characterization of endogenous haircut is related to the study of endogenous leverage initiated by Geanakoplos and Zame [1997] and developed by Fostel and Geanakoplos [2012]. Haircut in these papers is pinned down by the price of arrow securities, subject to additional constraints. In my model, repo contract is the only contract traded in the market and haircut is determined by a necessary condition for equilibrium rollover. This complements the extensive literature on credit cycles starting from Kiyotaki and Moore [1997a] and Bernanke and Gertler [1989], with recent development including Adrian and Shin [2010], Brunnermeier and Sannikov [2012] and Gertler and Kiyotaki [2013], where borrowing constraint is exogenous and always binding.

Another problem about the repo market is that investors can build leverage through rehypothecation (Singh and Aitken [2010]). The fact that a collateral asset can be used several times implies that lending using repo is not secured. Rather, only net position is secured (see Bottazzi et al. [2012]). The additional issues rehypothecation introduces to the repo market, such as novation (see Duffie [2010]), is left for further research. Unsecured lending in a long-term relationship as in Kehoe and Levine [1993] is also not modeled.

Disruption in the repo market between money market mutual funds and broker-dealers also played a major role in the crisis. I abstract from related issues and focus instead on the repo market between dealer banks. Gorton and Metrick [2012b] showed that during the financial crisis, money market mutual funds did not reduce net lending to the repo market as a whole. The disruption in the repo market took place in the bilateral repo market between broker-dealers or hedge funds and broker-dealers. Unsecured lending though such markets as the Fed funds market (see Afonso and Lagos [2012] for a model about the market) is not allowed. Operational risk such as settlement fails in the repo market (see Fleming and Garbade [2005]) is also not modeled.

Section 2 introduces in more detail institutional features the model aims to capture and stylized facts of the crisis that motivates the exercise. Section 3 introduces the model. In section 4, I formally define the dynamics equilibrium of the model. Section 5 discusses efficiency in the model. In section 6, I characterize the equilibrium with rollover of the repo contract, study the efficiency gain
of using repo contracts and characterize the efficiency loss from maturity mismatch and collateral risk. Section 7 studies the stability of the shadow banking system by looking at the transition dynamics triggered by a small but unexpected increase in collateral risk. Section 8 concludes the paper.

2 Shadow banking and the repo market: institutional features and stylized facts

In this section, I explain in more detail institutional features and stylized facts of collateral risk, the repo market and the shadow banking system.\textsuperscript{7}

**Shadow banking and repo lending between financial intermediaries** Unlike many financial markets where either the demand side or the supply side involves mainly agents from the real sector, many participants in the repo market are financial intermediaries who could be on either side of the market. Broker-dealers lend to each other in the repo market during the process of credit intermediation in the shadow banking system. Hedge funds and broker-dealers implement arbitrage strategies with each other using securities lending contracts. The trade among them reduces the cost of investment and arbitrage for agents.

Credit intermediation in the shadow banking system typically starts from loan origination and loan warehousing and ends with wholesale funding provided by such institution as money market mutual funds. Before the loans reach the end cash suppliers, they need to be packaged into asset backed securities (ABS), collateralized debt obligations (CDO) and asset backed commercial papers (ABCP), which typically involves securities issuance, warehousing, tranching and intermediation.\textsuperscript{8} The intermediate stages take time and rely on financing through the interbank lending market. A chain of credit intermediation comprised of broker-dealers and other intermediaries of the shadow banking system is formed during the process. This chain of intermediation within the shadow banking system relies heavily on short-term collateralized loan, such as repo. Repo financing in those intermediate steps is not directly financed by end cash suppliers, but by other broker-dealers

\textsuperscript{7}See for example Gorton and Metrick [2012a] and Copeland et al. [2012] for more details of the institutional features related to the repo market and the financial crisis.

\textsuperscript{8}See Pozsar, Adrian, Ashcraft, and Boesky [2013]
and financial intermediaries. This market-based intermediation is what distinguishes a shadow banking system from a traditional banking system.

**Maturity mismatch, solvency and liquidity of financial intermediaries** Agents in the repo and securities lending market rely on short-term debt to finance investment of longer maturity.\(^9\) The maturity mismatch links the liquidity of the short-term lending market to solvency of the borrowers. In the case of Lehman Brothers, the crisis starts from the asset side but not from short-term financing per se. The CDOs it was initiating, which is illiquid long-term investment, started losing money and became hard to sell way before the crisis. To *wait* for the investment to turn around, the bank had to rollover the debt. In the end, they lost the race and ran out of collateral and into bankruptcy. The bankruptcy of Lehman Brothers is not necessarily a self-fulfilling run. According to a WSJ report: “Six weeks before it went bankrupt, Lehman Brothers Holdings Inc. was effectively out of securities that could be used as collateral to back the short-term loans it needed to survive.” And they had to rely on “Repo 105”, a way to borrow against collateral without exposing its high leverage to the public, as early as the end of 2007. Even without a run, the bank may have had to default as they ran out of collateral.\(^10\)

Speculation and arbitrage through securities lending also typically involve maturity mismatch. Convergence trade, a strategy that goes long on one asset and goes short on a similar one, typically involves maturity transformation as difference in liquidity is often associated with the spread across similar assets and it takes time to realize the arbitrage profit. If it takes longer than expected to realize those gain from trade, the arbitraging or speculating agents may run out of financing resources and into insolvency, through the mechanism similar to the one in the case of Lehman Brothers. The downfall of Long-Term Capital Management\(^11\) and MF Global\(^12\) are both related to this.

In all these cases, solvency of a financial intermediary depends on maturity of project investment and liquidity of the repo or securities lending market. The solvency and liquidity of a financial intermediary can be better understood by looking at the equilibrium of a model.

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\(^9\) Adrian et al. [2009] show that about 65% of outstanding repos of primary dealers are overnight repos.

\(^10\)“Repos Played a Key Role in Lehman’s Demise.”,

http://online.wsj.com/article/SB1000142405274870344710457511815065179066.html

\(^11\)http://en.wikipedia.org/wiki/Long-Term_Capital_Management#Downturn

\(^12\)http://en.wikipedia.org/wiki/MF_Global
Collateral risk in the 2007-2008 financial crisis

According to the home-price-index of New York Fed, the growth rate of housing prices slowed down before 2007 and turned negative close to the end of 2007. Figure 1 illustrates the year-over-year changes in housing prices in US at the county level. In August 2007, 50% counties experienced negative price changes and at the end of 2007, more than 75% of counties started to show negative housing price change. The grey area in the figure marked the great recession.

As the housing price went on downward spiral, riskiness of mortgage back securities increased. Figure 2 shows the market price index for a credit default swap contract that provided insurance against the default risk of a pool of mortgage backed securities issued in early 2006. The discrepancy between the par value, 100, and the actual price index measures market belief about the riskiness of the mortgage backed securities. The figure shows the index for tranches with AAA rating to tranches with BBB- rating. Before mid 2007, the market’s belief about the riskiness of all tranches

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13 http://www.newyorkfed.org/home-price-index/
were literally zero. Riskiness of tranches of lower ratings increased first, at the beginning of 2007 and then in July 2007, riskiness of AAA tranches increased from zero to a positive number and kept increasing. The sudden changes in the price index imply that the shift in market beliefs came as an unexpected shock.

Why was there a sudden shift in the market belief about the collateral risk? Dang et al. [2009, 2012] relate this to the information sensitivity of the debt contract and the lemons problem of subprime mortgages. However, the link between the changes in housing prices and the riskiness of mortgage backed securities indicates that the risk is more likely to be related to the unexpected collapse of the housing market in the whole country, rather than the lemons problem that arises from the quality deterioration of a fraction of mortgages. And the collapse of the housing market is an information so widely publicized that asymmetric information on this fact became unlikely. So in this paper, I take the stand that the crisis is triggered by unexpected increase in collateral risk rather than market failure because of the lemons problem.

**Balance sheet adjustments of financial intermediaries and liquidity of bilateral repo market**  
He et al. [2010] estimated that on the asset side, hedge funds and broker-dealers reduce
holdings of securitized assets by approximately $800 billion during the 2008 crisis. But not only the size of broker-dealers’ asset holding shrank, but the portfolio composition changed as well. Before the crisis in November 2007, credit and mortgage-related asset made up of 32% of the total value of trading assets of Goldman Sachs, Morgan Stanley and Merrill Lynch. After crisis in March 2009, the number is 23%.\textsuperscript{15} The flight to such safe asset as treasury bills may be related to the increasing market risk of the securities and is also consistent with their reduced activity in the repo market where the haircut for the risky collateral asset increased sharply.\textsuperscript{16} Gorton and Metrick [2012b] found through Flow of Funds data that both the Repo asset and liability of broker-dealers shrank during the crisis, indicating the freeze of the repo market and broker-dealers’ reduced activity in the market, as illustrated in figure 3. These evidences imply that not only financial intermediaries’ balance sheet but also the portfolio composition of the balance sheet in the shadow banking system may contribute to and be affected by the financial crisis.

Krishnamurthy et al. [2012] shows that funding from cash providers like Money Market Mutual Funds did not change dramatically during the crisis. The implies that the dramatic change happened in the bilateral repo market between broker-dealers and hedge funds, which is where increasing haircut is reported in Gorton and Metrick [2012a,b], Hördahl and King [2008]. This is consistent with the finding in Gorton and Metrick [2012b], where they find that repo lending of investors other than money market mutual funds shrunk dramatically during the crisis. (See figure 3 for dynamics in repo lending and borrowing. See figure 4 for dynamics in haircut.)

3 The model

The model is set in continuous time. The economy starts at $t = 0$ and lasts forever. There is a continuum of agents of constant measure. At any moment, there is a constant inflow of entrants, $\eta$, and a constant outflow of exits.

There is a durable consumption good in the economy. And there are some productive trees. A

\textsuperscript{15}See Table 7 of He et al. [2010]. The definition of trading asset is “a collection of securities held by a firm that are held for the purpose of reselling for a profit. Trading assets are recorded as a separate account from the investment portfolio.” (http://www.investopedia.com/terms/t/trading-assets.asp)

\textsuperscript{16}For example, Gorton and Metrick [2012a] documented the devaluation of BBB asset backed securities and sharp increase in haircut in bilateral repo market during the crisis.
Figure 3: The freeze of the repo market. (from Gorton and Metrick [2012b])
tree bear some consumption goods at its maturity date. The maturity date of a tree is follows an idiosyncratic Poisson process.

Agents are ex ante homogeneous. They are endowed with $a_0 \in \mathbb{R}^{++}$ units of collateral trees and $m_0 \in \mathbb{R}^{++}$ units of consumption good when they enter the economy.

An agent’s expected payoff at time $t$ is $E_t \int_t^T c_u e^{-\rho(u-t)} du$, where $c_u du$ is the measure of apples she consumes between $u$ and $u + du$, $\rho$ is the discount factor and $T$ is the random moment when she leaves the economy.

A collateral tree matures with Poisson rate $\mu \in \mathbb{R}^{++}$. If a collateral tree matures at date $t$, it bears $y \cdot \omega_t$ apples at maturity date $t$, with $y \in \mathbb{R}^{++}$. $\omega_t$ is the aggregate state of the economy at date $t$. It represents the aggregate risk that the quality of collateral asset may deteriorate, or the aggregate collateral risk. There are two aggregate states, $\omega_t \in \{0, 1\}$. When $\omega_t = 1$, every tree bears $y$ apples if it matures at $t$ and when $\omega_t = 0$, every tree bears no apples. I assume the economy is in the good state initially, $\omega_0 = 1$. And the bad state, $\omega = 0$, is assumed to be an absorbing state. The arrival of the bad state follows a Poisson process with rate $\chi \in \mathbb{R}^{++}$. As the likelihood of devaluation shock can be small or large, the model applies to collateral asset of both high quality and low quality.

Another type of trees in the economy represent investment opportunities. An investment op-
portunity is a long-term technology that transforms consumption good at investment date to more consumption good at maturity date. The maturity date arrives with Poisson rate $\pi \in \mathbb{R}_{++}$. The investment is one shot and doesn’t require additional resources. And if an agent does not produce using the long-term technology the moment the investment opportunity arrives, they lose it. With $i$ units of consumption good as input at the investment date, the output at maturity is $\frac{\theta i^\alpha}{\pi} f(i)$, where $f : \mathbb{R}_{++} \to \mathbb{R}_{++}$. In the benchmark model, I assume the output of the technology takes the functional form, $f(i) = \theta i^\alpha$, with productivity parameter $\theta \in \mathbb{R}_{++}$ and $\alpha \in (0, 1)$, so it is concave in input and the marginal output at zero input is infinity, $f''(\cdot) < 0$ and $f''(0) = \infty$. I assume only the agent who invests in a long-term technology has the skill to manage it. The output from the technology if other agents own the project is zero. And other agents cannot take the output from it away from its investor.

The long-term investment opportunity is endowed to agents with delay, which represents the search friction to find a profitable investment opportunity. After entering the economy, every agent may receive at most one investment opportunity at a random date. The arrival date of an agent’s investment opportunity follows an idiosyncratic Poisson process, arrival rate $\lambda \in \mathbb{R}_{++}$.

An agent leaves the market after their asset matures and their long-term project matures. I assume for simplicity that agents lose their chance to find an investment opportunity after their collateral asset matures.

Figure 5 describes a realization of the life cycle of an agent in Autarky. Upon entry, the agent decides her consumption and storage. The reason why she stores some consumption good is that she wants to make investment in the long-term technology. When the investment opportunity arrives, she draws consumption good from storage to invest in the project. After that, she waits for the project to mature. She leaves the market after her asset and project matures.

Since storing consumption is not productive and delays consumption, a financial system can improve allocation and efficiency by allocating consumption good in storage to those agents who need more consumption good for their investment.

The repo market is where agents waiting for their long-term projects can earn interest payment through lending and agents with long-term projects can increase their investment through borrowing against their asset. The repo market is competitive. A repo contract has three components, interest rate $R_t \in \mathbb{R}_+$, haircut $h_t \in [0, \infty)$ and maturity $dt$. According to the contract, a borrower puts
down as collateral $h_t$ units of asset for each consumption good she borrows to the lender at the moment of signing the contract. At maturity date $t + dt$, if no party defaults on the contract, then the borrower pays $1 + R_t dt$ units of consumption good for each unit she borrows at $t$ and the lender delivers $h_t$ units of collateral asset back to the borrower for each unit of borrowing. $^{17}$Given the contract, an agent with asset holding $a$ can borrow up to $\frac{a}{h_t}$.

At the maturity date of a repo contract, the borrower can choose to default or repay the debt. There are two ways to repay the debt. She can repay the debt using her own consumption good that she stored using her endowment or she obtains when her trees mature. Or she can repay the debt by borrowing from other lenders. This is what I refer to as repo rollover. As long as a borrower has not borrowed up to $\frac{a}{h_t}$, she can choose to rollover her debt.

For lenders, as in Acharya and Bisin [2013], I assume the repo market is an opaque over-the-counter market so the repo contract is not conditional on such information as borrowers’ balance sheet. $^{18}$ At any moment, one repo contract clears the whole market. I assume that every lender is assigned to one borrower in the market clearing process. $^{19}$ After the repo market clears, lenders

$^{17}$Default does not incur any loss to the defaulting agent other than the collateral asset in the repo contract. For example, for an agent with a long-term investment, default does not affect her return from the investment because the investment generates only private return for the borrower that no one else can control.

$^{18}$Trading delay due to search friction, as in Duffie et al. [2005], Lagos and Rocheteau [2009], Afonso and Lagos [2012], is also ignored here.

$^{19}$A lender can only lend to one borrower, a borrower can borrow from several lenders.
meet their counter-party. At that moment, lenders can see borrowers’ information and whether they will default or not. Then lenders can decide whether to carry out the repo contract or reject the borrower and wait for next period.

Since a lender is matched with one borrower, the counter-party risk, the risk that a borrower may default, is undiversified. Denote the probability that the counter-party defaults on a repo contract signed at \( t \) to be \( \delta_t dt \). Counter-party default is a shock to the lender’s portfolio. For a lender with \( a \) units of asset, \( s \) units of apple in repo account, she will hold \( a + h_t s \) units of asset but no funding left in repo account when default happens. If default happens with probability 1, the contract is observationally equivalent to a spot transaction in which the borrower sells the asset to the lender, at unit price \( \frac{1}{h_t} \).

To simplify analysis, I impose the following restrictions on agents’ strategies. For a borrower, she is not allowed to borrow or lend in the repo market after she defaults. For a lender, she is not allowed to move additional funding from storage to the repo market after she loses all her funding to the repo market when she meets a defaulting borrower or she herself defaults on a repo contract\(^{20} \).

With these two simplifications, we focus on the borrowing decision of agents who need to finance their long-term investment and lending decisions of those lenders who have not met a defaulting borrower. Additionally, I assume that interest payment from repo lending must be consumed and cannot be accumulated for repo lending or storage.

### 3.1 Discussion

Although if all borrowers default with probability 1, the market is equivalent to a market for trading asset, agents cannot choose to buy or sell the asset and lend or borrow against the asset. What is missing is an additional market for firesale asset. Then lenders do not worry about counter-party default risk and buyers of the asset can optimize her portfolio based on her risk exposure. While the current setup is in line with the observation that the repo market is much more liquid than the market for trading assets, I will study the effect of introducing an additional market in the extension.

We also assume that the default over the collateral does not affect agents’ payoff from her project

\(^{20}\)When losing her funding of amount \( s \) to the repo market, she has \( s(1 + h) \) additional collateral asset. But she can still borrow from the repo market.
investment. Default does not lead to bankruptcy. And collateral delivery is the only requirement if a borrower defaults. The separation of default from bankruptcy allows the agent to buy and sell the collateral asset through repo contracts (and default). And it is consistent with the exemption of repo contracts from “automatic stay” (Garbade [2006]). The separation also allows the model to characterize credit derivative markets. I also assume that default does not incur loss to the borrower other than the collateral she puts down because other investors cannot separate he from her long-term investment. This is consistent with moral hazard frictions lenders may face.

Another property of the repo contract is that dividend from the collateral asset before the maturity of the repo contract belongs to the lender. This is not an issue for overnight repo as no dividend is generated from an asset over night. Likewise, since the maturity of a repo contract is assumed to be infinitesimal in my model, there will not be any dividend payment from the asset before the maturity of the contract.

I assume that lenders face undiversified counterparty risk. Although banks in the real world is large enough to diversify the idiosyncratic counterparty risk in normal times, diversification would be impossible when default is triggered by certain aggregate shock. My assumption would be more useful when we study the dynamic response of the financial system to aggregate shocks. Another reason I make this assumption is that I am modeling the repo market between financial intermediaries. As financial intermediaries borrow and lend in large scales and the total number of financial intermediaries is limited, undiversified counterparty risk is more relevant for the inter-bank repo market.

4 Equilibrium definition

In this section, I give the formal definition of the dynamic equilibrium with a certain initial distribution of agents and initial aggregate state. In the equilibrium definition subsection, I first formalize the individual agent’s problem, after which I define the law of motion of the economy. And in the end I define the equilibrium.
4.1 An agent’s problem

An agent’s problem depends on whether the repo market functions or not, which in turn depends on the aggregate state.

When $\omega_t = 0$, the repo market is not functioning as the supply of valuable collateral asset in this state is 0.

When $\omega_t = 1$, the repo market is functioning. Then an agent’s problem depends on her portfolio, her lending/borrowing history and whether she is still looking for an investment opportunity or she has already invested in a project. In that case, the life cycle of an agent who manages to find a long-term project is illustrated in Figure 6. After entry, the agent allocates her consumption good endowment to consumption, storage and repo lending. Before she finds her long-term technology, she continue making decisions on consumption, storage and repo lending. When she finds her long-term project, she withdraws consumption good from storage and repo lending, borrow from the repo market and invests in the long-term project. Before her long-term investment matures, she does not have consumption good left so she decides whether to rollover her debt or default. If her long-term investment or her asset matures when she is still rolling over her borrowing, she repays her outstanding debt and consumes the remaining consumption and wait for the rest of her trees to mature. After all her asset matures and she consumes all the consumption she owns, she leaves the market.

To solve an agent’s problem when $\omega_t = 1$, I go backward. I first solve her problem after project investment, after which I solve her problem at the long-term investment, after which her problem before the investment, after which her problem at the beginning of her life. Then I explain an investor’s problem when $\omega_t = 0$. Table 1 summarizes the value function and policy functions for an agent’s problem when she faces different situations. Now I will explain an agent’s problem in different situations.

$^{21}$When lending to other agents, the agent may meet a default borrower with Poisson rate $\delta$. If that happens, her lending turns into additional asset holding and she loses her capacity to continue lending.
<table>
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<td>$V_t(a, m, s)$</td>
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<td>$\omega_t = 1$, deactivated before LT investment</td>
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<td>consumption $\tilde{c}_A$</td>
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Table 1: Value functions and policy functions
Figure 6: life cycle of an agent with access to the repo market

**Situation 1: \( \omega_t = 1 \), agent’s problem after LT investment**

\[
W_t(a, m, s, i) = \max_{c, m', s'} c + EW_t(a, m', s', i),
\]

\[
\text{s.t. } \quad c + m' + s' \leq s + m,
\]

where the first constraint is agent’s budget constraint, the second one is the collateral constraint which imposes that borrowing cannot exceed \( \frac{a^s}{1+n} \) and the last two constraints are non-negativity constraints for consumption and storage allocation. Agents choose consumption \( c \), storage \( m' \) and repo lending \( s' \) to maximize payoff from consumption and continuation value \( EW_t(a, m', s', i) \), which
depends on random events between $t$ and $t + dt$.

$$EW_t(a, m, s, i) = \mu dt (a + m + s + f(i)) + \pi dt \left[ \frac{\mu}{\rho + \mu + \chi} a + m + s + \frac{\rho + \pi}{\pi} f(i) \right] + \chi dt [m + f(i)]$$

$$+ \delta_t dt \mathbb{I}_{s>0} e^{-\rho dt} \max_{z \in [0,1]} [zW_{t+dt}(a + h_t s, m, 0, i) + (1 - z)W_{t+dt}(a, m + s, 0, i)]$$

$$+ \left[1 - (\mu + \pi + \chi + \delta_t dt \mathbb{I}_{s>0}) dt\right] e^{-\rho dt} \max_{d \in [0,1]} \begin{pmatrix}
\frac{\mu y(a + h_t s)}{\rho + \mu + \chi} + m + f(i) \\
+ (1 - d) (R_t dt \mathbb{I}_{s>0} W_{t+dt}(a, m, s, i)) \mathbb{I}_{s>0} \\
+ (1 - d) W_{t+dt}(a, m, (1 + R_t dt)s, i) \mathbb{I}_{s \leq 0}
\end{pmatrix}.$$  

The continuation value depends on several random events. With probability $\mu dt$, the agent’s asset matures. In this case, she repays $-s$ apples to her lenders, if $s < 0$, or withdraws $s$ apples from inter-bank lending, consumes $a + m + s$ apples and waits for her long-term project to mature, which delivers expected payoff $f(i)$. With probability $\pi dt$, the agent’s long-term investment matures. In this case, she repays her debt or draws lending from the repo market, consumes $m + s + \frac{\rho + \pi}{\pi} f(i)$ apples and waits for her collateral asset holding to mature, which delivers expected payoff $\frac{\mu}{\rho + \mu + \chi} a$. With probability $\chi dt$, the aggregate devaluation shock hits the economy and the collateral asset becomes worthless. So if $s < 0$, she default on her debt, consumes $m$ apples she withdraw from her storage account and waits from her long-term investment to mature. If $s > 0$, on seeing the aggregate shock hitting the economy, she withdraws her apples from the repo market and consumes the $s$ apples immediately. The next term is the expected payoff from the event that counter-party defaults. This term only shows up if the agent is lending in the repo market. So if $s > 0$, then with probability $\delta_t dt$, the agent meets a defaulting borrower. In this case, her payoff depends on whether she is willing to lend to the borrower after knowing that the borrowing is going to default. If she decides to lend to the defaulting borrower, her lending $s$ will turn into $h_t s$ units of additional asset holding when the repo contract matures in $dt$ period. So her continuation value with this choice is $e^{-\rho dt} W(a + h_t s, m, 0, i)$. If she decides not to lend to the defaulting borrower, she makes an additional asset holding when the repo contract matures in $dt$ period. So her continuation value with this choice is $e^{-\rho dt} W(a, m + s, 0, i)$. With the residual probability, $[1 - (\mu + \pi + \chi + \delta_t dt \mathbb{I}_{s>0}) dt]$, the agent meets a non-defaulting borrower and she decides whether to default or not. If she defaults, her continuation value is $e^{-\rho dt} \left[ \frac{\mu y(a + h_t s)}{\rho + \mu + \chi} + m + f(i) \right]$, where the first component is the expected payoff from asset holding, $m$ is payoff from consumption of apples withdrawn from the storage
account and \( f(i) \) is the expected payoff from long-term investment. If she does not default, she will get \( R_t \) apples as interest payment so her continuation value is \( e^{-\rho dt} W(a, m, (1 + R_t dt)s, i) \). Refraining from lending is ruled out here as long as interest rate \( R_t \) is positive. If \( s < 0 \), she needs to borrow from the repo market. Similarly, \( e^{-\rho dt} \left\{ \frac{\nu y[a+h_t s]}{\rho + \mu + \chi} + m + f(i) \right\} \) is her payoff from default and \( e^{-\rho dt} W(a, m, (1 + R_t dt)s, i) \) is her payoff from rolling over her debt.

Debt rollover is an option only if \( (1 + R_t dt)s \geq -\frac{a}{h_t} \). So if the state variable \( s = -\frac{a}{h_t} \), the agent’s borrowing constraint is already binding, then she can’t but default. So

\[
EW_t \left( a, m, -\frac{a}{h_t}, i \right) = m + f(i) \tag{3}
\]
which is equal to \( m + f(i) \) plus some term of the same order of magnitude as the infinitesimal period \( dt \).

**Situation 2: \( \omega_t = 1 \), agent’s problem at long-term investment**

\[
U_t(a, m, s) = \max_{c,m',s',i} c + W_t(a, m', s', i) \tag{4}
\]
\[
c + s' + m' + i \leq s + m,
\]
\[
s' \geq -\frac{a}{m'},
\]
\[
c, m', i \geq 0,
\]
where agents choose consumption \( c \), storage \( m' \), lending or borrowing through the repo market, \( s' \), and investment in the long-term project, \( i \), to maximize her payoff from consumption and continuation value \( W_t(a, m', s', i) \). The constraints the agent faces are resource constraint, borrowing constraint and non-negativity constraints of the choice variables, \( c, m' \) and \( i \).

**Situation 3: \( \omega_t = 1 \), agent’s problem before long-term investment**

\[
V_t(a, m, s) = \max_{c,s',m',z,d} c + EV_t(a, m', s'), \tag{5}
\]
\[
c + s' + m' \leq s + m,
\]
\[
s' \geq -\frac{a}{m'},
\]
\[
c, m' \geq 0,
\]
which is similar to agents’ problem after project investment, equation 1.

\[
EV_t(a, m, s) = \mu dt (a + m + s) + \lambda dt U_t(a, m, s) + \chi dt V_t^A(m)
\]

\[
+ \delta t \mathbb{I}_{s>0} dt e^{-\rho dt} \max_{z \in [0,1]} \left\{ z V_t^d(a + h_t s, m) + (1 - z) V_{t + dt}(a, m, s) \right\}
\]

\[
+ [1 - (\mu + \lambda + \chi + \delta t \mathbb{I}_{s>0}) dt] \max_{d \in [0,1]} e^{-\rho dt} \left\{\left( dV_{t + dt}(a + h_t s, m, 0) \right) + (1 - d) \left( R_t dt s + V_{t + dt}(a, m, s) \right) \mathbb{I}_{s>0} \right\}
\]

The continuation value has a similar expression to the continuation value 2. The additional random event is the event of finding a long-term project. With probability \(\lambda dt\), the agent finds a long-term project in \(dt\) period. The continuation value contingent on the event is \(U_t(a, m, s)\). With probability \(\mu dt\), the agent’s asset matures.\(^{22}\) And if a lender lends to a defaulting borrower, she will be deactivated, with continuation value, \(V^d_{t + dt}(a + h_t s, m)\).

If \(s = -\frac{a}{h_t}\), the agent’s borrowing constraint is already binding, then she can’t but default. So

\[
EV_t \left( a, m, -\frac{a}{h_t} \right) = V_t^A(m)
\]

After default, the agent has no asset left. So her continuation value is the same as what she would have under Autarky with storage \(m\).

The problem of an agent deactivated because of lending to a defaulting borrower is

\[
V^d_t(a, m) = \max_{c, m'} c + EV^d_t(a, m'),
\]

s.t.
\[
c + m' \leq m,
\]
\[
c, m' \geq 0,
\]

where the continuation value is,

\[
EV^d_t(a, m) = \mu dt (a + m) + \lambda dt U_t(a, m, 0) + \chi dt V_t^A(m) + [1 - (\mu + \lambda + \chi) dt] e^{-\rho dt} V^d_t(a, m).
\]

\(^{22}\)The collateral asset of an agent is assumed to mature at the same time. This is not exactly consistent with the assumption the maturity of collateral asset is idiosyncratic across agents because the asset an agent receive from other agents is assumed to mature at the same time her own asset.
The agent finds her long-term project with probability $\lambda dt$ and her continuation value is $U_t(a, m, 0)$.

**Situation 4: $\omega_t = 0$**

The value function, $V_A(m)$ and policy functions, $\tilde{c}_A(m)$, $\tilde{m}_A(m)$, solve the problem of an agent’s problem before she finds her long-term projects.

$$V_A(m) = \max_{c,m'} c + EV_A(m'),$$  \hspace{1cm} (10)

subject to

$$c + m' \leq m,$$

$$c, m' \geq 0,$$

where the agents choose consumption $c$ and storage for the next period $m'$ to maximize her payoff from consumption and continuation value $E\tilde{V}_A(m')$, subject to resource constraint and non-negativity constraints on consumption and storage. The continuation value depends on the random events that may happen during $dt$ period.

$$EV_A(m) = \lambda dt \max_{0 \leq i \leq m} \{m - i + f(i)\} + \mu dt m + [1 - (\lambda + \mu) dt] e^{-\rho dt} V^A(m).$$

With probability $\lambda dt$, the agent finds her long-term investment in that case chooses optimally investment in the long-term project and consumption so as to maximize her expected payoff $c + f(i)$, where $c = m - i$, subject to non-negativity constraints of consumption and investment. With probability $\mu dt$, her collateral asset matures, bears no apple at maturity and she loses her chance to find a long-term investment opportunity in the future. In that case, she consumes away the apples in the storage account and leaves the market. With the residual probability $1 - (\lambda + \mu) dt$, nothing happened during the $dt$ period so her continuation value is $e^{-\rho dt} V^A(m)$.

**Laws of motion for the distribution of agents**

The exact law of motion is left to the appendix. Here, we illustrate the laws of motion on the equilibrium path where $\omega_t = 1, \forall t$ using Figure 7. All new entrants enter as active lenders and are counted in the distribution $F_0t$. A lender leaves the pool of active lenders in three situations. If her collateral asset matures before she finds her long-term project, she leaves the economy after consuming all her consumption good. If she meets a defaulting borrower, she enters the pool of
Figure 7: Law of motion of an agent’s state before the realization of collateral shock \( \omega_t = 1 \).

Deactivated lenders with distribution \( F_{dt} \). If she finds her investment opportunity, she enters the pool of borrowers with distribution \( F_{1t} \). For an agent in the pool of deactivated lenders, she leaves the economy if her collateral asset matures before she finds a project, or she enters the pool of borrowers when she finds her long-term project. For an agent in the pool of borrowers, they exit the economy after both her project and collateral asset mature. They will stop borrowing when their collateral asset or projects mature, or when they default on their loan.

**Equilibrium definition**

**Definition 4.1.** An equilibrium with initial distribution \( F_{10}, F_{d0} \) and \( F_{00} \) and initial state \( \omega_0 = 1 \), is repo contract term \( \{R_t, h_t\}_{t \geq 0} \), default rate, \( \{\delta_t\}_{t \geq 0} \), agents’ policy functions and value functions and aggregate law of motion such that,

(i) given \( R, h \) and \( \lambda \), agents’ policy functions and value function solve their problems if \( \omega_t = 1 \).

(ii) agents policy functions and value functions solve their problem if \( \omega_t = 0 \).

(iii) the contract \( (R, h) \) clears the repo market given the distribution of agents and agents’ decision function,

\[
\int \tilde{s}_{1t}(a, m, s, i) dF_{1t}(a, m, s, i) + \int \tilde{s}_{0t}(a, m, s) dF_{0t}(a, m, s) = 0, \text{ if } \omega_t = 1, \forall t.
\]

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(iv) the distribution of agents is endogenously determined by laws of motion.

(v) agents’ expectation on default rate $\delta_t$ is consistent with the actual default rate,

$$\delta_t dt = \frac{\int_{\{(a,m,s):s\leq 0\}} d_{1t}(a,m,s,i)sdF_{1t}(a,m,s) + \int_{\{(a,m,s):s\leq 0\}} d_{0t}(a,m,s)sdF_{0t}(a,m,s)}{\int_{\{(a,m,s):s\leq 0\}} sdF_{1t}(a,m,s) + \int_{\{(a,m,s):s\leq 0\}} sdF_{0t}(a,m,s)}, \text{ for } \omega_t = 1.$$ 

$\delta_t dt$ is the probability that a lender’s counterparty may default on their borrowing. Since the lender is ignorant of the borrower’s portfolio, the lender’s counterparty can be thought of as a random draw from the pool of borrowers, weighted by borrowers’ funding demand. Suppose an agent borrows to invest in a long-term project and she also has invested all her apples in storage and in interbank lending. So when the her initial borrowing matures in $dt$ period, the cumulative probability that her project or collateral asset matures is $(\pi + \mu) dt$, which is negligible. She has two choices, to rollover her debt or to default. If every borrower chooses to default immediately after initial borrowing, $\delta_t dt = 1$. $\delta_t dt < 1$ when some agents choose to rollover their debt in equilibrium. In the next two subsections, we will characterize the properties of equilibrium with debt rollover and equilibrium with only defaulting repo contracts.

**Definition 4.2.** A stationary equilibrium with initial state $\omega_0 = 1$ is an equilibrium with initial distributions $F_{10}$, $F_{d0}$ and $F_{00}$ and initial state $\omega_0 = 1$, such that $F_{0t}(a,m,s) = F_{00}(a,m,s)$, $F_{d0}(a,m) = F_{d0}(a,m)$, $F_{10}(a,m,s,i) = F_{10}(a,m,s,i)$, $\forall t$.

5 Efficiency

To understand the welfare loss from the market incompleteness, we study properties of the efficient allocation in this section.

The social planner’s choice is aggregate consumption flow, $C_\tau$, for each newly born agent at moment $\tau$, aggregate storage, $m_\tau$, investment in long-term tech, $i_\tau$, for each agent with an opportunity to invest in the long-term technology.

The social planner’s problem at moment $t$ is
In the objective function of the planner, $\eta dt$ is the measure of agents born during $\tau$ and $\tau + dt$ and $e^{-\rho(\tau-t)}$ is the discount factor for future welfare gain. The planner chooses allocation, $\{C_\tau, m_\tau, i_\tau\}_{\forall \tau \geq t}$, to maximize the objective function, subject to nonnegative constraints of the choice variables and the resource constraint. The right hand side of the resource constraint includes apples from maturing long-term projects from past investment, maturing collateral from agents born before $\tau$, production from short-term technology and storage technology. $\frac{\rho + \pi}{\mu + \lambda} i_\tau$ is the measure of projects found between $\tau$ and $\tau + dt$. And $\eta ds$ is the measure of agents born between moment $s$ and $s + ds$. Similarly, $\lambda \frac{n}{\mu + \lambda} ds$ is the measure of projects created between moment $s$ and $s + ds$. Similarly, $\mu \omega_{\tau}$ is the measure of projects found between $\tau$ and $\tau + dt$.

**Proposition 5.1.** The efficient allocation $\{C_\tau, m_\tau, i_\tau\}_{\forall \tau \geq t}$ is

\[
  i_\tau = i^*, \text{ such that } \theta \alpha i^{*\alpha-1} = 1,
\]

\[
  m_\tau = 0.
\]

And the efficiency of the economy is characterized solely by investment allocation $i^*$

\[
  C_\tau = \left[ \frac{\rho + \pi}{\mu} i^{*\alpha} - i^* \right] \frac{\lambda \eta}{\mu + \lambda} + \lambda \omega_{\tau} \eta.
\]

As a benchmark for comparison, in Autarkic allocation, where the repo market is shut down, individual investment, $i$, is constrained to be equal to the amount of apples agents are endowed with and store using the storage technology until they find the project. If the initial endowment is small
enough so that the marginal return from storing all endowment is greater than 1, \( \frac{\lambda \delta \alpha m_{0}^{-1} + \mu a}{\rho + \mu + \lambda} > 1 \), the expected return from investment is \( \frac{\lambda \delta \alpha m_{0}^{-1} + \mu a}{\rho + \mu + \lambda} \). Otherwise, the storage is such that the expected marginal return from storage is equal to 1, \( \frac{\lambda \delta \alpha m_{0}^{-1} + \mu a}{\rho + \mu + \lambda} = 1 \). So the marginal return from investment in the long-term technology is greater than \( \frac{\rho + \lambda}{\lambda} \). Therefore, there exists a wedge between marginal return of project and the marginal utility of consumption in Autarkic allocation, which is greater than \( \frac{\rho + \lambda}{\lambda} \). The lower bound of the wedge increases with the search friction to find the long-term project.

In the efficient allocation, in contrast, the wedge between marginal return of project investment and the marginal utility of consumption is 0. The social planner, not subject to enforceability constraints, can allocate consumption goods from maturity projects to new investment. So the optimal investment allocation does not depend on the aggregate risk on the value of collateral. The efficient investment in the long-term project tree does not depend on the value of collateral asset. As we will see, the efficiency gain from using repo contracts in this environment comes from the transfer of output from maturity projects to investment in new projects.

Another difference between Autarkic allocation and the first best allocation is the allocation to storage, which can be interpreted as liquidity hoarding. While storage of each agent in Autarky is equal to investment in their long-term technology, the aggregate storage is 0 in efficient allocation. In other words, it is socially wasteful for agents to hoard liquidity when the social planner is not constrained by market incompleteness. The wedge and inefficiency in Autarkic economy also comes from the fact that the return on storage technology is low so it is costly for agents to use the storage technology. If the return on storage technology is equal to the discount factor, the lower bound of the wedge in Autarky would zero. This is the case, for example, if the storage technology is a fiat currency and the monetary authority is following Friedman rule. 23

6 Equilibrium characterization

With collateral risk and the state-contingent repo contract, default does not happen in equilibrium as long as not all agents default. Then we add maturity mismatch to the analysis. In the full equilibrium, I will show that maturity mismatch results in repo rollover and equilibrium default.

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23 See Lagos and Wright [2005], Berentsen et al. [2007], Williamson and Wright [2010].
I will also show that default triggers more default in the equilibrium because of the undiversified counterparty risk.

### 6.1 Characterization of an equilibrium with repo rollover

With short-term repo contract not contingent on liquidity arrival, agents borrow from the repo market when they find their long-term investment opportunity and then they rollover their debt to wait for their long-term investment or collateral asset to mature. But as long as the interest rate $R$ is positive, they cannot rollover their debt infinitely. At certain point they will reach their debt limit, $\frac{a}{h}$. The dynamics of debt holding in the equilibrium with debt rollover is illustrated in Figure 8(a). Liquidity arrives when the borrower’s collateral asset or project matures, which is a random date that could be earlier or later than the moment of reaching debt limit. The borrower will keep rolling over her debt until she repay her debt using consumption good from her trees at maturity or she defaults when she reaches her debt limit.

In this equilibrium, there is a trade-off between default and investment in the long-term technology. This trade-off is illustrated in Figure 8(b). As initial borrowing $b'$ increases, the moment of reaching debt limit moves to the left of the time line. So the probability that she reaches her debt limit before she receives liquidity from maturing trees decreases. If the agent takes on more initial debt, she gains more from long-term investment but she finds it less likely to receive apples from long-term investment or collateral asset before she defaults when she reaches her debt limit.
The dynamics of other choice variables in the equilibrium with stationary distribution are as follows. Before an agent finds her long-term investment, she keeps lending a constant amount of apples to other agents through repo market and store a constant amount of apples in storage technology. She consumes all interest payment from lending. If she finds a long-term technology, she invests all her apples in storage and repo lending and from her initial borrowing to the long-term technology and stops consuming until the project or her asset matures. If she meets a defaulting agent before her investment, all her lending turns into asset and her consumption drops to zero.

**Necessary conditions for an equilibrium with debt rollover**

**Lemma 6.1.** A necessary and sufficient condition for a borrower with debt holding $b$ to rollover her debt until reaching her debt limit is $\frac{\mu}{\rho + \mu + \chi} y h \geq \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R}$ and $R < \rho + \chi + \mu + \pi$.

Through debt rollover, a borrower can avoid losing collateral asset to lenders. The benefit of avoiding default is higher when haircut is high enough. Lemma 6.1 gives the necessary and sufficient condition for borrowers to rollover their debt holding.

For equilibrium rollover to take place, a lender must be willing to lend to defaulting borrowers and she does not default on repo lending to non-defaulting borrowers. This means that when she meets a defaulting borrower, her continuation value from lending to a defaulting borrower, $e^{-\rho dt} V^d(a + h s, m)$, must be no less than her continuation value from waiting for next lending opportunity, $e^{-\rho dt} V(a, m, s)$. And when she meets a non-defaulting borrower, her continuation value from waiting for debt repayment, $e^{-\rho dt} V(a, m, s(1 + Rd t))$, must be no less than her continuation value from defaulting $e^{-\rho dt} V^d(a + h s, m)$.

**Lemma 6.2.** A sufficient and necessary condition for lenders’ strategy in the equilibrium with debt rollover to be optimal is: $V(a, m, s) = V^d(a + h s, m)$.

Lemma 6.2, Lemma 6.1 and the conditions for initial borrowing to be strictly between 0 and the debt limit in Lemma 6.4 are conditions in additional to other conditions in the equilibrium definition.

Lemma 6.2 implies that

$$\mu (y h - 1) s = s R + \lambda [U(a, m, s) - U(a + h s, m, 0)]$$
which shows that haircut compensates three losses from lending to a defaulting borrower instead of a non-defaulting borrower: the loss of interest payment, the loss of funding for long-term investment when the devaluation shock hits the economy and the difference in the continuation value when she invests in long-term technology. The second component reflects the heterogeneity in preferences of lenders waiting for their long-term investment and borrowers already with a long-term project. When the arrival rate of the aggregate shock increases, the heterogeneity increases and haircut may be more likely to satisfy the condition for borrowers to rollover their debt in Lemma 6.1.

Equilibrium characterization

**Lemma 6.3.** If the repo contract satisfies the condition in Lemma 6.1, the value function \( W(a, 0, -b, i) \) for \( b \in \mathbb{R}_{++} \) in the equilibrium with debt rollover is characterized by the following differential equation

\[
\rho W(a, 0, -b, i) = \pi \left[ \frac{\rho + \pi}{\pi} f(i) + \frac{\mu}{\rho + \mu + \chi} ya - b - W(a, 0, -b, i) \right]
+ \chi [f(i) - W(a, 0, -b, i)] + \mu [f(i) + ya - b - W(a, 0, -b, i)]
+ \frac{\partial W(a, 0, -b, i)}{\partial b} b R
\]

and boundary condition

\[
W(a, 0, -\frac{a}{h}, i) = f(i).
\]

The solution to the differential equation is

\[
W(a, 0, -b, i) = \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} \left( \frac{1}{\tilde{R} h} - \frac{y a}{\rho + \mu + \chi} \right) \left( \frac{hb}{a} \right)^{(\rho + \chi + \mu + \pi)/\tilde{R}}
- \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} b + f(i) + \frac{\mu}{\rho + \mu + \chi} ya
\]

The initial borrowing at the moment the agent receives an investment opportunity is pinned down by problem (4). The trade-off between losing asset through default and more investment in the long-term project leads to the following result about initial borrowing.

**Lemma 6.4.** If \( h > \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} \frac{\rho + \mu + \chi}{\mu y} \), \( f'(s + m) > \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} \), and \( f'(s + m + \frac{a}{h}) < \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} y h - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} \), an investor’s initial borrowing, \( b \), is solved by

\[
\frac{\rho + \chi + \mu + \pi}{\tilde{R}} \left( \frac{hb}{a} \right)^{(\rho + \chi + \mu + \pi - \tilde{R})/\tilde{R}} = \frac{f'(s + m + b) - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} y h - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}}}{\frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}} y h - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - \tilde{R}}}.
\]
**Initial borrowing** is defined as $b = 0$ if $f(s + m) \leq \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R}$ and $b = \frac{a}{1 + h}$ if $f'(s + m + \frac{a}{h}) \geq \frac{\mu}{R} \frac{\rho + \chi + \mu + \pi}{\rho + \mu + \chi} y h - \frac{\mu + \pi}{R}$.

The first condition for **Lemma 6.4** is the sufficient and necessary condition for debt rollover stated in **Lemma 6.1**. If the marginal return from long-term investment when an agent takes on no initial borrowing is lower than $\frac{\mu + \pi}{R} \frac{\rho + \chi + \mu + \pi}{\rho + \mu + \chi} y h - \frac{\mu + \pi}{R}$, she will find it not profitable to take on any initial borrowing. If the marginal return from long-term investment when she takes on her initial borrowing up to the debt limit is higher than $\frac{\mu}{R} \frac{\rho + \chi + \mu + \pi}{\rho + \mu + \chi} y h - \frac{\mu + \pi}{R}$, she will borrow up to her debt limit.

Given the initial borrowing, the duration between an agent’s initial borrowing and reaching debt limit, $T(a, b)$, depends on her initial borrowing and asset holding. It is solved by equation $\frac{a}{R} = b e^{RT(a, b)}$. So $T(a, b) = \frac{1}{R} \ln \left( \frac{a}{R b} \right)$. Given $T(a, b)$, the probability that an agent is able to pay back the debt is

$$\int_0^{T(a,b)} (\mu + \pi) e^{-(\chi + \mu + \pi)t} dt = \frac{\mu + \pi}{\chi + \mu + \pi} \left[ 1 - e^{-(\chi + \mu + \pi)T(a,b)} \right] = \frac{\mu + \pi}{\chi + \mu + \pi} \left[ 1 - \left( \frac{hb}{a} \right)^{\frac{\chi + \mu + \pi}{R}} \right].$$

So the probability of default is $\frac{\chi}{\chi + \mu + \pi} + \frac{\mu + \pi}{\chi + \mu + \pi} \left( \frac{hb}{a} \right)^{\frac{\chi + \mu + \pi}{R}}$, which is increasing in the ratio between initial borrowing and debt limit, $\hat{b} = \frac{hb}{a}$. From equation (14), $\hat{b}$ is increasing in the marginal return from project investment, $f'(s + m + b)$ and decreasing in haircut and dividend from an asset. This reflects the tradeoff between return from project investment and losing asset through default. So the default probability of an agent is increasing the productivity of the long-term technology and decreasing in haircut and the dividend of an asset.

The default probability of an agent after initial borrowing also depends on the liquidity of an agent’s portfolio. For an agent with portfolio $(a + hs', m, s - s')$, the following corollary shows that the default probability is increasing in $s'$.

**Corollary 6.1.** Default triggers more default. Counter-party default increases default probability of lenders in the future. For an agent with portfolio $(a + hs', m, s - s')$,

$$\frac{\partial b}{\partial s'} > 0$$

Corollary (6.1) is derived from **Lemma (6.4)**. The portfolio of an agent with portfolio $(a, m, s)$ turns into $(a + hs, m, 0)$ after she meets a defaulting agent. So according to the corollary, an agent whose counterparty defaults before she starts her long-term investment is more likely to default than...
an agent who has not met a default counterparty when she starts her long-term investment. This implies that counterparty default has an externality on other agent’s default probability. Figure (9) illustrates the externality. Agent $i$ borrows from agent $j$ and deliver her asset holding to agent $j$ upon default. This increases agent $j$’s default probability when she borrow from agent $k$ among other agents to invest in her long-term project. So agent $j$ is more likely to deliver her asset to agent $k$ upon default. As we can see from the inter-temporal chain of reactions, the default of one agent increases the default probability of those agents who lend to her, which may affect the default probability of those who lend to agent $j$’s. This intertemporal chain reminds us of Kiyotaki and Moore [1997b], which study the propagation of shocks through credit chains. In my model, shocks to agents’ portfolio, making their portfolio less liquid, are contagious. The increase in default risk probability passes on from a defaulting borrower to her lenders, or to say, borrowers-to-be. The risk taking of an individual borrower therefore adds to the risk of the whole system. I will study the contagion more in the section on dynamics.

**Proposition 6.1.** A stationary equilibrium with debt rollover must satisfy conditions in Lemma
6.1 and Lemma 6.2. The equilibrium can be summarized by interest rate \( R \), haircut \( h \), default rate \( \delta \), the portfolio choice of active lenders, \((m, s)\), the initial borrowing of active lenders, \( b_1 \) and the initial borrowing of deactivated lenders, \( b_0 \), the following system of equations,

\[
\frac{(hb_1)}{(a_0 + hs)} = \frac{R}{\rho + \chi + \mu + \pi} \frac{f'(m_0 + b_1) - \frac{\mu + \pi}{\rho + \mu + \pi} \mu + \psi}{\rho + \chi + \mu + \pi},
\]

\[
\frac{(hb_0)}{(a_0 + hs)} = \frac{R}{\rho + \chi + \mu + \pi} \frac{f'(m + b_0) - \frac{\mu + \pi}{\rho + \mu + \pi} \mu + \psi}{\rho + \chi + \mu + \pi},
\]

\[
\mu (yh - 1) s = sR + \lambda \left[U(a, m, s) - U(a + hs, m, 0)\right],
\]

\[
\frac{1}{\mu + \lambda + \delta} s = B,
\]

\[
(m, s) \in \arg \max_{m + s \leq m_0, m \geq 0, s \geq 0} V(a, m, s).
\]

where \( B \) denotes total borrowing, \( B^d \) denotes the borrowing from defaulting borrowers.

\[
B = \frac{\lambda}{\mu + \lambda + \delta} b_1 \int_0^{T_1} e^{(R - \mu - \pi)s} ds + \frac{\delta}{\mu + \lambda} \frac{\lambda}{\mu + \lambda + \delta} b_0 \int_0^{T_0} e^{(R - \mu - \pi)s} ds,
\]

and

\[
B^d = \frac{\lambda}{\mu + \lambda + \delta} b_1 e^{(R - \mu - \pi)T_1} + \frac{\delta}{\mu + \lambda} \frac{\lambda}{\mu + \lambda + \delta} b_0 e^{(R - \mu - \pi)T_0},
\]

where \( T_0 = \frac{1}{\mu} \ln \left( \frac{hb_0}{a_0 + hs} \right) \) and \( T_1 = \frac{1}{\mu} \ln \left( \frac{hb_1}{a_0} \right) \).

Equation (15) and (16) are the optimality condition for the initial borrowing of active and deactivated lenders when they find their long-term projects. Equation (17) is the condition for lenders to be willing to rollover the debt. Equation (18) pins down the equilibrium default rate, which is the ratio of the demand of funding from defaulting borrowers and the total demand of borrowers. Equation (19) is the market clearing condition. Equation (20) pins down the portfolio choice of active lenders.

### 6.2 The effect of collateral risk

As the equilibrium of the full model depends on two key features, collateral risk and maturity mismatch, let us take one step back before looking at the comparative statics of the full model. In this subsection, we study the effect of collateral risk on liquidity hoarding and cash in the repo
market by studying a state contingent debt contract with an exogenous borrowing constraint. By shutting down maturity mismatch, we can focus on the effect of collateral risk on the equilibrium outcome. In the next subsection, we will go one step further to characterize the whole equilibrium with maturity mismatch because of using short-term repo contracts.

6.2.1 Equilibrium with state contingent repo contracts

The repayment of the repo contract I consider in subsection is contingent on the arrival of liquidity when the borrower’s collateral asset or long-term project matures.\(^{24}\) With state contingent repo contracts, there are two symmetric equilibria, an equilibrium where no agents will default when they start borrowing, and an equilibrium where all agents default immediately when they start borrowing.

**Proposition 6.2.** Suppose \( \min \left\{ f'(m_0 + a_0), \frac{\mu y f}{\rho + \chi + \mu} h \right\} > \frac{(\mu + \pi)(1 + R)}{\rho + \chi + \mu + \pi} \), then there exists a unique stationary equilibrium with the state-contingent repo contract characterized by the following system of equations,

\[
\begin{align*}
R &= \frac{X}{\mu + \pi} \frac{\lambda}{\rho + \lambda} f'(m), \\
\frac{m_0 - m}{\mu + \lambda} &= \frac{\lambda}{\mu + \lambda} \frac{a_0}{(1 + R)h} \mu + \frac{1}{\mu + \pi},
\end{align*}
\]

Proof of the proposition is left to the appendix. The assumption in the proposition makes sure that an agent who borrow to invest in her long-term project would borrow against all her collateral asset because the return from project investment is high enough. According to proposition 6.2, interest rate increases in equilibrium liquidity hoarding of agents waiting for their own investment opportunities, taking parameter values as given. In the next proposition, we do some comparative statics.

**Proposition 6.3.** Under the assumption in Proposition 6.2,

1. \( \frac{\partial m}{\partial \pi} > 0, \frac{\partial R}{\partial \pi} > 0, \frac{\partial \pi}{\partial \pi} > 0; \)

2. \( \frac{\partial m}{\partial \pi} > 0, \frac{\partial R}{\partial \pi} < 0, \frac{\partial \pi}{\partial \pi} > 0. \)

\(^{24}\)Without loss of generality, I assume that the liquidity at the maturity of either the project or the collateral asset is enough to repay all the debt.
Proposition 6.3 gives the comparative statics with respect to collateral risk and the maturity of the project. When collateral risk increases, agents waiting for their investment opportunity refrain from inter-bank lending and increases storage, as consumption good in the storage would be the only resource available when the economy is in the bad state. As a result, funding for the repo market dries up. So the interest rate of the repo contract increase and investment in long-term investment drops. When the maturity of the project decreases, liquidity from maturing projects flow more frequently to lenders and subsequently to project investment and liquidity hoarding. With more funding available to lenders, the tension between liquidity hoarding and project investment is tempered. Consequently, both liquidity hoarding and investment increases and interest rate drops.

6.3 Market liquidity, solvency and balance sheet of financial intermediaries

In this subsection, I examine the comparative statics of the rollover equilibrium keeping fixed expected payoff from dividend payment from collateral. This way, we can see clearer the effect of collateral risk on the rollover equilibrium.

Figure 10 illustrates how collateral risk affects liquidity of the repo market and solvency of financial intermediaries. Figure 11 illustrates how collateral risk affects the aggregated balance sheet of financial intermediaries. When the arrival rate of devaluation shock increases, banks hoard more liquidity in their storage account and refrain from lending to other banks. This increases interest rate and haircut in the market. These are consistent with the analytical results we derive in an environment without maturity mismatch. With maturity mismatch between investment and liability, borrowers are more likely to default, facing the higher haircut and interest rate. The increase in default rate would further increases the haircut and interest rate.

The disfunctioning of the repo market is reflected in the aggregate long-term investment, as collateral risk, the aggregated long-term investment and therefor the aggregate output of the economy drops. And as we can see from Figure 11, repo lending contributes less to long-term investment, more investment uses funding from storage.

In figure 11, the increase in storage, which represents liquidity hoarding, and decrease in repo lending as collateral risk increases is not completely substitutional. As collateral risk increases,

\[ y_{\mu+\mu+\chi} = \frac{y_{\mu+\mu+\chi}}{\mu+\mu+\chi}. \]

Parameter values used in the comparative statics and transition dynamics is listed in Table 2 in the appendix.
default is more likely, rollover is harder and creation of private IOU to share return from project investment becomes more difficult. As a result, the sum of repo lending and storage, which is what we call short-term asset, decreases. Notice that with the state-contingent repo contract, the sum of repo lending and reserve is constant. The decrease in reflects the increase in default rate. When debt rollover is harder, the value of repo lending drops and illiquid collateral asset held by agents waiting for their investment opportunities increases.

While comparative statics in the model resemble a Kiyotaki-Moore type of model assuming exogenous borrowing constraint, the constraint in my model is endogenous and depends on exogenous collateral risk and the contract structure. For example, the haircut for the over-night repo contract would be higher than that for the state-contingent repo contract is different. The endogenous haircut is especially relevant when we study equilibrium dynamics, where the borrowing constraint implied from haircut will move endogenously along the dynamic path. We now turn to characterize the transition dynamics triggered by an unexpected shock to the riskiness of collateral asset.
Figure 11: Credit risk and the aggregated balance sheet of the securitized banking system.
7 Equilibrium dynamics

In this section, I characterize the transition dynamics when agents’ expectations about the collateral risk shifts towards a more pessimistic one. The exercise is intended to capture the transition dynamics triggered by a shock similar to what we experience during the great recession. More importantly, it will help us understand the stability of the shadow banking system to shocks to collateral risk.

I assume the economy before $t=0$ is the steady state equilibrium with $\omega_t = 1$ with the parameter values given in Table 2. At $t = 0$, $\chi$ switches from 0.01 to 0.02. The algorithm to compute the transition dynamics is in the appendix.

Figure 12 shows the transition dynamics for the liquidity of the repo market and solvency of financial intermediaries. Figure 13 illustrates the transition dynamics for the aggregated balance sheet of financial intermediaries in the securitized banking system. And figure 14 illustrates the transition dynamics of output and investment, the impact of the financial crisis on the real economy.\(^{27}\)

The stability can be measure by two dimensions: initial response and persistence of the impact.

**Initial response of the shadow banking system**

Let’s first look at the initial response of the economy to the shift in expectation. At the moment the expectation shifts, haircut increases discontinuously and overshoots to a level even higher than the haircut in the new steady state. This means that the debt limit borrowers face given their collateral asset holding drops discontinuously. As a result, borrowers with debt holding above the new debt limit but below the old debt limit are forced to default at $t$. The mass of default drains up the funding available in the repo market. So the aggregate amount of repo lending drops and the massive initial default shows up in the discontinuous drop in repo lending in Figure 13. The overshooting of haircut increases the probability of the initial default. The figure shows that about 25% of outstanding repo borrowing defaults at the moment the shock hits the economy.

The hike in haircut reflects the fact that the massive default drains liquidity from the market as lenders’ liquid funding is replaced by illiquid collateral asset from defaulting borrowers. With

\(^{27}\)In figure 13 and figure 14, I normalized the values of the variables in the initial steady state to one.
less funding available for both liquidity hoarding and repo lending, the interest rate overshoots and aggregate storage of investors drops.

As new entrants bring in more liquidity over time, the interest rate and haircut drops and liquidity hoarding increases. But the crisis is not over yet. The counterparty default risk remains high. The overshooting in default rate drains liquidity of the financial system and triggers more default in the future. This leads us to the persistence of the crisis.

**Persistence of the impact of the shock**

Because counterparty default is contagious, massive initial default and high default rate related to the debt overhang for borrowers who start borrowing before the crisis, we can see that the overshooting in drop in investment, repo borrowing or lending and depressed liquidity hoarding persists for a long period. If we think of a period as a quarter, it would still take more than 10 years for the economy to recover from the crisis and recover to the output level in the new steady state.

The disruption in the financial system, has a big impact on the investment and output of the economy. From figure 14, total investment drops by 20% immediately after the shock to collateral risk and eventually drops by about 12%. The aggregate output responds with delay because it takes time for investment in new projects to yield output. Because of the contagion effect and the initial massive default, the decrease in output also overshoot. The maximum drop is about 7%, after which the output recovers a little and the output at the new steady state is about 5% lower than the old steady state.

The fluctuation on the transition path is related to the massive initial default and the outstanding repo borrowing when the shock hits the economy. Before the shock, investors borrow more without worrying too much about default because of lower interest rate and lower haircut. When collateral risk increases unexpectedly, borrowers with outstanding repo borrowing appear to be over-borrowing and therefore likely to default under the higher haircut and interest rate. The hike in default rate leads to fluctuations in market liquidity and the balance sheet of the shadow banking system. Since there are two types of borrowers with different initial borrowings and debt limit in the steady state before the shock to expectation hits the economy, the distribution of borrowers is a combination of two continuous distributions and has two peaks. This is why we can see two
distinct peaks in default rate at the beginning of the transition path. The peaks help us see more clearly the effect of intertemporal contagion through default. This can be observed mostly clearly from the transition dynamics of the interest rate, which is very sensitive to changes in market conditions. The two initial peaks leads to another two peaks in the interest rate. The time lag reflects the average time it takes lenders facing counterparty default to find their own investment and starts borrowing and the peaks reflects the distortion of counterparty default on investors’ borrowing decisions.

As we can see, the transition dynamics has a lot to do with the mass of default at the moment of expectation shift and debt overhang problem troubling investors with outstanding repo borrowing that needs to be rolled over at the moment of expectation shift. Without the asset purchasing of the Federal Reserve, the mass of default would trigger over-shooting in haircut and interest rate and misallocation and fluctuation of lenders’ funding between repo lending and reserve. After the breakout of crisis in 2007, the Federal Reserve stepped in to provide liquidity to the dealer banksAdrian et al. [2009]. In daily news, many argue whether bailing out “too big to fail” dealer banks is a good idea. From this exercise, we can see a counterfactual equilibrium dynamics for what would have happened after the shift in expectation when the Federal Reserve Bank had not stepped in.

8 Conclusion

In this paper, I build a dynamic model to study the efficiency and stability of the shadow banking system. I show that collateral risk leads to increase in counterparty default risk in the equilibrium with repo rollover. Counterparty default drains liquidity from the repo market and reduces output. By studying the dynamic equilibrium triggered by a shock to collateral risk, I show that the shift in collateral risk could be an important contributor to the disruptions in the repo market and the shadow banking system we observed during the 2007-2008 financial crisis. And the shadow banking system is vulnerable to shifts in market participants’ perception of the collateral risk in two senses. First, a small shift in the market belief could trigger a massive initial default. Second, the effect of the shock is long lasting. The exercise shows that the externality of counterparty default has important implications on the efficiency and stability of the shadow banking system and related
Figure 12: Equilibrium dynamics of the liquidity of the repo market and solvency of financial intermediaries, when collateral risk increases unexpectedly.
Figure 13: Equilibrium dynamics of the aggregate balance sheet of the securitized banking system, when collateral risk increases unexpectedly.
Figure 14: Equilibrium dynamics of investment and output, when collateral risk increases unexpectedly.
government policies.

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Table 2: Parameter values used in numerical exercises

A  Laws of motion

First if $\omega_t = 1$, the law of motion of $F_{0t}(a, m, s)$ is

$$dF^1_{0t}(a, m, s) = -\int \left( \begin{array}{c} a' \leq a, \\ \tilde{m}_{0t}(a', m', s') \leq m, \\ \tilde{s}_{0t}(a', m', s') \leq s \end{array} \right) (\mu + \lambda) dt (1 - \tilde{d}_{0t}(a', m', s')) dF^1_{0,t-dt}(a', m', s')$$

$$-\int \left( \begin{array}{c} a' \leq a, \\ m' \leq m, \\ s' \leq s \end{array} \right) \tilde{d}_{0t}(a', m', s') dF^1_{0,t-dt}(a', m', s')$$

$$+\int \left( \begin{array}{c} a' + h_t s' \leq a, \\ \tilde{m}_{0t}(a', m', s') \leq m, \\ \tilde{s}_{0t}(a', m', s') \leq s \end{array} \right) \tilde{d}_{0t}(a', m', s') dF^1_{0,t-dt}(a', m', s')$$

$$+ \eta dt \left( \begin{array}{c} a \leq a_0, \\ \tilde{m}_{0t}(a_0, 0, m_0) \leq m, \\ \tilde{s}_{0t}(a_0, 0, m_0) \leq s \end{array} \right)$$

The first component in equation 23 is the outflow of maturing asset and agents receiving an investment opportunity, conditional the agent does not default. Agents with maturing asset consumes and leave the economy. Agents with incoming projects flow to the measure $F_{1t}$. The second component is the outflow of defaulting agents from type $(a', m', s')$ to other types. The third component is the inflow of agents who meet defaulting borrowers at $t - dt$ but do not accept them. The
fourth component is the inflow from defaulting agents. The last component is the inflow from new comers at \( t - dt \).

\[
dF^1_{dt}(a, m) = - \int \mathbb{I} \left( \begin{array}{c}
a' \leq a, \\
\hat{m}_{dt}(a', m') \leq m, \\
\hat{s}_{dt}(a', m') \leq s
\end{array} \right) (\mu + \lambda) dt F^1_{0, t - dt}(a', m')
\]

\[+ \int \mathbb{I} \left( \begin{array}{c}
a' + (1 + h)\hat{s}_{dt}(a', m', s') \leq a, \\
\tilde{m}_{dt}(a', m', s') \leq m, \\
\tilde{s}_{dt}(a', m', s') \leq s
\end{array} \right) \delta_{l_{s'} > 0} dt \tilde{z}_{dt}(a', m', s') dF_{0, t - dt}(a', m', s')
\]

(24)

Similarly, if \( \omega_t = 1 \), the law of motion of \( F^1_t(a, m, s, i) \) is
\[
\begin{align*}
    dF_{11}(a, m, s, i) &= -\int_\mathbb{Z} \left( \begin{array}{l}
a' \leq a, \\
\bar{m}_{11}(a', m', s') \leq m, \\
\bar{s}_{11}(a', m', s') \leq s, \\
i' \leq i
\end{array} \right) \left( \mu + \pi \right) dt (1 - \bar{d}_{11}(a', m', s', i')) dF_{1,1-t-dt}(a', m', s', i') \\
- \int_\mathbb{Z} \left( \begin{array}{l}
a' \leq a, \\
m' \leq m, \\
s' \leq s, \\
i' \leq i
\end{array} \right) \bar{d}_{11}(a', m', s', i') dF_{1,1-t-dt}(a', m', s', i') \\
+ \int_\mathbb{Z} \left( \begin{array}{l}
a' + h_i \bar{s}_{11}(a', m', s', i') \leq a, \\
\bar{m}_{11}(a', m', s', i') \leq m, \\
\bar{s}_{11}(a', m', s', i') \leq s, \\
i' \leq i
\end{array} \right) \delta_{1} \mathbb{I}_{s' \geq 0} dt \bar{z}_{11}(a', m', s', i') dF_{1,1-t-dt}(a', m', s', i') \\
+ \int_\mathbb{Z} \left( \begin{array}{l}
a' \leq a, \\
\bar{m}_{11}(a', m', s', i') \leq m, \\
\bar{s}_{11}(a', m', s', i') \leq s, \\
i' \leq i
\end{array} \right) \bar{d}_{11}(a', m', s', i') dF_{1,1-t-dt}(a', m', s', i') \\
+ \int_\mathbb{Z} \left( \begin{array}{l}
a' + (1 + h) s' \leq a, \\
\bar{m}_{11}(a', m', s', i') \leq m, \\
\bar{s}_{11}(a', m', s', i') \leq s, \\
i' \leq i
\end{array} \right) \bar{d}_{11}(a', m', s', i') dF_{1,1-t-dt}(a', m', s', i') \\
+ \lambda dt \int_\mathbb{Z} \left( \begin{array}{l}
a' \leq a, \\
\bar{i}_{11}(a', m', s') \leq i, \\
\bar{m}_{11}(a', m', s') \leq m, \\
\bar{s}_{11}(a', m', s') \leq s
\end{array} \right) dF_{0,1-t-dt}(a', m', s') \\
+ \lambda dt \int_\mathbb{Z} \left( \begin{array}{l}
a' \leq a, \\
\bar{i}_{11}(a', m', 0) \leq i, \\
\bar{m}_{11}(a', m', 0) \leq m, \\
\bar{s}_{11}(a', m', 0) \leq s
\end{array} \right) dF_{d,1-t-dt}(a', m')
\end{align*}
\]

(25)

If \( \omega_{1-t-dt} = 0 \), \( F_{0t}^1(\infty, \infty, \infty) = 0 \), \( F_{dt}^1(\infty, \infty) = 0 \), \( F_{dt}^1(\infty, \infty, \infty) = 0 \), and

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\[ dG_0(t) = - (\mu + \lambda) dt \int_{\{\tilde{m}(m') \leq m\}} dG_{0t-\Delta t}(m') + \eta dt \{ \tilde{m}_0A \leq m \} \]  

(27)

\[ dG_1(t, i) = - \pi dt \int_{\{\tilde{m}_{10}(m') \leq m\}} dG_{1,t-\Delta t}(m', i) + \lambda dt \int \mathbb{I} \{ \tilde{i}_{10}(m') \leq i, \ \tilde{m}_{10}(m') \leq m \} dG_{0t-\Delta t}(m') \]  

(28)

If \( \omega_{t-\Delta t} = 1 \), then conditional on the \( \omega_t = 0 \),

\[ G_0(t) = \int \mathbb{I} \{ m' \leq m \} [1 - (\mu + \lambda) dt] dF_{0,t-\Delta t}(a', m', s') \]  

(29)

\[ G_1(t, i) = \int \mathbb{I} \{ m' \leq m, \ \tilde{i}' \leq i \} [1 - (\mu + \pi) dt] dF_{1,t-\Delta t}(a', m', s', i') \]  

(30)

B Efficiency

Proof for Proposition 5.1:
Proof.

\[ S_t = \max_{\{C_t, m_t, i_t\}_t \geq t} \int_t^\infty C_t e^{-\rho(t-t)} dt, \]

\[ C_t d\tau + (\lambda d\tau) \frac{\eta}{\mu + \lambda} i_t + m_t \leq \int_{-\infty}^\tau \frac{\rho + \pi}{\mu} \theta i_t e^{-\pi(t-s)} \lambda \frac{\eta}{\mu + \lambda} ds + e^{-\rho(t-t)} \gamma_t \]

s.t.

\[ + \int_{-\infty}^\tau a_t \omega_t (\mu d\tau) e^{-\mu(t-s)} \eta ds + (\eta d\tau) M_t + m_{t-d\tau}, \]

\[ C_t, m_t, i_t \geq 0. \]

FOC

\[ C_t : \quad e^{-\rho(t-t)} d\tau - e^{-\rho(t-t)} \gamma_t d\tau + \gamma_{ct} = 0 \]

\[ m_t : \quad -e^{-\rho(t-t)} \gamma_t + e^{-\rho(t+\mu-t)} \gamma_{t+\mu} + \gamma_{mt} = 0 \]

\[ i_t : \quad -\lambda d\tau \frac{\eta}{\mu + \lambda} e^{-\rho(t-t)} \gamma_t + \int_{-\infty}^\tau \theta \alpha i_t^{\alpha-1} (\rho + \pi) d\tau e^{-\pi(t-s)} \lambda \frac{\eta}{\mu + \lambda} e^{-\rho(s-t)} \gamma_s ds + \gamma_{it} = 0 \]

\[ -1 + \theta \alpha i_t^{\alpha-1} \int_{-\infty}^\tau (\rho + \pi) e^{-(\rho+\pi)(s-t)} ds = 0 \]

\[ \theta \alpha i_t^{\alpha-1} = 1 \]

Substituting the resource constraint to the objective function ...

\[ C_t - \eta m_t + \lambda \frac{\eta}{\mu + \lambda} i_t = \int_{-\infty}^\tau \left[ \frac{\rho + \pi}{\mu} \theta i_t e^{-\pi(t-s)} \right] \lambda \frac{\eta}{\mu + \lambda} ds + \int_{-\infty}^\tau a_t e^{-\mu(t-s)} \eta ds \]

\[ = \frac{\rho + \pi}{\pi} \theta i_t^* \lambda \frac{\eta}{\mu + \lambda} + a\eta \]

\[ C^* - \eta m^* = \left[ \frac{\rho + \pi}{\pi} \theta i_t^{*\alpha} - i^* \right] \frac{\lambda \eta}{\mu + \lambda} + a\eta \]

\[ \square \]

C the effect of collateral risk

C.1 Equilibrium with state-contingent collateralized debt contract and exogenous haircut

The contract is contingent on liquidity arrival: at the date of borrowing, the borrower put down $h$ units of collateral for each unit of consumption good she borrows, when the borrower has liquidity
to repay the debt, the repayment is $1 + R$, the borrower does not repay the debt, the lender will keep the collateral.

The borrower’s expected payoff when she invests in the long-term project is

$$U(a_0, m, s) = \max_b f(m + s + b) + \frac{\mu y}{\rho + \chi + \mu} a_0 - \frac{(\mu + \pi)(1 + R)b}{\rho + \chi + \mu + \pi}$$

s.t. $0 \leq b \leq \frac{a_0}{(1 + R)h}$

FOC:

$$f'(m + s + b) - \gamma_1 - \gamma_0 - \frac{(\mu + \pi)(1 + R)}{\rho + \chi + \mu + \pi} = 0$$

If the optimal borrowing is such that $f'(m + s + b) > \frac{(\mu + \pi)(1 + R)}{\rho + \chi + \mu + \pi}$, then $b = \frac{a_0}{(1 + R)h}$. Otherwise, $f'(m + s + b) = \frac{(\mu + \pi)(1 + R)}{\rho + \chi + \mu + \pi}$.

Assume that $\theta$ is high enough and $a_0$ and $m_0$ is small enough so that, $f'(m + s + b) > \frac{(\mu + \pi)(1 + R)}{\rho + \chi + \mu + \pi}$, or $f'(m + s + \frac{a_0}{h}) > \frac{(\mu + \pi)(1 + R)}{\rho + \chi + \mu + \pi}$, then,

$$U(a_0, m, s) = f \left( m + s + \frac{a_0}{h} \right) + \left[ \frac{\mu y}{\rho + \chi + \mu} - \frac{\mu + \pi}{\rho + \chi + \mu + \pi} \right] a_0$$

The value function of an agent before she finds her long-term project is

$$(\rho + \mu + \chi + \lambda)V(a_0, m, s) = \mu(a_0 y + m_0) + \chi \frac{\lambda}{\rho + \lambda} f(m) + (\mu + \pi)Rs$$

$$+ \lambda \left\{ f \left( m + s + \frac{a_0}{h} \right) + \left[ \frac{\mu y}{\rho + \chi + \mu} - \frac{\mu + \pi}{\rho + \chi + \mu + \pi} \right] a_0 \right\}$$

$$(\rho + \mu + \chi + \lambda)V(a_0, m, m_0 - m) = \mu(a_0 y + m_0) + \chi \frac{\lambda}{\rho + \lambda} f(m) + (\mu + \pi)R(m_0 - m)$$

$$+ \lambda \left\{ f \left( m_0 + \frac{a_0}{h} \right) + \left[ \frac{\mu y}{\rho + \chi + \mu} - \frac{(\mu + \pi)(1 + R)}{\rho + \chi + \mu + \pi} \right] a_0 \right\}$$

Optimal choice of portfolio implies that $\frac{\partial V(a_0, m, m_0 - m)}{\partial m} = 0$.

$$R = \frac{\chi}{\mu + \pi} \frac{\lambda}{\rho + \lambda} f'(m)$$
The equilibrium is solved by the following system of equations,

\[ R = \frac{\lambda}{\mu + \mu + \lambda} f'(m), \]
\[ \frac{m_0 - m}{\mu + \lambda} = \frac{1}{\mu + \lambda (1 + R)h} \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi}} f'(m), \]

the last equation being the market clearing condition.

With the system of equations, the net supply to the market for the state contingent contract, \( \Gamma \), can be reduced to a function of storage holding \( m \).

\[ (\mu + \lambda)\Gamma(m) = m_0 - m - \frac{\lambda}{\mu + \pi} \frac{a_0}{(1 + R)h} \]
\[ = m_0 - m - \frac{\lambda}{\mu + \pi} \frac{a_0}{h} \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi}} f'(m) \]

\( \Gamma(0) = 0 \) and \( (\mu + \lambda)\Gamma(m_0) = -\frac{\lambda}{\mu + \lambda} \frac{a_0}{h} \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi}} f'(m_0) < 0. \)

\[ (\mu + \lambda)\Gamma'(m) = -1 + \frac{\lambda}{\mu + \pi} \frac{a_0}{h} \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi}} f''(m) \left\{ 1 + \frac{\lambda}{\mu + \pi} \frac{\lambda}{\mu + \pi} f'(m) \right\} < 0 \]

So if the equilibrium exists, it is unique.

Comparative statics

\[ (\mu + \lambda) \frac{\partial \Gamma}{\partial \lambda} = \frac{\lambda}{\mu + \pi} \frac{a_0}{h} \frac{1}{1 + \frac{\lambda}{\mu + \pi} f'(m)} \left( \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi} f'(m)} \right)^2 > 0 \]

\[ (\mu + \lambda) \frac{\partial \Gamma}{\partial \mu} = -\frac{\lambda}{\mu + \pi} \frac{a_0}{h} \frac{1}{1 + \frac{\lambda}{\mu + \pi} f'(m)} \left\{ -\frac{1}{\mu + \pi} - \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi} f'(m)} \right\} \]

\[ = \frac{\lambda}{(\mu + \pi)^2} \frac{a_0}{h} \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi} f'(m)} \frac{1}{1 + \frac{\mu + \lambda}{\mu + \pi} f'(m)} > 0 \]
D Characterization of equilibrium with debt rollover

D.1 The problem of agents with a long term project

Given the price of asset at fire sale, implied by $h$, the maximum amount of borrowing is $\frac{a_0}{h}$. At the moment of default, the borrower’s continuation value is $f(i) + \frac{\mu}{\rho + \mu + \chi}(a_a - a_0)$. For a borrower with asset $a$ decides to default when she puts down $a_0$ units of asset as collateral, the value function can be solved by the following differential equation. It is easy to verify that an agent will hold zero storage after her long-term investment. So we denote $W(a, b, 0, i) = \max_{0 \leq a_0 \leq a} \tilde{W}(a, b, a_0, i)$

\[
\rho \tilde{W}(a, b, a_0, i) = \pi \left[ \frac{\rho + \pi + \chi}{\rho + \mu + \chi} f(i) + \frac{\mu}{\rho + \mu + \chi} ya - b - \tilde{W}(a, b, a_0, i) \right] \\
+ \chi [f(i) - W(a, b, a_0, i)] \\
+ \mu [f(i) + ya - b - W(a, b, a_0, i)] + \frac{\partial W(a, b, a_0, i)}{\partial b} bR \\
\tilde{W}(a, a_0, i) = f(i) + \frac{\mu y}{\rho + \mu + \chi} (a_a - a_0)
\]

where $\tilde{W}(a, a_0, i)$ is the continuation when she defaults.

**Lemma D.1.**

\[
\tilde{W}(a, b, a_0, i) = \left[ \frac{\mu + \pi - \chi}{\rho + \chi + \mu + \pi} - \frac{1}{R h} \right] a_0 \left( \frac{b R}{a_0} \right)^{(\rho + \chi + \mu + \pi)/R} \\
- \frac{\frac{\mu + \pi - \chi}{\rho + \chi + \mu + \pi} b + f(i) + \frac{\mu}{\rho + \mu + \chi} ya}{R}
\]

**Proof.**

\[
(\rho + \chi + \mu + \pi) \tilde{W}(a, b, a_0, i) = (\rho + \chi + \mu + \pi) f(i) - (\mu + \pi) b + \mu \frac{\rho + \mu + \chi + \pi}{\rho + \mu + \chi} ya + \frac{\partial W(a, b, a_0, i)}{\partial b} bR \\
\tilde{W}(a, b, a_0, i) = C_0 b^{(\rho + \chi + \mu + \pi)/R} + C_1 b + C_2 \left( f(i) + \frac{\mu}{\rho + \mu + \chi} ya \right) + C_3. \text{ Then } \frac{\partial \tilde{W}(a, b, a_0, i)}{\partial b} = \frac{\rho + \chi + \mu + \pi}{R} C_0 b^{(\rho + \chi + \mu + \pi - R)/R} + C_1. \\
(\rho + \chi + \mu + \pi) \left[ C_0 b^{(\rho + \chi + \mu + \pi)/R} + C_1 b + C_2 \left( f(I) + \frac{\mu}{\rho + \mu + \chi} ya \right) + C_3 \right] = (\rho + \chi + \mu + \pi) f(I) - (\mu + \pi - \chi) b + \mu \frac{\rho + \mu + \chi + \pi}{\rho + \mu + \chi} ya \\
+ bR \left[ \frac{\rho + \chi + \mu + \pi}{R} C_0 b^{(\rho + \chi + \mu + \pi - R)/R} + C_1 \right]
\]

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Lemma D.2. The partial derivative of the value function is

\[
\begin{align*}
\frac{\partial \tilde{W}(a, b, a_0, i)}{\partial y} &= \frac{\mu}{\rho + \mu + \chi} \left[ a - a_0 \left( \frac{hb}{a_0} \right)^{(\rho + \chi + \mu + \pi)/R} \right] \\
\frac{\partial \tilde{W}(a, b, a_0, i)}{\partial a_0} &= \left[ \frac{\mu}{\rho + \mu + \chi} y - \frac{\mu + \pi}{\rho + \mu + \chi - R} \right] \frac{1}{h} \frac{\rho + \chi + \mu + \pi}{R} \left( \frac{hb}{a_0} \right)^{(\rho + \chi + \mu + \pi)/R} \\
\frac{\partial \tilde{W}(a, b, a_0, i)}{\partial a} &= \frac{\mu}{\rho + \mu + \chi} y \\
\frac{\partial \tilde{W}(a, b, a_0, i)}{\partial b} &= \left[ \frac{\mu}{\rho + \mu + \chi} y h - \frac{\mu + \pi}{\rho + \mu + \chi - R} \right] \frac{\rho + \chi + \mu + \pi}{R} - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R} \\
\frac{\partial \tilde{W}(a, b, a_0, i)}{\partial i} &= f'(i)
\end{align*}
\]

From Lemma D.2, we can see that the borrower will rollover her debt up the debt limit as long as \( \frac{\mu}{\rho + \mu + \chi} y - \frac{\mu + \pi}{\rho + \mu + \chi - R} \frac{1}{h} > 0 \). And in this case, \( W(a, -b, i) = \tilde{W}(a, b, a, i) \).
Lemma D.3. agents

- will borrow against all their asset if $\frac{\mu}{\rho + \mu + \chi} y - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R} h > 0$ or $h > \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R} \frac{\rho + \mu + \chi}{\mu y}$.
- will not borrow if $h < \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R} \frac{\rho + \mu + \chi}{\mu y}$
- are indifferent between borrowing or not if $h = \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R} \frac{\rho + \mu + \chi}{\mu y}$.

D.2 The problem of agents when they find a long term project

Next consider the optimization problem at the moment the agent finds an investment opportunity.

$$U(a, m, s) = \max_{c \geq 0, 0 \leq b \leq \frac{a_0}{1 + h}} c + W(a, -b, s + m - c + b)$$

The first order condition of the problem is

$$\frac{dW(a, -b, s + m - c + b)}{db} = 0,$$

$$\frac{\partial}{\partial b} W + \frac{\partial}{\partial i} W = 0.$$

From the first order condition we have the following lemma.

**Lemma D.4.** The optimal choice of project investment and initial borrowing of an agent with portfolio $(a, s, m)$ is solved by equation

\[
\left( \frac{hh}{a} \right) \left( \frac{\rho + \chi + \mu + \pi - R}{R} \right) = \frac{R}{\rho + \chi + \mu + \pi} \frac{f'(s + b)}{\frac{\mu + \pi}{\rho + \mu + \chi} yh - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R} R}
\]

Let $\hat{b} = \frac{hh}{\alpha + h s'}$. $b = \frac{a + h s'}{h}, s - s' + b = s + \frac{a}{h} \hat{b} - (1 - \hat{b}) s'$. Let

\[
\Gamma = \left[ \frac{\mu}{\rho + \mu + \chi} yh - \frac{\mu + \pi}{\rho + \chi + \mu + \pi - R} \right] \left( \frac{\rho + \chi + \mu + \pi}{R} \hat{b} \left( \rho + \chi + \mu + \pi - 2R \right) / R \right)
\]

\[
\Gamma = f'' \left( s + \frac{a}{h} \hat{b} - (1 - \hat{b}) s' \right) < 0
\]

Then from Implicit Function Theorem, we have the following result.
Lemma D.5. *Counter-party default increases default probability of lenders in the future*

\[ \frac{db}{ds} > 0 \]

Since default probability is an increasing function of \( \hat{b} \), we know from this lemma that counter-party default that transforms liquid funding to collateral asset will increase the default probability when the agent starts borrowing.

D.3 The problem of agents waiting for the investment opportunity

In the stationary environment, the value functions of agents waiting for their investment opportunities, given the optimal portfolio choice on storage \( m \) and lending \( s \), can be written as follows,

\[
(\rho + \mu + \chi + \lambda + \delta) V(a, m, s) = sR + \mu (ya + s + m) + \delta V^d(a + hs, m) + \chi V^A(m) + \lambda U(a, m, s),
\]

\[
(\rho + \mu + \chi + \lambda) V^d(a + hs, m) = \mu [y(a + hs) + m] + \chi V^A(m) + \lambda U(a + hs, m, 0).
\]

\[
(\rho + \lambda)V^A(m) = \lambda \theta m^\alpha.
\]

Lemma D.6. *The following condition must be satisfied in equilibrium:*

\[
\mu (yh - 1) s = sR + \lambda [U(a, s, m) - U(a + hs, 0, m)]
\]

Proof. Suppose instead \( V^d(a + hs, m) > V(a, m, s) \), then lenders will default on the loan by keeping the collateral. So \( V^d(a + hs, m) \leq V^d(a, m, s) \). Suppose \( V^d(a + hs, m) < V(a, m, s) \), then lenders won’t be willing to lend to the defaulting borrowers when they observe that the borrowers are going to default. Rollover is not possible.

Given the value functions, the optimal portfolio choice is solved by the following problem

\[
\max_{s, m \in \mathbb{R}_+} V(a, m, s)
\]

\[
s.t. s + m \leq m_0
\]
E Equilibrium dynamics with a constant $\chi$

Given the initial condition, the distribution of lenders and borrowers, we need to solve the sequence of default rate $\delta_t$, interest rate $R_t$, haircut $h_t$, the optimal portfolio choice of active lenders between repo lending $s_t$ and storage $m_t$.

Active lenders’ portfolio choice

Given the value functions,

$$V_t(a, m, s) = \max_{\tilde{c}, \tilde{m}, \tilde{s}} sR_t dt + \tilde{c} + \mu dt ay + \tilde{m} + \tilde{s}) + \delta_t dt e^{-\rho dt} V_{t+dt} (a + h_t \tilde{s}, \tilde{m}, 0)$$

$$V_t(a, m, s) = \max_{\tilde{c}, \tilde{m}, \tilde{s}} + \chi dt \frac{\lambda}{\rho + \lambda} f(\tilde{m}) + \lambda dt \tilde{W}_t (a, \tilde{m}, \tilde{s})$$

$$+ [1 - (\mu + \delta_t + \chi) dt] e^{-\rho dt} V_{t+dt} (a, \tilde{m}, \tilde{s})$$

s.t. $\tilde{c} + \tilde{m} + \tilde{s} \leq m + s$

$\tilde{c}, \tilde{m}, \tilde{s} \geq 0$

We focus on the parameter space where consumption allocation before agents exit the market is always 0. So $\tilde{c} = 0$, and $m + s = \tilde{m} + \tilde{s} = m_0$. Since $m_0$ is a constant, $\frac{\partial V_t(a, m, s)}{\partial s} = R_t dt$, $U_t(a, \tilde{m}, m_0 - \tilde{m})$ does not depend on $\tilde{m}$ given $w$, and $\tilde{m}$ is solved by the following problem

$$\max_{0 \leq \tilde{m} \leq m_0} \delta_t dt e^{-\rho dt} V_{t+dt} (a + h_t (m_0 - \tilde{m}), \tilde{m}, 0) + \chi dt \frac{\lambda}{\rho + \lambda} f(\tilde{m}) - R_t dt \tilde{m}$$

FOC

$$\delta_t ( -h_t \frac{\partial}{\partial \tilde{m}} + \frac{\partial}{\partial \tilde{m}} ) V_{t+dt} (a + h_t (m_0 - \tilde{m}), \tilde{m}, 0) + \chi \frac{\lambda}{\rho + \lambda} f'(\tilde{m}) - R_t + \gamma_{\tilde{m} \geq 0} - \gamma_{\tilde{m} \leq w} = 0$$

$$\frac{\partial}{\partial a} V_{t+dt}^d (a + h_t (m_0 - \tilde{m}), \tilde{m}) = \int_0^{\infty} \left[ \mu y + \lambda \frac{d}{da} U_{t+s} (a', m', 0) \right] e^{-(\mu + \lambda + \rho) s} ds$$

$$\frac{d}{da} U_t (a, m, 0) = \int_0^{T_t(b)} \left[ \mu y + \pi \frac{\mu y}{\rho + \chi + \mu} \right] e^{-(\chi + \mu + \rho) \tau} d\tau$$

$$= \frac{\mu y}{\rho + \chi + \mu} \left[ 1 - e^{-(\rho + \mu + \chi) T_t(b)} \right]$$
\begin{align*}
\frac{\partial}{\partial a} V^d_{t+dt} (a + h_t(m_0 - \tilde{m}), \tilde{m}) \\
= \int_0^\infty \left[ \mu y + \lambda \frac{\mu y}{\rho + \chi + \mu} \left( 1 - e^{-(\rho + \pi + \mu + \lambda) T_{t+s}(b_0(a + (1+h_t)(w-\tilde{m}), \tilde{m}))} \right) \right] e^{-(\mu + \lambda + \chi + \rho)s} ds
\end{align*}

\begin{align*}
\frac{\partial}{\partial m} V^d_{t+dt} (a + h_t(m_0 - \tilde{m}), \tilde{m}) \\
= \int_0^\infty \left[ \mu + \lambda \frac{\partial}{\partial m} U_{t+s}(a + h_t(w - \tilde{m}), \tilde{m}, 0) + \chi \frac{\lambda}{\rho + \lambda} f'(\tilde{m}) \right] e^{-(\mu + \lambda + \chi + \rho)s} ds
\end{align*}

Haircut and indifference condition

\begin{align*}
V^d_t(a, m) &= \int_0^\infty \left[ \mu(ay + m) + \lambda U_{t+s}(a, m, 0) + \chi \frac{\lambda}{\rho + \lambda} f(m) \right] e^{-(\mu + \lambda + \chi + \rho)s} ds \\
&= \frac{\mu(ay + m) + \chi \frac{\lambda}{\rho + \lambda} f(m)}{\mu + \lambda + \chi + \rho} + \int_0^\infty \lambda U_{t+s}(a, m, 0) e^{-(\mu + \lambda + \chi + \rho)s} ds
\end{align*}

\begin{align*}
V_t(a, \tilde{m}_t, \hat{s}_t) = \int_0^\infty \left[ \hat{s}_{t+\tau} R_{t+\tau} + \delta_{t+\tau} V^d_{t+\tau}(a + h_t\hat{s}_{t+\tau}, \tilde{m}_{t+\tau}) + \mu(ay + m_0) \right] \\
&+ \chi \frac{\lambda}{\rho + \lambda} f(\tilde{m}_{t+\tau}) + \lambda U_{t+\tau}(a, m_0) \\
e^{-\int_0^\tau \delta_{s+u}(u + \chi + \lambda + \rho) du} e^{-(\mu + \lambda + \rho + \tau)} d\tau
\end{align*}

Haircut must be such that given the optimal choice \((\hat{m}_t, \hat{s}_t)\)

\begin{align*}
V_t(a, \hat{m}_t, \hat{s}_t) &= V^d_t(a + h_t\hat{s}_t, m_t)
\end{align*}

Initial borrowing of investors with LT projects

\begin{align*}
U_t(a, m, s) &= \max_{0 \leq b \leq m_t \frac{R}{\rho}} R_{t+s} dt + f(m + s + b) \\
&+ \int_0^{T_t(b)} \left[ \chi \cdot 0 + \mu \left( ay - be_{t+s}^{R_{t+s} ds} \right) + \pi \left( \frac{\mu y}{\rho + \mu + \chi} - be_{t+s}^{R_{t+s} ds} \right) \right] e^{-(\chi + \mu + \pi + \rho)s} d\tau
\end{align*}

where \(T_t(b) = \inf \left\{ \tau : be_{t+s}^{R_{t+s} ds} h_{t+\tau} = a \right\} \). The maximization problem is equivalent to the following problem
\[
\max_{0 \leq b \leq \frac{a}{h_{n+\tau}}} f\left(m + s + b\right) + \int_{0}^{T_{i}(b)} \frac{\mu + \mu + \pi}{\rho + \mu + \chi} \mu\alpha e^{-\left(\beta + \mu + \rho\right)\tau} d\tau
\]

A sufficient condition for equation \( b e \int_{0}^{T_{i}+s} d\tau = \frac{a}{h_{n+\tau}} \) to have at most one solution is that \( e \int_{0}^{T_{i}+s} d\tau [R_{t+\tau} h_{t+\tau} + \dot{h}_{t+\tau}] > 0 \).

\[
\frac{\partial T_{i}(b)}{\partial b} = -\frac{h_{t+\tau}}{b [R_{t+\tau} h_{t+\tau} + \dot{h}_{t+\tau}]}
\]

**Market clearing condition**

Density of deactivated lenders with portfolio \( (a + s_{t}(1 + h_{\tau}), m_{\tau}, 0) \) is \( \delta_{t} n_{t} e^{-\left(\mu + \lambda\right)(t-\tau)} \). The measure of active lenders at \( t \), \( n_{t} \).

\[
\text{Demand}_{t} = \int_{0}^{\infty} \int_{0}^{\infty} \lambda \delta_{t-\tau-s} n_{t-\tau-s} e^{-\left(\mu + \lambda\right)s} b_{t-\tau-s} \left(a + s_{t-\tau-s} h_{t-\tau-s}, m_{t-\tau-s}, 0\right) e^{(R-\pi-\mu)\tau}
\]

\[
\mathbb{I} \{\tau \leq T_{t-\tau} (b_{t-\tau-s} (a + s_{t-\tau-s} h_{t-\tau-s}, m_{t-\tau-s}, 0))\} d\sigma d\tau
\]

\[
+ \int_{0}^{\infty} \lambda n_{t-\tau-s} b_{t-\tau-s} (a, m_{0}, 0) e^{(R-\pi-\mu)\tau} \mathbb{I} \{\tau \leq T_{t-\tau} (b_{t-\tau-s} (a, m_{0}, 0))\} d\tau
\]

\[
\text{Supply}_{t} = n_{t} \delta_{t}
\]

**Default rate**

Updation \( \delta_{t} \) depends on the measure of demand from borrowers and the flow of demand from defaulting borrowers.

\[
\delta_{t} = \frac{\text{Demand of defaulting borrower}_{t}}{\text{Demand}_{t}}
\]

Distribution of lender’s portfolio, \( F(a, m, s) \), a function of timing of counterparty default. \( n_{t} = \int_{0}^{\infty} \eta e^{-\left(\mu + \lambda\right)\tau} - \int_{0}^{T_{i}(b)} \delta_{t-\tau-s} d\sigma d\tau \). The measure of agents in default cohort \( \tau (\tau < t) \) at moment \( t \) is \( \delta_{t} n_{t} e^{-\left(\mu + \lambda\right)(t-\tau)} \). And agents in default cohort \( \tau \) have portfolio, \( (a + s_{t}(1 + h_{\tau}), m_{\tau}, 0) \).
Numerical algorithm to compute the transition path

Suppose the shock to expectation arrives at \( t = 0 \).

1. Guess a sequence of interest rate, haircut and default rate, \( \{ R_t, h_t, \delta_t \} \forall t \geq 0 \).

2. Given the sequence, \( \{ R_t, h_t, \delta_t \} \forall t \geq 0 \), solve for the policy functions and value functions of agents on the transition path.

3. Given the \( h_0 \), solve for the mass of initial default. And then given the policy function of agents, solve the distribution of agents along the transition path.

4. Given the distributions of agents and the policy functions, update the default rate of borrowers.

5. Given the distributions of agents and the policy functions, solve for the net demand of repo borrowing. Update the interest rate according to the net demand of repo borrowing.

6. Given the value functions of agents, update haircut.

7. Go back to step 2 with the updated sequence of interest rate, haircut and default rate, until convergence.