ESTIMATING DYNAMIC DISCRETE CHOICE MODELS VIA CONVEX ANALYSIS

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Abstract. Using results from convex analysis, we characterize the identification and estimation of dynamic discrete-choice models based on the random utility framework. Based on these insights, we propose a new two-step estimator for these models, which is easily applicable to models in which the utility shocks may not derive from an extreme-value distribution, and may be mutually correlated with each other and with the state variables. Monte Carlo results demonstrate the good performance of this estimator, and we provide a short application based on Rust’s (1987) bus engine replacement model.

1. Introduction

Empirical research utilizing dynamic discrete choice models of economic decision-making has flourished in recent decades, with applications in all areas of applied microeconomics including labor economics, industrial organization, public finance, and health economics. Existing identification results for these models allow for quite general specifications of the additive choice-specific utility shocks. However, in practice, almost all applications of these models maintain the restrictive assumption that the utility shocks are distributed i.i.d. type I extreme value, independently of the state variables, leading to choice probabilities which take the multinomial logit form. No doubt this is due to the computational convenience of the logit model, because in that case a number of structural components of the model have convenient, analytical closed forms.

Date: First draft: April 2013. This version: October 2013. Comments welcome.
The authors thank Ben Connault, Thierry Magnac, Emerson Melo, Bob Miller, John Rust, and participants at the CEMMAP conference on inference in game-theoretic models (June 2013) and a UCLA econometrics mini-conference for helpful comments. Galichon’s research has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n°313699 and from FiME, Laboratoire de Finance des Marchés de l’Energie (www.fime-lab.org).
We might imagine scenarios where utility shocks are correlated across different choices, or that the realisation of utility shocks depends on what the current states are. Capturing these modelling assumptions compels us to look beyond the multinomial logic form, which then calls for a tractable method to estimate dynamic discrete-choice (DDC) models with different sets of distributional assumptions. Moreover, this methodological improvement would also make possible the task of evaluating the robustness of estimation to different distributional assumptions.

This paper aims to bridge this gap and the contribution is twofold. First, we show how the powerful tools of convex analysis can be used to describe the empirical content of dynamic discrete-choice models. Based upon these insights, we exploit the convex nature of the problem to propose a new estimator for dynamic discrete-choice models which can accommodate any distribution of the utility shocks. Our findings expand the set of dynamic discrete-choice models suitable for applied work far beyond those with extreme-value distributed utility shocks.

In discrete choice models, the social surplus function (McFadden, 1978) provides us with the mapping from payoffs to the probabilities with which a choice is chosen at each state (conditional choice probabilities). Recognizing that the social surplus function is convex, we utilize the idea that the convex conjugate of the social surplus function gives us the inverse mapping - from choice probabilities to payoffs. Specifically, the subdifferential of the convex conjugate is a correspondence that maps from the observed choice probabilities to an identified set of payoffs. In this way, we use the language of convex analysis to succinctly characterise the empirical content of discrete choice models.

For dynamic discrete choice models, the identified payoff consists of the per-period payoff term and the continuation values, collectively known as the choice-specific value functions. Even though the set of choice-specific value functions consistent with a vector of observed choice probabilities is typically larger than a singleton (i.e. they are “partially identified”), we proceed to demonstrate the point identification of the per-period payoffs. The non-trivial aspect of this results lies in showing that we can, without loss of generality, isolate a
particular choice-specific value function from its identified set, and recover the per-period payoffs from this choice-specific value function.

Not only is the convex conjugate of the social surplus function a useful theoretical object; it also gives us a new and practical way to non-parametrically estimate payoffs when utility shocks are no longer distributed i.i.d. type I extreme value. We show how the conjugate along with its set of subgradients can be efficiently computed by means of linear programming. This linear programming formulation has the structure of an optimal assignment problem similar to the Shapley-Shubik assignment game. This surprising connection enables us to employ insights developed in optimal matchings to dynamic discrete choice models.

Section 2 contains our main results regarding the empirical content of dynamic discrete-choice models, as seen through the lens of convex analysis. Based on these results, we propose, in Section 3, a two-step estimation approach for these models. We also emphasize here the surprising connection between dynamic discrete-choice and optimal matching models. In Section 4 we discuss computational details for our estimator, focusing on the use of linear programming to compute (approximately) the convex conjugate function from the dynamic discrete-choice model. Monte Carlo experiments (in Section 5) show that our estimator performs well in practice, and we apply the estimator to Rust’s (1987) bus engine replacement data (Section 6). Section 7 concludes. The Appendix contains proofs and also a brief primer on basic results from convex analysis.

2. Basic Model

2.1. The framework. In this section we review the basic dynamic discrete-choice setup, as encapsulated in Rust’s (1987) seminal paper. The state variable is $x_t \in \mathcal{X}$ which, for convenience, we assume to be finite discrete-valued. Agents choose actions $y_t \in \mathcal{Y}$ from a finite space $\mathcal{Y} = \{0, 1, \ldots, D\}$.

The single-period utility which an agent derives from choosing $y_t$ in period $t$ is

$$\bar{u}(y_t, x_t) + \epsilon_{y_t}$$

where $\epsilon_{y_t}$ denotes the utility shock pertaining to action $y_t$, which differs across agents. Across agents and time periods, the set of utility shocks $\{\epsilon_{y_t}\}_{y_t \in \mathcal{Y}}$ is distributed according
to a joint distribution function \( Q_t(\cdots; x_t) \) which can depend on the current values of the state variables \( x_t \).

Following Rust (1987), and most of the subsequent papers in this literature, we maintain the following conditional independence assumption (which rules out serially persistent forms of unobserved heterogeneity\(^1\)):

**Assumption 1** (Conditional Independence). \((x_t, \epsilon_t)\) is a controlled first-order Markov process, with transition

\[
Pr(x_{t+1}, \epsilon_{t+1}|y_t, x_t, \epsilon_t)
\]

\[=Pr(\epsilon_{t+1}|x_{t+1}, y_t, x_t, \epsilon_t) \cdot Pr(x_{t+1}|y_t, x_t, \epsilon_t)
\]

\[=Pr(\epsilon_{t+1}|x_{t+1}) \cdot Pr(x_{t+1}|y_t, x_t).
\]

We let \( \Pi = (\Pi^1, \ldots, \Pi^D) \) denote the (stationary) Markov transition matrix where \( \Pi^y_{ij} = Pr(x_{t+1} = j|y_t = y, x_t = i) \), that is, the \( ij \)-th entry of the matrix \( \Pi^y \) denotes the probability that the state transitions from state \( i \) to state \( j \) when the action taken is \( y \).

The discount rate is \( \beta \). Agents are dynamic optimizers who solve

\[
y^*_t \in \arg \max_{y_t \in \mathcal{Y}} \left\{ \bar{u}(y_t, x_t) + \epsilon_{y_t} + \beta \mathbb{E} \left[ \bar{V}(x_{t+1}, \epsilon_{t+1}) \big| x_t, y_t \right] \right\},
\]

where, under standard conditions, the value function \( \bar{V} \) is recursively defined as

\[
\bar{V}(x_t, \epsilon_t) = \max_{y_t \in \mathcal{Y}} \left\{ \bar{u}(y_t, x_t) + \epsilon_{y_t} + \beta \mathbb{E} \left[ \bar{V}(x_{t+1}, \epsilon_{t+1}) \big| x_t, y_t \right] \right\}.
\]

\( V(x_t) \), the ex-ante value function, is defined as:

\[
V(x_t) = \mathbb{E} \left[ \bar{V}(x_{t+1}, \epsilon_{t+1}) \big| x_t \right].
\]

The expectation above is conditional on \( x_t \). In the literature, \( V(x) \) is called the ex-ante (or integrated) value function, because it measures the continuation value of the dynamic optimization problem before the agent observes his shocks \( \epsilon \), so that the optimal action is still stochastic from the agent’s point of view.

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\(^1\)See Norets (2009), Kasahara and Shimotsu (2009), Arcidiacono and Miller (2011), and Hu and Shum (2012).
We now define the choice-specific value functions as consisting of two terms: the per-period payoff and the discounted continuation payoff.

\[ w_{y_t}(x_t) \equiv w(y_t, x_t) = \bar{u}(y_t, x_t) + \beta \mathbb{E}[V(x_{t+1})|x_t, y_t], \]

In turn, we can define the conditional choice probabilities (CCP’s):

\[ p(y|x) \equiv P_r(y_t = y|x_t = x) = P_r\{y = \arg\max_{y \in \mathcal{Y}} \{w(y_t, x_t = x) + \epsilon_{y_t}\}\} \]

### 2.2. Convex analysis approach.

We now recast these results using convex analysis. This will allow us to characterize the relationship between the observables (which are the choice-probabilities at different states \( p(y|x) \) and the transition probabilities \( \Pi \) from one state to another), and the unobserved functions of interests which we want to identify and estimate (which are ultimately the per-period utilities \( \bar{u}_y(x) \), and along the way, the ex-ante value function \( V(x) \)). For convenience, we will suppress the time subscript.

First, we introduce the indirect expected utility of a decision maker facing the \(|\mathcal{Y}|\)-dimensional vector of choice-specific values \( w.(x) \):

\[ G(w.(x)) = \mathbb{E}\left[ \max_{y \in \mathcal{Y}} \{w_y(x) + \epsilon_y\}|x\right] \]

where the expectation is assumed to be finite and is taken over the conditional distribution of the utility shocks, \( Q_t \). This function \( G : \mathbb{R}^{|\mathcal{Y}|} \to \mathbb{R} \), is called the “social surplus function” in McFadden’s (1978) random utility framework, and can be interpreted as the expected welfare of a representative agent in the dynamic discrete-choice problem.

We also introduce the convex conjugate convex conjugate of \( G \), which we denote as \( G^* \):\(^2\)

\(^2\)Details of convex conjugates are expounded in the Appendix. Elsewhere in economics, convex conjugates are also encountered in production theory. When \( f \) is the convex cost function of the firm (decreasing returns to scale in production), then the convex conjugate of the cost function, \( f^* \), is in fact the firm’s optimal profit function.
Definition 1 (Convex Conjugate). We define $G^*$, the Legendre-Fenchel conjugate function of $G$ (a convex function), by

$$G^*(p) = \sup_{w \in \mathbb{R}^\mathcal{Y}} \left\{ \sum_{y \in \mathcal{Y}} p_y w_y - G(w) \right\}$$  \hspace{1cm} (2)$$

if $p$ is a probability over the set $\mathcal{Y}$, that is $p_y \geq 0$ and $\sum_{y \in \mathcal{Y}} p_y = 1$, and $G^*(p) = +\infty$ otherwise. Note that $p_y$ is the $y$-th component of the $|\mathcal{Y}|$-dimensional vector.

Next, we explore a particular dual relationship between $w.(x)$ and $p(\cdot|x)$.

This dual relationship is made precise in Proposition 2 below.

2.3. Duality between choice probabilities and choice-specific value functions. We begin by reiterating a well-known that the derivatives of the social surplus function $G$ at a vector of utilities $\vec{u}(x)$, is the vector of choice probabilities $p(\cdot|x)$:

**Proposition 1** (The Williams-Daly-Zachary (WDZ) Theorem).

$$p(\cdot|x) = \nabla G(w.(x)).$$

This result, which is analogous to Roy’s Identity in discrete choice models, is expounded in McFadden (1981) and Rust (1994; Thm. 3.1)). It characterizes the vector of choice probabilities corresponding to optimal behavior in a discrete choice model as the gradient of the social surplus function. (For completeness, we include a proof in the Appendix.)

The WDZ theorem provides a mapping from the choice-specific value functions (which are unobserved by researchers) to the observed choice probabilities $p(\cdot|x)$. The following result, which is the basis of our identification and estimation strategy, provides an “inverse” correspondence from the observed choice probabilities back to the unobserved $w.(x)$, which is a necessary step for identification and estimation.

**Proposition 2.** The following pair of equivalent statements identify $w.(x)$:

(i) $p$ is in the subdifferential of $G$ at $w$

$$p(\cdot|x) \in \partial G(w.(x)), \hspace{1cm} (3)$$
(ii) $w$ is in the subdifferential of $\mathcal{G}^*$ at $p$

$$w.(x) \in \partial \mathcal{G}^*(p.(|x|)).$$ (4)

The definition and properties of the subdifferential of a convex function is provided in Appendix A. Proposition 2 describe a dual relationship between the observed choice probabilities $p$ and the unobserved choice-specific value functions $w$. Part (i) is, of course, connected to the WDZ theorem above, and hence encapsulates an optimality requirement that the vector of observed choice probabilities $p$ be derived from optimal discrete-choice decision making for some unknown vector $w$ of choice-specific value functions.

Part (ii) of this proposition, which describes the “inverse” mapping from conditional choice probabilities to choice-specific value functions, does not appear to have been exploited in the literature on discrete choice. It relates to Galichon and Salanié (2012) who use convex analysis to estimate matching games with transferable utilities. It specifically states that the vector of choice-specific value functions can be identified from the corresponding vector of observed choice probabilities $p$ as the subgradient of the convex conjugate function $\mathcal{G}^*(p)$.

It is related to, and perhaps a more general statement of, results in existing papers (cf. Hotz and Miller (1993), Rust (1994), Magnac and Thesmar (2002), Arcidiacono and Miller (2012, Lemma 5)). Eq. (4) is also constructive, and suggests a procedure for computing the choice-specific value functions corresponding to observed choice probabilities; we will fully elaborate this procedure in subsequent sections.

Before proceeding, we discuss the example of the logit model, for which the functions and relations above reduce to familiar expressions.

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$\mathcal{G}$ is differentiable at $w$ if and only if $\partial \mathcal{G}(w)$ is single-valued. In that case, part (i) of Prop. 2 reduces to $p = \nabla \mathcal{G}(w)$. If, in addition, $\nabla \mathcal{G}$ is one-to-one, then we immediately get $w = (\nabla \mathcal{G})^{-1}(p)$, or $\nabla \mathcal{G}^*(p) = (\nabla \mathcal{G})^{-1}(p)$, which is the case of the classical Legendre transform. However, as we show below, $\nabla \mathcal{G}(w)$ is not typically one-to-one in discrete choice models, so that the statement in part (ii) of Prop. 2 is more suitable.

Clearly, Proposition 2 also applies to static random utility discrete-choice models, with the $w.(x)$ being interpreted as the utility indices for each of the choices. As such, Eq. (4) relates to results regarding the mapping between choice probabilities and utilities in static discrete choice models (e.g. Berry (1994); Haile, Hortacsu, and Kosenok (2008)). Similar results have also arisen in the literature on stochastic learning in games (Hofbauer and Sandholm (2002); Cominetti, Melo and Sorin (2010)).
Example 1 (Logit). When the distribution $Q$ of $\epsilon$ obeys an extreme value type I distribution, it follows from Extreme Value theory that $G$ and $G^*$ can be obtained in closed form:\footnote{Relatedly, Arcidiacono and Miller (2011, pp. 1839-1841) discuss computational and analytical solutions for the $G^*$ function in the generalized extreme value setting.}

\begin{align*}
G(w) &= \log\left(\sum_{y \in Y} \exp(w_y)\right) + \gamma \\
G^*(p) &= \sum_{y \in Y} p_y \log p_y - \gamma,
\end{align*}

where $\gamma \approx 0.57$ (Euler’s constant). Hence in this case, $G^*$ is the entropy of distribution $p$. The subdifferential of $G^*$ is characterised as follows: $w \in \partial G^*(p)$ if and only if $w_y = \log p_y - K$, for some $K \in \mathbb{R}$.

2.4. Empirical Content of Dynamic Discrete Choice Model. The transition probabilities $\Pi$ and CCPs $p$ can be consistently estimated from data directly and, thus, $(\Pi, p)$ are assumed to be known in the analysis of identification. In addition, we also assume that $(\beta, Q_\epsilon)$ are known to the econometrician. The object of interest is the per-period payoff function $\bar{u}$, and the main goal of this non-parametric identification exercise is to estimate $\bar{u}$ using ideas from convex analysis.

To summarize the empirical content of the model, we start with the ex-ante value function $V$, which solves the following equation

$$
V(x) = \sum_{y \in Y} p(y|x) \left( \bar{u}_y(x) + \mathbb{E}[\epsilon_y|y, x] + \beta \sum_{x'} p(x'|x, y) V(x') \right) \tag{5}
$$

found eg. in Pesendorfer and Schmidt-Dengler (2008), where we write $p(x'|x, y) = Pr(x_{t+1} = x'|x_t = x, y_t = y)$. Letting $Y^*$ denote the (random) optimal alternative, we get

$$
G(w, (x)) = \mathbb{E}[w_{Y^*}(x) + \epsilon_{Y^*}|x] = \sum_{y \in Y} Pr(Y^* = y|x) (w_y(x) + \mathbb{E}[\epsilon_y|Y^* = y, x]) \tag{6}
$$

where the argument of $G$ is the $|Y|$-dimensional vector of choice specific value functions. Hence, if we compare this with the previous equation (5), we obtain

$$
V(x) = G(w, (x)), \text{ where } \bar{u}_y(x) = w_y(x) - \beta \mathbb{E}[V(x')|x, y]. \tag{7}
$$

$$
\bar{u}_y(x) = w_y(x) - \beta \mathbb{E}[V(x')|x, y]. \tag{8}
$$
Eqs. (7) and (8), along with the “inverse mapping” between choice probabilities and choice-specific value function \( w(x) \in \partial G^*(p(.|x)) \), present the relations between the unobserved functions \( \bar{u}_y(x), V(x) \) and \( w_y(x) \) and the observed choice probabilities \( p(y|x) \), via the convex functions \( G \) and \( G^* \), which are known and computable given distributional assumptions regarding the random utility shocks \( \epsilon_y \). In that sense, these equations summarize the empirical content of the dynamic discrete-choice model. Proposition 1 describes the structure of random utility discrete-choice models, while Eq. (8) presents the recursive restrictions of the dynamic discrete-choice model. These equations echo, in perhaps the most general form, analogous derivations in the existing papers on identification and estimation of dynamic discrete-choice models, including Hotz and Miller (1993), Magnac and Thesmar (2002) and Bajari, Chernozhukov, Hong, and Nekipelov (2007).

In addition, it can be shown (cf. Eq. (24) in Appendix A) that \( G(w) + G^*(p.) = \sum_{y \in Y} p_y w_y \) if and only if \( p. \in \partial G(w.) \). Therefore, Eq. (7) can be rewritten as:

\[
V(x) = \sum_{y \in Y} p(y|x)w_y(x) - G^*(p(.|x)).
\]

Combining this with Eq. (6), we see that the general expression for the convex conjugate function \( G^* \), is

\[
G^*(p(.|x)) = -\sum_{y} p(y|x)E[\epsilon_{y^*}|Y^* = y, x], \tag{9}
\]
corresponding to the weighted conditional expectations of the utility shocks \( \epsilon_y \) conditional on choosing the option \( y \).

3. Estimation procedure

Based upon the derivations in the previous section, we present an estimation procedure, which follows two steps. From the key proposition above, we know that identification and estimation of the model boil down to evaluating the unknown functions \( G \) and \( G^* \). We propose an estimation algorithm which only requires computing the \( G^* \) function.

3.1. First step. In the first step, we use convex analysis to recover the vector of choice-specific value functions \( w_0^0(x) \in \partial G^*(p(.|x)) \) at each vector of observed choice probabilities \( p(.)|x \) for each value of \( x \). The following proposition, which was derived in Galichon and
Salanié (2012, Proposition 2), characterizes the \( G^* \) function as an optimum of a well-studied mathematical program: the “mass transportation” problem (cf. Villani (2003)).

**Proposition 3.** Assume that \( Q_\epsilon \) is such that \( \{\epsilon_k - \epsilon_0\}_k \) has full support on the real line. Let \( p = (p_y)_{y \in \mathcal{Y}} \) be a vector of choice probabilities. Then the function \( G^*(p) \) is the value of the mass transportation problem in which the distribution \( Q_\epsilon \) of utility shocks \( \epsilon = \{\epsilon_y\}_{y \in \mathcal{Y}} \) is matched optimally to the distribution of actions \( y \) given by the multinomial distribution \( p \), when the cost associated to a match of \( (\epsilon, y) \) is given by

\[
c(y, \epsilon) = -\epsilon_y
\]

where \( \epsilon_y \) is the utility shock from taking the \( y \)-th action. That is,

\[
G^*(p) = \sup_{w(y) + z(\epsilon) \leq c(y, \epsilon)} \{E_p[w(Y)] + E_{Q_\epsilon}[z(\epsilon)]\}
\]  

(10)

which, by the Monge-Kantorovich duality, coincides with its dual

\[
G^*(p) = \min_{Y \sim p, \epsilon \sim Q_\epsilon} E[c(Y, \epsilon)]
\]  

(11)

and \( w \in \partial G^*(p) \) if and only if there exists \( z \) such that \((w, z)\) solves (10).

In Eq. (11) above, the minimum is taken across all joint distributions of \((Y, \epsilon)\) with marginal distribution equal to, respectively, \( p \) and \( Q_\epsilon \). It follows from the proposition that the main problem of identification of the choice-specific value functions \( w(x) \) can be recast as a mass transportation problem (Villani (2003)), in which the set of optimizers to Eq. (10) yield vectors of choice-specific value functions \( w \in \partial G^*(p) \).

Moreover, the mass transportation problem can be interpreted as an optimal matching problem. Using a marriage market analogy, consider a setting in which a matched couple consisting of a “man” (with characteristics \( y \sim p \)) and a “woman” (with characteristics \( \epsilon \sim Q_\epsilon \)) obtain a joint marital surplus \(-c(y, \epsilon) = \epsilon_y\). Accordingly, Eq. (11) is an optimal matching problem in which the joint distribution of characteristics \((y, \epsilon)\) of matched couples is chosen to maximum the aggregate marital surplus.
Proof of Proposition 3. This startling connection between the $G^*$ function and a matching model follows from manipulation of the variational problem in Equation (2) defining $G^*$:

$$
G^*(p) = \sup_{w \in \mathbb{R}^Y} \left\{ \sum_y p_y w_y - \mathbb{E}_Q \left[ \max_{y \in Y} (w_y + \epsilon_y) \right] \right\}.
$$

$$
= \sup_{w \in \mathbb{R}^Y} \left\{ \sum_y p_y w_y + \mathbb{E}_Q \left[ \min_{y \in Y} (-w_y - \epsilon_y) \right] \right\}.
$$

Defining $c(y, \epsilon) \equiv -\epsilon_y$, one can rewrite the above as

$$
G^*(p) = \sup_{w(y) + z(\epsilon) \leq c(y, \epsilon)} \left\{ \mathbb{E}_p [w(Y)] + \mathbb{E}_Q [z(\epsilon)] \right\}.
$$

As is well-known from the results of Monge-Kantorovich (Villani (2003), Thm. 1.3), this is the dual-problem for a mass transportation problem. The corresponding primal problem is

$$
G^*(p) = \min_{Y \sim p, \epsilon \sim \hat{Q}} \mathbb{E} [c(Y, \epsilon)] \int_{Y} \int_{\epsilon}
$$

which is equivalent to (16)-(18). \hfill \square

In the case when $Q_\epsilon$ is a discrete distribution, the mass transportation problem in the above proposition reduces to a linear-programming problem which coincides with the assignment game of Shapley and Shubik (1971). This connection suggests a convenient way for efficiently computing the $G^*$ function (along with its subgradient). These computational details are the focus of Section 4 below.

3.1.1. Normalization. The proposition above shows how we can tractably compute the subdifferential $\partial G^*(p(\cdot|x))$ at each vector of observed choice probabilities $p(\cdot|x)$, for any given distribution of the unobserved variable. The true choice-specific value function $w(x)$ lies in this set of subgradients $\partial G^*(p(\cdot|x))$.

From examining the social surplus function $G$, we see that if $w(x) \in \partial G^*(p(\cdot|x))$, then it is also true that $w(x) - K(x) \in G^*(p(\cdot|x))$, where $K(x) \in \mathbb{R}^{|Y|}$ is vector taking values of $K(x)$ across all $Y$ components. In the context of the mass transportation problem in Proposition 3, the solutions are non-unique. If $w$ solves mass transportation problem
(Equation 10), then \( w_* - K \) solves Equation 10 as well. Indeed, the choice probabilities are only affected by the differences in the levels offered by the various alternatives.\(^6\)

For our estimation procedure, we tackle this indeterminacy problem by isolating a particular \( w^0(x) \) among those satisfying \( w(x) \in \partial G^*(p(\cdot|x)) \), in particular:

\[
G(w^0(x)) = 0. \quad (12)
\]

The next proposition shows that when the distribution of the unobservable heterogeneity \( \epsilon \) has full support, then the normalization implied by Eq. (12) defines \( w^0 \) unambiguously. We stress that the utility vector \( w^0(x) \) which satisfies Eq. (12) need not satisfy Eqs. (7) or (8). However, as expressed in our next result, all \( w(x) \) satisfying (3) are of the form \( w^0(x) - K(x) \), for a vector of (state-dependent) constants \( K(x) \); hence, the “true” \( w(x) \) – that which satisfies all the Eqs. (3), (7) and (8) – will differ from \( w^0(x) \) by a constant term \( K(x) \). Moreover, this proposition also pins down the value of this constant term.

**Theorem 1.** Assume that the distribution \( Q_\epsilon \) of the utility shocks \((\epsilon_0, \ldots, \epsilon_K)\) is such that \( \{\epsilon_k - \epsilon_0\}_k \) has full support on the real line. Let \( w(x) \) be the true choice-specific value function. Then:

- (i) There exists a unique \( w^0(x) \in \partial G^*(p(\cdot|x)) \) such that \( G(w^0(x)) = 0 \),
- (ii) \( w(x) \in \partial G^*(p(\cdot|x)) \) if and only if there exists \( K(x) \) such that \( w(x) = w^0(x) - K(x) \),
- (iii) \( K(x) = -V(x) \).

The proof of this theorem is in the Appendix. By appealing to the convexity of the problem, Propositions 2-3 show that we can draw connections between estimating dynamic discrete choice models and the problem of mass transportation (or assignment games in the finite case). This formulation in terms of mass transportation gives us a tractable way to compute \( G^*(p) \) and \( \partial G^*(p) \), for any distribution of \( \epsilon \). Theorem 1 then identifies a unique element \( w^0 \) from the set \( \partial G^*(p) \).

\(^6\) This indeterminacy issue arises also in static discrete choice models, but the typical remedy in that simpler setting is to set the utility for one of the alternatives to zero.
3.2. Second step. From the first step, we obtained $w_0^0(x)$ such that $w_0(x) = w_0^0(x) + V(x)$. Now in the second step, we use the recursive structure of the dynamic model (encapsulated in Eq. (8)), along with a normalization on the per-period payoffs, to jointly pin down the values of $w_0(x)$ and $V(x)$. Finally, once $w_0(x)$ and $V(x)$ are known, the per-period utilities can be obtained from $ar{u}_y(x) = w_0^0(x) - \beta \mathbb{E}[V(x')|x,y]$

In order to nonparametrically identify $ar{u}_y(x)$, we need to impose a normalization. Following Bajari, Chernozhukov, Hong, and Nekipelov (2009), we assume:

**Assumption 2** (Per-period utility normalization). $\forall x$, $\bar{u}(y_0, x) = 0$.

With this assumption, we get

$$0 = w_0^0(x) + V(x) - \beta \mathbb{E}[V(x')|x,y = y_0]. \quad (13)$$

Let $W$ be the column vector whose general term is $(w_0^0(x))_{x \in \mathcal{X}}$, let $V$ be the column vector whose general term is $(V(x))_{x \in \mathcal{X}}$, and let $\Pi^0$ be the $|\mathcal{X}| \times |\mathcal{X}|$ matrix whose general term $\Pi^0_{ij}$ is $Pr(x_{t+1} = j|x_t = i, y = y_0)$. Equation (13), rewritten in matrix notation, is

$$W = \beta \Pi^0 V - V$$

and for $\beta < 1$, matrix $I - \beta \Pi$ is a diagonally dominant matrix. Hence, it is invertible and Equation (13) becomes

$$V = (\beta \Pi^0 - I)^{-1} W. \quad (14)$$

The right hand side of this equation is uniquely estimated from the data. After obtaining $V(x)$, $\bar{u}_y(x)$ can be nonparametrically identified by

$$\bar{u}_y(x) = w_0^0(x) + V(x) - \beta \mathbb{E}[V(x')|x,y], \quad (15)$$

where $w_0^0(x)$ is as in Theorem 1, and $V$ is given by (14).

As a sanity check, one recovers $\bar{u}_{y_0}(. ) = W + V - \beta \Pi^0 V = 0$. Also, when $\beta \to 0$, one recovers $\bar{u}_y(x) = w_0^0(x) - w_{y_0}^0(x)$ which is the case in standard static discrete choice.

Eqs. (14) and (15) above, showing how the per-period utilities can be recovered from the choice-specific value functions via a system of linear equations, echoes similar derivations
in the existing literature (e.g. Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008), Arcidiacono and Miller (2011, 2013)). Hence, the innovative aspect of our estimator lies not in the second step, but rather in the first step, in which we show how the choice-specific value functions can be recovered for any assumed distribution of the utility shocks \((\epsilon_0, \ldots, \epsilon_K)\) conditional on \(x\). In the next section, we focus on computational aspects of this first step.

3.3. **Comparison with existing estimation approaches.** Our estimation procedure is distinctive in several ways. The estimation procedures proposed in much of the literature on identification and estimation of dynamic models require one of the two following steps. First, some procedures require “inverting” the mapping between choice probabilities and choice-specific value functions (e.g. Hotz and Miller (1993), Magnac and Thesmar (2002)). Proposition 2(ii) provides perhaps the most specification characterization of this inverse mapping, equating the set of choice-specific value functions consistent with a vector of observed choice probabilities to the subdifferential of \(G^*\), the convex conjugate of the social surplus function.\(^7\)

Second, existing procedures also rely on a small class of distributions for the utility shocks – primarily those in the extreme-value family, as in Example 1 above – because these distributions yield analytical (or near-analytical) formulas for the choice probabilities and \(\{\mathbb{E}[\epsilon_y | y, x]\}_y\), the vector of conditional expectation of the utility shocks for the optimal choices, which is required in order to recover the per-period utility functions.\(^8\) Our approach, however, works for any assumed distribution for the utility shocks;\(^9\) it is based on computing the \(G^*\) function (which can be done by linear programming, as we shall show in following

\(^7\)This remark is also relevant for static discrete choice models. In fact, the random-coefficients multinomial demand model of Berry, Levinsohn, and Pakes (1995) does not have a closed-form expression for the choice probabilities, thus necessitating a simulation-based inversion procedure. In ongoing work (Chiong, Galichon, Shum (2013)), we are exploring the estimation of random-coefficients discrete-choice demand models using our approach.


\(^9\)Norets and Tang (2013) propose another estimation approach for binary dynamic choice models in which the choice probability function is not required to be known.
section). This easily accommodates different choices for $Q$, the (joint) distribution of the utility shocks $(\epsilon_0, \ldots, \epsilon_K)$ conditional on $X$. Therefore, our findings expand the set of dynamic discrete-choice models suitable for applied work far beyond those with extreme-value distributed utility shocks.

4. Computation Details for First Step

In Section 3.1, we show that the problem of identification in DDC models can be formulated as an optimal transportation problem. In this section, we consider how this may be implemented in practice. In showing how to compute $G^*$, we exploit the connection, alluded to above, between this function and the assignment game, a model of two-sided matching with transferable utility which have been used to model marriage and housing markets (such as Shapley and Shubik (1971) and Becker (1973)). In particular, we use a linear programming formulation of the optimal transportation problem, which coincides with the classic assignment game.

4.1. Linear programming approach. Let $\hat{Q}_\epsilon$ be a discrete approximation to the distribution $Q_\epsilon$. Specifically, consider a $S$-point approximation to $Q$, where the support is $\text{Supp}(\hat{Q}_\epsilon) = \{\epsilon^1, \ldots, \epsilon^S\}$. Let $Pr(\hat{Q}_\epsilon = \epsilon^s) = q_s$. The best $S$-point approximation is such that the support points are equally weighted, $q_s = \frac{1}{S}$, i.e. the best $\hat{Q}$ is a uniform distribution.\(^{10}\) Therefore, let $\hat{Q}$ be a uniform distribution whose support can be constructed by drawing $S$ points from the distribution $Q_\epsilon$. It is also known that $\hat{Q}_\epsilon$ converges to $Q_\epsilon$ almost uniformly as $S \to \infty$, so that the approximation error from this discretization will vanish when $S$ is large. Under these assumptions, Problem (10)-(11) has a Linear Programming

\(^{10}\)cf. Kennan (2006).
formulation as

\[
\max_{\pi \geq 0} \sum_{y,s} \pi_{ys} \epsilon_y^s \tag{16}
\]

\[
\sum_{s=1}^{S} \pi_{ys} = p_y, \ \forall y \in Y
\]

\[
\sum_{y \in Y} \pi_{ys} = q_s, \ \forall s \in \{1, ..., S\}. \tag{17}
\]

For this discretized problem, the set of \( w \in \partial G^* (p) \) is the set of vectors \((w)_y\) of Lagrange multipliers corresponding to constraints (17). To see how we recover \( w^0 \), the specific element in \( \partial G^* (p) \) as defined in Theorem 1, we begin with the dual problem

\[
\min_{\lambda, z} \sum_{y \in Y} p_y \lambda_y + \sum_{s=1}^{S} q_s z_s
\]

s.t. \( \lambda_y + z_s \geq \epsilon_y^s \). \tag{19}

Consider \((\lambda, z)\) a solution to (19). By duality, \( \lambda \) is a vector of Lagrange multipliers associated to constraint (17), and \( z \), as Lagrange multipliers associated to constraint (18).\(^{11}\) We have \( G^* (p) = \sum_{y \in Y} p_y \lambda_y + \sum_{s=1}^{S} q_s z_s \), which implies that \( G (\lambda) = - \sum_{s=1}^{S} q_s z_s. \)\(^{12}\) Also, because two elements \( \lambda, w^0 \in \partial G^* (p) \) yield the same value for \( G^* (p) \), we have \( \sum_{y \in Y} p_y \lambda_y - G (\lambda) = \sum_{y \in Y} p_y w^0_y - G (w^0) \). Hence, because \( w^0 \) satisfies \( G (w^0) = 0 \), we get

\[
w^0_y = \lambda_y - G (\lambda) = \lambda_y + \sum_{s=1}^{S} q_s z_s.
\]

This quantity converges to the true value of \( w^0_y \) when \( S \) is large enough.\(^{13}\)

\(^{11}\)Because the two linear programs (16) and (19) are dual to each other, the Lagrange multipliers of interest \( \lambda_y \) can be obtained by computing either program. In practice, for the simulations and empirical application below, we computed the primal problem (16).

\(^{12}\)This uses Eq. (24) in Appendix A, which (in our setup) states that \( G^* (p) + G (\lambda) = p \cdot \lambda \), for all Lagrange multiplier vectors \( \lambda \in \partial G^* (p) \).

\(^{13}\)In Appendix C, we present an alternative approach to computing the \( G^* \) function, based on “power diagrams”.
4.1.1. Discretization of $Q_{\epsilon}$ and a second type of indeterminacy issue. Thus far, we have proposed a procedure for computing $G^{*}$ (and the choice-specific value functions $w_0$) by discretizing the otherwise continuous distribution $Q_{\epsilon}$. However, as because the support of $\epsilon$ is discrete, $w_0^y$ will generally not be unique. This is due to the non-uniqueness of the solution to the dual of the LP problem in Eq. (16), and corresponds to Shapley and Shubik’s (1971) well-known results on the multiplicity of the core in the finite assignment game. Applied to discrete-choice models, it implies that when the support of the utility shocks is finite, the utilities from the discrete-choice model will only be partially identified. In this section, we discuss this partial identification, or indeterminacy, problem further.

Recall that

$$
G^{*}(p) = \sup_{w(y) + z(\epsilon) \leq c(y,\epsilon)} \{ E_p[w(Y)] + E_Q[z(\epsilon)] \}
$$

(20)

where $c(y,\epsilon) = -e_y$. In Proposition 3, this problem was shown to be the dual formulation of an optimal assignment problem.

We call identified set of payoff vectors, denoted by $I(p)$, the set of vectors $w$ such that

$$
\Pr \left( w_y + e_y \geq \max_{y'} \{ w_{y'} + e_{y'} \} \right) = p(y)
$$

(21)

and we denote by $I_0(p)$ the normalized identified set of payoff vectors, that is the set of $w \in I(p)$ such that $G(w) = 0$. Note that if $Q$ were to have full support, $I_0(p)$ would contain only one element so $I_0(p) = \{ w^0 \}$ as in Theorem 1. Instead, when the distribution $Q$ is discrete, the set $I_0(p)$ contains a multiplicity of vectors $w$ which satisfy (3). One has:

**Proposition 4** (Identified set). The following holds:

(i) The set $I(p)$ coincides with the set of $w$ such that there exists $z$ such that $(w, z)$ is a solution to (20). Thus

$$
I(p) = \left\{ w : \exists z, \begin{array}{l}
   w(y) + z(\epsilon) \leq c(y,\epsilon) \\
   E_p[w(Y)] + E_Q[z(\epsilon)] = G^{*}(p)
\end{array} \right\}.
$$

Note that Theorem 1 requires $\epsilon$ to have full support.
(ii) The set $I_0(p)$ is determined by the following set of linear inequalities

$$I_0(p) = \left\{ w : \exists z, \ E_p[w(Y)] = \mathcal{G}^*(p) \right\}.$$  

This result allows us to easily derive identification bounds using the characterization of the identified set using linear inequalities. Indeed, for each $y \in Y$, we can obtain upper (resp. lower) bounds on $w_y$ by maximizing (resp. minimizing) $w_y$ subject to the linear inequalities characterizing $I_0(p)$, which is a linear programming problem.

Furthermore, when the dimensionality of discretization ($S$) is high, the core typically shrinks to a singleton, and the core collapses to $\{w_0\}$. A detailed discussion of this (along with counterexamples) is provided in Gretsky, Ostroy, and Zame (1999, section 6). In our Monte Carlo experiments below, we provide evidence for the size of this indeterminacy problem under different levels of discretization.

5. Monte Carlo Evidence

In this section, we illustrate our estimation framework using a dynamic model of resource extraction. To illustrate how our method can tractably handle any general distribution of the unobservables, we use a distribution in which shocks to different choices are correlated. We will begin by describing the setup.

At each time $t$, let $x_t \in \{1, 2, \ldots, 20\}$ be the state variable denoting the size of the resource pool. There are three choices,

$y_t = 0$: The pool of resources is extracted fully. $x_{t+1} \mid x_t, y_t = 0$ follows a multinomial distribution on $\{1, 2, 3, 4\}$ with parameter $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$. The per-period payoff is $u(y_t = 0, x_t) = 0.5 \sqrt{x_t} - 2 + \epsilon_0$.

$y_t = 1$: The pool of resources is extracted partially. $x_{t+1} \mid x_t, y_t = 1$ follows a multinomial distribution on $\{\max\{1, x_t - 10\}, \max\{2, x_t - 9\}, \max\{3, x_t - 8\}, \max\{4, x_t - 7\}\}$ with parameter $\pi$. The per-period payoff is $u(y_t = 1, x_t) = 0.4 \sqrt{x_t} - 2 + \epsilon_1$. 
\( y_t = 2 \): Agent waits for the pool to grow and does not extract. \( x_{t+1} | x_t, y_t = 2 \) follows a multinomial distribution on \( \{x_t, x_t + 1, x_t + 2, x_t + 3\} \) with parameter \( \pi \). We normalized the per-period payoff to be \( u(y_t = 2, x_t) = \epsilon_2 \).

The joint distribution of the unobserved state variables is given by \( (\epsilon_0 - \epsilon_2, \epsilon_1 - \epsilon_2) \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 1 \end{pmatrix} \right) \). Other parameters we fix and hold constant for the Monte Carlo study are the discount rate, \( \beta = 0.9 \) and \( \pi = (0.3, 0.35, 0.25, 0.10) \).

**Figure 1.** Comparison between the estimated and true per-period utilities.

5.1. **Asymptotic performance.** As a preliminary check of our estimation procedure, we show that we are able to recover the per-period utilities using the actual conditional choice probabilities implied by the underlying model. We discretized the distribution of \( \epsilon \) using \( S = 1000 \) support points. As is clear from Figure 1, the estimated per-period payoffs (plotted as dots) as a function of states matched the actual utility functions very well.

5.2. **Finite sample performance.** To test the performance of our estimation procedure when there is sampling error in the CCPs, we generate simulated panel data of the following form: \( \{y_{it}, x_{it} : i = 1, 2, \ldots, N; t = 1, 2, \ldots, T\} \) where \( y_{it} \in \{0, 1, 2\} \) is the dynamically
optimal choice at $x_{it}$ after the realisation of simulated shocks. We vary the number of cross-section observations $N$ and the number periods $T$, and for each combination of $(N, T)$, we generate 100 independent datasets.\footnote{In each dataset, we initialised $x_{i1}$ with a random state in $X$.}

For each replication or simulated dataset, the root-mean-square error (RMSE) and $R^2$ are calculated, showing how well the estimated $\bar{u}_y(x)$ fits the true utility function. The averages are reported in Table 1.


table
<table>
<thead>
<tr>
<th>Design</th>
<th>RMSE($y = 0$)</th>
<th>RMSE($y = 1$)</th>
<th>$R^2(y = 0)$</th>
<th>$R^2(y = 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 100, T = 100$</td>
<td>31.0570</td>
<td>0.2582</td>
<td>-0.0482</td>
<td>0.7687</td>
</tr>
<tr>
<td>$N = 100, T = 500$</td>
<td>4.1041</td>
<td>0.1318</td>
<td>0.7019</td>
<td>0.9215</td>
</tr>
<tr>
<td>$N = 100, T = 1000$</td>
<td>0.2922</td>
<td>0.1053</td>
<td>0.9320</td>
<td>0.9458</td>
</tr>
<tr>
<td>$N = 200, T = 100$</td>
<td>9.7096</td>
<td>0.1813</td>
<td>0.4363</td>
<td>0.8525</td>
</tr>
<tr>
<td>$N = 200, T = 200$</td>
<td>4.7124</td>
<td>0.1434</td>
<td>0.6858</td>
<td>0.9110</td>
</tr>
<tr>
<td>$N = 500, T = 100$</td>
<td>1.2857</td>
<td>0.1170</td>
<td>0.8734</td>
<td>0.9363</td>
</tr>
<tr>
<td>$N = 500, T = 500$</td>
<td>0.0976</td>
<td>0.0812</td>
<td>0.9706</td>
<td>0.9716</td>
</tr>
<tr>
<td>$N = 1000, T = 100$</td>
<td>0.1076</td>
<td>0.0901</td>
<td>0.9647</td>
<td>0.9647</td>
</tr>
<tr>
<td>$N = 1000, T = 1000$</td>
<td>0.0716</td>
<td>0.0644</td>
<td>0.9841</td>
<td>0.9817</td>
</tr>
</tbody>
</table>

Table 1

5.3. **Size of the identified set of payoffs.** As mentioned in Section 4.1.1, using a discrete approximation to the distribution of the unobserved state variable introduces a partial identification problem: the identified choice-specific value functions might not be unique. Using simulations, we next show that the identified set of choice-specific value functions (which we will simply refer to as “payoffs”) shrinks to a singleton as $S$ increases, where $S$ is the number of support points in the discrete approximation of $Q_\epsilon$. For $S$ ranging from 100 to 1000 (which is the number of points used in the previous Monte Carlo exercises), we plot in Figure ??, the differences between the largest and smallest choice-specific value function
for \(y = 3\) across all values of \(p \in \Delta^3\) (using the linear programming procedures described in Section 4.1.1).

\[\begin{align*}
0 & \quad 0.02 \\
0.04 & \quad 0.06 \\
0.08 & \quad 0.1 \\
0.12 & \quad 0.14 \\
0.16 & \quad 0.16 \\
\end{align*}\]

\text{Number of discretized points, } S

\text{Upper bound}

\text{Size of the identified set of payoff for choice } y=3

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The identified set of payoffs shrinks to a singleton across \(\Delta^3\).}
\end{figure}

For each value of \(S\), we plot the values of the differences \(\max_{w \in \partial G^*(p)} w - \min_{w \in \partial G^*(p)} w\) across all values of \(p \in \Delta^3\). In the boxplot, the central mark is the median, the edges of the box are the 25th and 75th percentiles the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.

As is evident, even at small \(S\), the identified payoffs are very close to each other in magnitude. At \(S = 1000\), where computation is near-instantaneous, for most of the values in the discretised grid of \(\Delta^3\), the core is a singleton; when it is not, the difference in the estimated payoff is less than 0.01. Similar results hold for the choice-specific value functions for choices \(y = 1\) and \(y = 2\), which are plotted in the Appendix. To sum up, it appears that this indeterminacy issue in the payoffs is not a worrisome problem for even very modest values of \(S\).

\[\text{Figure 2. The identified set of payoffs shrinks to a singleton across } \Delta^3.\]

\[\text{For each value of } S, \text{ we plot the values of the differences } \max_{w \in \partial G^*(p)} w - \min_{w \in \partial G^*(p)} w \text{ across all values of } p \in \Delta^3. \text{ In the boxplot, the central mark is the median, the edges of the box are the 25th and 75th percentiles the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.}\]

\[\text{As is evident, even at small } S, \text{ the identified payoffs are very close to each other in magnitude. At } S = 1000, \text{ where computation is near-instantaneous, for most of the values in the discretised grid of } \Delta^3, \text{ the core is a singleton; when it is not, the difference in the estimated payoff is less than 0.01. Similar results hold for the choice-specific value functions for choices } y = 1 \text{ and } y = 2, \text{ which are plotted in the Appendix. To sum up, it appears that this indeterminacy issue in the payoffs is not a worrisome problem for even very modest values of } S.}\]

\[\text{The analogous plots of the largest and smallest choice-specific value functions for } y = 1 \text{ and } y = 2 \text{ are Figures ?? and ?? in the Appendix.}\]
In this section, we apply our estimation procedure to the bus engine replacement dataset first analyzed in Rust (1987). In each week $t$, Harold Zurcher (bus depot manager), chooses $y_t \in \{0, 1\}$ after observing the mileage $x_t \in X$ and the realised shocks $\epsilon_t$. If $y_t = 0$, then he chooses not to replace the bus engine, and $y_t = 1$ means that he chooses to replace the bus engine. The states space is $X = \{0, 1, \ldots, 29\}$, that is, we divided the mileage space into 30 states, each representing a 12,500 increment in mileage since the last engine replacement.\footnote{17} Harold Zurcher manages a fleet of 104 identical buses, and we observe the decisions that he made, as well as the corresponding bus mileages at each time period $t$. The duration between $t + 1$ and $t$ is a quarter of a year, and the dataset spans 10 years. Figures 5 and 6 in the Appendix summarize the frequencies and mileage at which replacements take place in the dataset.

Firstly, we can directly estimate the probability of choosing to replace and not to replace the engine for each state in $X$. Also directly obtained from the data is the Markov transition probabilities for the observed state variable $x_t \in X$, estimated as:

$$
\hat{Pr}(x_{t+1} = j|x_t = i, y_t = 0) = \begin{cases} 
0.7405 & \text{if } j = i \\
0.2595 & \text{if } j = i + 1 \\
0 & \text{otherwise}
\end{cases}
$$

$$
\hat{Pr}(x_{t+1} = j|x_t = i, y_t = 1) = \begin{cases} 
0.7405 & \text{if } j = 0 \\
0.2595 & \text{if } j = 1 \\
0 & \text{otherwise}
\end{cases}
$$

\footnote{17} This grid is coarser compared to Rust’s (1987) original analysis of this data, in which he divided the mileage space into increments of 5,000 miles. However, because replacement of engines occurred so infrequently (there were only 61 replacement in the entire ten-year sample period), using such a fine grid size leads to many states that have zero probability of choosing replacement. Our procedure – like all other CCP-based approaches – fail when the vector of conditional choice probability lies on the boundary of the simplex.
For this analysis, we assumed a normal mixture distribution of the error term, specifically, $$\epsilon_t \sim 0.5 \cdot N(0, 1) + 0.5 \cdot N(0, \frac{1}{1+0.1x})$$. We chose this mixture distribution in order to allow the utility shocks to depend on mileage – which accommodates, for instance, maintenance costs which may be more volatile and unpredictable at different levels of mileage. At the same time, these specifications for the utility shock distribution showcase the flexibility of our procedure in estimating dynamic discrete choice models for any general error distribution. For comparison, we repeat this exercise using an error distribution that is homoskedastic, i.e., its variance does not depend on the state variable $$x_t$$. The result appears to be robust to using different distributions of $$\epsilon_{t0} - \epsilon_{t1}$$. We set the discount rate $$\beta = 0.9$$.

To non-parametrically estimate $$\bar{u}(y = 0, x)$$, we normalized $$\bar{u}(y = 1, x)$$ to 0 for all $$x \in X$$. Hence, our estimates of $$\bar{u}(y = 0, x)$$ should be interpreted as the magnitude of maintenance costs relative to replacement costs, with positive values implying that replacement costs exceed maintenance costs. The estimated per-period payoff from choosing $$y = 0$$ (don’t replace) relative to $$y = 1$$ (replace engine) are plotted in Figure 3. We only present estimates for mileage within the range $$x \in [9, 25]$$, because only in this range are there enough replacements to allow for reliable estimates (cf. footnote 17 and Figure 6).
Within this range, the estimated utility function is roughly constant, and ranges within the narrow band between 9 and 9.5. That utility is positive means that replacement costs are higher than maintenance costs. In other words, our results suggest that the average maintenance cost to be constant when the mileage goes beyond the rule-of-thumb cutoff point of 100,000 miles. It is worth noting that Rust (1987) mentioned: “According to Zurcher, monthly maintenance costs increase very slowly as a function of accumulated mileage.”

To get an idea for the effect of sampling error on our estimates, we bootstrapped our estimation procedure. For each of 100 resamples, we randomly drew 80 buses with replacement from the dataset, and re-estimated the per-period utilities \( \bar{u}(y = 0, x) \) using our procedure. The results are plotted in Figure 4. The evidence suggests that we are able to obtain fairly tight cost estimates for states that are reached often enough; that is, mileage states between \( x = 11 \) (137,500 miles) and \( x = 22 \) (275,000 miles). However, the estimates are quite noisy not only at the extreme of the range of \( x \) for which we are able to estimate utility functions (at \( x < 11 \) and \( x > 22 \)). This is because there are few replacements at these values of \( x \) (cf. Figure 6), which lead to large volatility in the utility estimates across resamples.

7. Conclusions

In this paper, we have shown how results from convex analysis can be fruitfully applied to study identification in dynamic discrete choice models; modulo the use of these tools, a large class of dynamic discrete choice problems with quite general utility shocks becomes no more difficult to compute and estimate than the Logit model encountered in most empirical applications. This has allowed us to provide a natural and holistic framework encompassing the papers of Rust (1987), Hotz and Miller (1993), and Magnac and Thesmar (2002). While the identification results in this paper are comparable to other results in the literature, the convex analysis approach appears new. Far more than providing a mere reformulation, this approach is powerful, and has significant implications in several dimensions:

First, by drawing the (surprising) connection between the computation of the \( G^* \) function and the computation of optimal matchings in the classical assignment game, we can apply
Figure 4. Boostrapped estimates of per-period payoffs $\tilde{u}(y = 0, x)$

We plot the values of the bootstrapped resampled estimates of $\tilde{u}(y = 0, x)$. In each boxplot, the central mark is the median, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the 5th and 95th percentiles.

the powerful tools developed to compute optimal matchings to dynamic discrete-choice models. Moreover, by reformulating the problem as an optimal matching problem, all existence and uniqueness results are inherited from the theory of optimal transportation. For instance, the uniqueness of a systematic utility rationalizing the consumer's choices follows from the uniqueness of a potential in the Monge-Kantorovich theorem.

We believe the present paper opens a more flexible way to deal with discrete choice models. While identification is exact for a fixed structure of the unobserved heterogeneity, one may wish to parameterize the distribution of the utility shocks and do inference on that parameter. The results and methods developed in this paper may also extend to dynamic discrete games, with the utility shocks reinterpreted as players' private information (see,  

\[ 18 \text{ While the present paper has used standard Linear Programming algorithms such as the Simplex algorithm, other, more powerful matching algorithm such as the Hungarian algorithm may be efficiently put to use when the dimensionality of the problem grows.} \]
e.g. Aguirregabiria and Mira (2007) or Pesendorfer and Schmidt-Dengler (2008)). However, we leave these directions for future exploration.

REFERENCES


Appendix A. Background results

A.1. Convex analysis for Discrete-choice Models. Here, we give a brief review of the main notions and results used in the paper. We keep an informal style and do not give proofs, but we refer to Rockafellar (1970) for an extensive treatment of the subject.

Let $u \in \mathbb{R}^{|Y|}$ be a vector of utility indices. For utility shocks $\{\epsilon_y\}_{y \in Y}$ distributed according to a joint distribution function $Q_{\epsilon}$, we define the social surplus function as

$$G(u) = \mathbb{E}[\max_y \{u_y + \epsilon_y\}],$$  \hspace{1cm} (22)

where $u_y$ is the $y$-th component of $u$. If $\mathbb{E}(\epsilon_y)$ exists and is finite, then the function $G$ is a proper convex function that is continuous everywhere. Moreover assuming that $Q_{\epsilon}$ is sufficiently well-behaved (for instance, if it has a density with respect to the Lebesgue measure), $G$ is differentiable everywhere.

Define the Legendre-Fenchel conjugate, or convex conjugate of $G$ as $G^*(p) = \sup_{u \in \mathbb{R}^{|Y|}} \{p \cdot u - G(u)\}$. Clearly, $G^*$ is a convex function as the supremum of affine functions. Note that the inequality

$$G(u) + G^*(p) \geq p \cdot u$$  \hspace{1cm} (23)

holds in general. The domain of $G^*$ consists of $p \in \mathbb{R}^{|Y|}$ for which the supremum is finite. In the case when $G$ is defined by (22), it follows from Norets and Takahashi (2013) that the domain of $G^*$ contains the simplex $\Delta^{|Y|}$, which is the set of $p \in \mathbb{R}^{|Y|}$ such that $p_y \geq 0$ and $\sum_{y \in Y} p_y = 1$.

The subgradient $\partial G(u)$ of $G$ at $u$ is the set of $p \in \mathbb{R}^{|Y|}$ such that

$$p \cdot u - G(u) \geq p \cdot u' - G(u')$$

holds for all $u' \in \mathbb{R}^{|Y|}$. Hence $\partial G$ is a set-valued function or correspondence. $\partial G(u)$ is a singleton if and only if $G(u)$ is differentiable at $u$; in this case, $\partial G(u) = \nabla G(u)$. 
One sees that \( p \in \partial G(u) \) if and only if \( p \cdot u - G(u) = G^*(p) \), that is if equality is reached in inequality (23):

\[
G(u) + G^*(p) = p \cdot u. \tag{24}
\]

By symmetry in (24), one sees that \( p \in \partial G(u) \) if and only if \( u \in \partial G^*(p) \). In particular, when both \( G \) and \( G^* \) are differentiable, then \( \nabla G^* = \nabla G^{-1} \).

**Appendix B. Proofs**

**Proof of Proposition 1.** Consider the \( y \)-th component, corresponding to \( \frac{\partial G(w, \epsilon)}{\partial w_y} \):

\[
\frac{\partial G(w, \epsilon)}{\partial w_y} = \frac{\partial}{\partial w_y} \int_y \max[w_y + \epsilon_y]dQ_{\epsilon} = \int \frac{\partial}{\partial w_y} \max[w_y + \epsilon_y]dQ_{\epsilon} = \int 1(w_y + \epsilon_y \geq w_{y'} + \epsilon_{y'}, \forall y' \neq y)dQ_{\epsilon} = p(y). \tag{27}
\]

(We have suppressed the dependence of \( x \) for convenience.)

**Proof of Proposition 2.** This follows directly from Fenchel’s inequality (see Rockafellar (1970), Theorem 23.5).

**Proof of Theorem 1.** In this proof we shall drop \( x \) from the notation for the sake of clarity.

For a vector \( w \) we shall denote \( Y(w, \epsilon) \) be the value of \( y \) which maximizes \( w_y + \epsilon_y \).

Because \( \epsilon \) has full support, the choice probabilities \( p \) will lie strictly in the interior of the simplex \( \Delta^3 \). Let \( \tilde{w} \in \partial G^*(p) \), and let \( w_y = \tilde{w}_y - G(\tilde{w}) \). One has \( G(w) = 0 \), and an immediate calculation shows that \( \partial G(w) = p \). Let us now show that \( w \) is unique. Consider \( w \) and \( w' \) such that \( G(w) = G(w') = 0 \), and \( p \in \partial G(w) \) and \( p \in \partial G(w') \). Assume \( w \neq w' \) to get a contradiction; then there exist two distinct \( y_0 \) and \( y_1 \) such that \( w_{y_0} - w_{y_1} \neq w'_{y_0} - w'_{y_1} \); without loss of generality one may assume

\[
w_{y_0} - w_{y_1} > w'_{y_0} - w'_{y_1}.
\]
Let $S$ be the set of $\varepsilon$'s such that

$$w_{y_0} - w_{y_1} > \varepsilon_{y_1} - \varepsilon_{y_0} > w'_{y_0} - w'_{y_1}$$

$$w_{y_0} + \varepsilon_{y_0} > \max_{y \neq y_0, y_1} w_y + \varepsilon_y$$

$$w'_{y_1} + \varepsilon_{y_1} > \max_{y \neq y_0, y_1} w'_y + \varepsilon_y$$

Because $\varepsilon$ has full support, $S$ has positive probability.

Let $\bar{w} = \frac{w + w'}{2}$. Because $p \in \partial \mathcal{G}(w)$ and $p \in \partial \mathcal{G}(w')$, one has $\mathcal{G}(\bar{w}) = 0$, thus

$$0 = \mathbb{E}[\tilde{w} Y(\bar{w}, \varepsilon) + \varepsilon Y(\bar{w}, \varepsilon)] = \frac{1}{2} \mathbb{E}[w Y(\bar{w}, \varepsilon) + \varepsilon Y(\bar{w}, \varepsilon)] + \frac{1}{2} \mathbb{E}[w' Y(\bar{w}, \varepsilon) + \varepsilon Y(\bar{w}, \varepsilon)]$$

$$\leq \frac{1}{2} \mathbb{E}[w Y(w, \varepsilon) + \varepsilon Y(w, \varepsilon)] + \frac{1}{2} \mathbb{E}[w' Y(w', \varepsilon) + \varepsilon Y(w', \varepsilon)]$$

$$= \frac{1}{2} (\mathcal{G}(w) + \mathcal{G}(w')) = 0$$

Hence equality holds term by term, and

$$w Y(w, \varepsilon) + \varepsilon Y(w, \varepsilon) = w Y(\bar{w}, \varepsilon) + \varepsilon Y(\bar{w}, \varepsilon)$$

$$w' Y(w', \varepsilon) + \varepsilon Y(w', \varepsilon) = w' Y(\bar{w}, \varepsilon) + \varepsilon Y(\bar{w}, \varepsilon)$$

For $\varepsilon \in S$, $Y(w, \varepsilon) = Y(\bar{w}, \varepsilon) = y_0$ and $Y(w', \varepsilon) = Y(\bar{w}, \varepsilon) = y_1$, and we get the desired contradiction.

Hence $w = w'$, and the uniqueness of $w$ follows.

Finally, to prove (iii), we will exploit the structure of the dynamic optimization problem in order to determine the value of $K$

From $\mathcal{G}(w^0(x)) = 0$ and $w^0(x) = w^0(x) - K(x)$, it follows that $\mathcal{G}(w(x)) = -K(x)$. But we also know that $V(x) = \mathcal{G}(w(x))$, therefore we get $K(x) = -V(x)$, so that

$$w(x) = w^0(x) + V(x).$$
Proof of Proposition 4. (i) follows from Proposition 3 and the fact that if \( w(y) + z(\epsilon) \leq c(y, \epsilon) \), then \( \mathbb{E}_p[w(Y)] + \mathbb{E}_Q[z(\epsilon)] = G^*(p) \) if and only if \((w, z)\) is a solution to the dual problem.

(ii) follows from the fact that \(-z(\epsilon) = \sup_y \{ w(y) - c(y, \epsilon) \} = \sup_y \{ w(y) + \epsilon_y \} \), thus \( \mathbb{E}_Q[z(\epsilon)] = 0 \) is equivalent to \( \mathbb{E}_Q[\sup_y \{ w(y) + \epsilon_y \}] = 0 \), that is \( G(w) = 0 \). □

Appendix C. Power cells approach for computing \( G^* \) function

Second, we can give geometric insights for the locus of \( \epsilon \) which lead to the choice of some given \( y \). For this, we need to reinterpret the utility shock \( \epsilon_y \) as a scalar product in a higher dimensional space – a classical trick. For \( y \in \mathcal{Y} \), let \( \iota_y \in \{0, 1\}^\mathcal{Y} \) the vector such that \((\iota_y)_{y'} = 1(y = y')\). Introduce \( S_y = \{\iota_y : y \in \mathcal{Y}\} \), which is nothing else than the canonical basis of \( \mathbb{R}^\mathcal{Y} \). Denoting \( \cdot \) the scalar product in \( \mathbb{R}^\mathcal{Y} \), one has \( \epsilon_y = \epsilon \cdot \iota_y \), and letting \( P \) be the distribution over \( S_y \) which gives probability \( p_y \) to point \( \iota_y \), problem (11) rewrites as

\[ G^*(p) = -\max_{\epsilon \sim Q_y} \mathbb{E}[\epsilon \cdot Z]. \]

Hence, \( -G^*(p) \) is the value of a Monge-Kantorovich problem with a quadratic surplus. This problem is very well studied, and by Brenier’s theorem, there exists a convex map \( V : \mathbb{R}^\mathcal{Y} \to \mathbb{R} \) such that the optimal coupling \((Z, \epsilon)\) is such that \( Z \in \partial V(\epsilon) \). As a result, \( Y^* \) is defined in (1) is related to \( \epsilon \) by

\[ \iota_{Y^*} \in \partial V^w(\epsilon) \]  \hspace{1cm} (28)

where \( V \) is a convex, piecewise linear function given by \( V^w(\epsilon) = \max_{y \in \mathcal{Y}} \{ \epsilon \cdot \iota_y - w_y \} \). Because \( V^w \) is a convex function, it is (Lebesgue-) almost everywhere differentiable, so if the distribution of \( \epsilon \) is absolutely continuous, then \( \nabla V^w(\epsilon) \) exists almost surely, and (28) rewrites as \( \iota_{Y^*} = \nabla V^w(\epsilon) \).

Define \( C^w_y \) as the set of \( \epsilon \) which lead to the choice of \( y \), that is

\[ C^w_y = \{ \epsilon \in \mathbb{R}^\mathcal{Y} : \iota_y \in \partial V^w(\epsilon) \}. \]

\( C^w_y \) are closed convex polytopes which are called Power Diagrams in combinatorial geometry, see Aurenhammer (1987). The probability of choice of \( y \) is hence \( Q(C^w_y) \), the mass assigned
by distribution $Q$ to set $C_w^y$. Routines in combinatorial geometry provide the computation of the area of $C_w^y$. Note that

$$Q(C_w^y) = \int_{C_w^y} dQ$$

which can also be approximated using simulation techniques.

Once $Q(C_w^y)$ is computed, we can use the following result to obtain $w^y$:

**Theorem 2.** When $Q_{\epsilon}$ has full support, Problem (10)-(11) reformulates as

$$G^\star(p) = -\min_{w \in \mathbb{R}^Y} \sum_{y \in Y} p_y w_y + Q(C_w^y).$$

Problem (29) is a convex optimization problem and can be solved using a gradient descent of the form

$$w_y^{t+1} = w_y^{t+1} + \delta \left( w_y - Q(C_w^t) \right).$$

**Appendix D. Additional Figures**
For each value of $S$, we plot the values of the differences $\max_{w \in \partial G^* (p)} w_1 - \min_{w \in \partial G^* (p)} w_1$ across all values of $p \in \Delta^3$. In the boxplot, the central mark is the median, the edges of the box are the 25th and 75th percentiles the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.
For each value of $S$, we plot the values of the differences $\max_{w \in \partial G^* (p)} w_2 - \min_{w \in \partial G^* (p)} w_2$ across all values of $p \in \Delta^3$. In the boxplot, the central mark is the median, the edges of the box are the 25th and 75th percentiles the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.
Figure 5

Figure 6