State dependent monetary policy *

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Abstract

Departures from Friedman’s rule occur when markets are incomplete and liquidity injections transfer wealth from rich to poor agents who need insurance. We develop a stylized model where the stochastic evolution of the wealth distribution drives the business cycle and evaluate how to regulate liquidity as a function of aggregate fluctuations. Within a class of state contingent policies we find that the one that maximizes welfare prescribes liquidity expansions in recessions (when insurance is most needed), and liquidity contractions otherwise. Interestingly, in spite of the sporadic liquidity expansions, this policy echoes Friedman’s principle: on average the liquidity supply shrinks.

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1 Introduction

Friedman’s (1969) conjecture that the optimal money supply should be shrinking at the rate of time preference holds true in a variety of contexts in the presence of complete markets.\footnote{See, for example, Chari, Christiano and Kehoe (1996).} However in incomplete markets models where liquidity is essential for trading Friedman’s rule may not be implementable, as in Bewley (1983), or not desirable as in Kehoe, Levine and Woodford (1990), Levine (1991), Berentsen, Camera and Waller (2005), Green and Zhou (2005), Lagos and Wright (2005), Deviatov (2006), Molico (2006), Deviatov and Wallace (2012) and Wallace (2012) who show that a constant monetary expansion may be optimal. In these models the underlying motive for the optimality of an expansionary policy is a demand for insurance, provided through the wealth redistribution caused by lump sum monetary transfers.

This paper studies the provision of liquidity in a model that features aggregate fluctuations. The novelty is to investigate how the policy should depend on the state of the business cycle, a question that is motivated by the fact that the demand for insurance varies along the business cycle. To the best of our knowledge the analysis of state dependent policy in an environment in which money is essential has not been previously analyzed. The simplicity of the underlying model allows us to cast some light on this difficult question. For expositional simplicity and tractability the model abstracts from sticky prices and other frictions. Within a class of state contingent policies we find that the one that maximizes welfare prescribes liquidity expansions in recessions (when insurance is most needed), and liquidity contractions otherwise. Interestingly, in spite of the beneficial effects of sporadic expansions, the state contingent policy echoes Friedman’s principle in the sense that “on average” liquidity is expected to contract.

To study how to regulate the supply of liquidity we use an analytically tractable model of production and savings that builds on Scheinkman and Weiss (1986). The economy is populated by two groups, each one composed of many agents. The two groups oscillate randomly between productive and unproductive spells, and at each instant only the agents of one group are productive. Since the state is not observable agents cannot issue private debt, i.e. they face a borrowing constraint. Unproductive agents can consume by acquiring goods in the competitive market, so that the liquid asset (“money”) is valued in equilibrium.

We extend Scheinkman and Weiss’s analysis, which assumes a constant money supply, by letting the government control the money supply through lump-sum transfers.\footnote{As in Levine (1991) we assume that the government does not know which agent is productive, so that the transfers are equal across agents. See Kehoe, Levine and Woodford (1990) for a thorough discussion of this assumption and in particular Levine (1991) for a careful derivation of the equal-treatment restriction from first principles.} We provide
an analytical characterization of the price of the liquid asset and of aggregate production in terms of the policy rule and the wealth distribution. A feature of the competitive equilibrium that we study is that rich-productive agents are relatively less interested in trading goods for assets than poor-productive agents. It follows that trade volumes, aggregate production and the asset price depend on the distribution of wealth, which evolves through time following the history of shocks. The time series generated by this economy displays a “business cycle” in spite of the fact that the production possibility frontier is constant through time.

We adopt an ex-ante welfare criterion and characterize the welfare implications of various policies. The analysis proceeds as follows. For a given policy rule we provide an analytical characterization of the equilibrium, which is summarized by a system of differential equations for the value of money in the productive and unproductive state and associated boundary conditions. This system completely defines the behavior of the economy. However, since the system cannot be solved in closed form, the welfare analysis involves a numerical evaluation across equilibria, each one indexed by a different policy rule. A useful characteristic of the model is that prices, values, and allocations are homogeneous in the two exogenous parameters of the model: the discount rate, $\rho$, and the Poisson rate at which agents switch their productive state, $\lambda$. Thus our problem has essentially a single parameter, namely the normalized discount rate $\rho/\lambda$, a feature that helps in the analysis.

Two types of rules are considered for the provision of liquidity. The first is a constant growth rate (positive, zero or negative), which can be thought of as the policy rule when nor the individual state nor aggregate variables are observable to the policy maker. This flat (or state independent) policy has been previously considered in a variety of environments where money is essential, which are briefly recalled below. The second rule allows to liquidity supply to depend on some aggregate features of the economy, such as aggregate production (or the asset price). The analysis of this “state dependent” policy is the main innovation of this paper. Policy affects the distribution of wealth and its real effects that involve two important margins: the first one is that an increase in the supply of liquidity provides insurance to agents who incur in a long spell of unproductive periods (who end-up having low liquidity and low consumption). The second margin is the classic cost of inflation: an expansionary policy lowers the return on liquid assets, lowering productive agents’ incentives to save and produce.

An optimal policy trades off the provision of insurance versus the cost of inflation. When policy cannot be state dependent we show that an expansionary policy is desirable if the insurance motive outweighs the production incentives. This happens when the normalized discount rate is sufficiently high; for example, this can happen when the average length of a productive state, given by $1/\lambda$, is high enough. This result is reminiscent of the optimal
(steady) expansions characterized by Kehoe, Levine and Woodford (1990), Levine (1991), Green and Zhou (2005), Deviatov (2006), Molico (2006), Deviatov and Wallace (2012). In contrast, a state dependent policy is able to decouple (at least partially) the provision of insurance from the incentives to produce, improving social welfare. The optimal state dependent policy is expansionary only in the states in which the unproductive agents is very poor, a situation that corresponds to a recession in the model. Otherwise, the policy prescribes a contraction of the liquidity supply. Interestingly, in an economy where a state independent rule optimally prescribes an expansionary policy, the state dependent policy only uses the expansions sporadically, so that the average money growth rate is negative. In particular we find that in states where output is sufficiently high the rate at which liquidity contracts is larger than the rate of time preference. This “overshooting” of the Friedman rule is necessary because the expected return on money involves future expected money growth so that in good times policy “compensates” agents for the future expected low returns due to monetary expansions. The behavior of the money supply around the unconditional “mean” state of the economy is very close to the rate of time preference. To the best of our knowledge our characterization of the ex-ante optimal state-dependent rule is novel in the literature, though it is related to the properties of ex-post monetary expansions that are discussed in e.g. Section 3 of Scheinkman and Weiss (1986). Our result also relates to Cavalcanti and Erosa (2008) who show that in an economy with aggregate shocks the return on money should depend on the history of shocks. In spite of its stylized nature our model features many of the ingredients that appear in “policy relevant” models such as Brunnermeier and Sannikov (2012), with whom we share the key features of time-varying (endogenous) risk premia, and the “redistributive” role of monetary policy in the presence of incomplete financial markets.

After a brief summary of the related literature, in the next subsection, Section 2 presents the set up of the model. Section 3 defines a monetary equilibrium, provides a characterization of the value of money and consumption as a function of the wealth distribution, whose law of motion and invariant distribution are also characterized. Section 4 defines an ex ante welfare criterion and studies the best rule for the supply of liquidity. Section 5 concludes.

Related literature

A few previous contributions discuss environments where a flat expansion of the money supply is desirable in an economy with incomplete markets and where money serves an essential role. Levine’s (1991) seminal paper considers an endowment economy where the agents’ (bounded) utility functions change randomly according to whether they are “buyers” or “sellers”, a state that follows an exogenous Markov process. In Levine’s model sellers sell their entire endowment, which amounts to a restriction on the agents’ marginal utilities over
the set of feasible trades. Because of this assumption policy can provide insurance at no cost since altering the relative price has no effect on welfare in the corner solution. Kehoe, Levine and Woodford (1990) extend Levine’s setting to allow for internal solutions where sellers do not necessarily sell all their endowment. For reasons of tractability, they restrict attention to equilibria in which what happens in each period is independent of history (two-state markov equilibria). Compared to these papers, our contribution is to analyze the question of the state dependent policy in the context of a production economy, and to focus on equilibria in which the decisions in each period depend on the whole history of shocks.

Our paper is also related to Molico (2006) which shows that mild monetary expansions can be beneficial in a search model of money. In his model, agents meet randomly bilaterally. Once agents meet, they exchange goods for money. The price paid by the buyer results from bargaining and depends on the amount of money held by each agent upon entering the pairwise meeting. Therefore, the distribution of money is generically non-degenerate and so monetary injections, via lump-sum, can improve the terms of trade for poor buyers. Related results in the context of search models of money are obtained by Green and Zhou (2005), Deviatov (2006), and Deviatov and Wallace (2012). Our model departs from Molico (2006) in a few important ways. First, he evaluates the benefits of expansions by comparing across stationary distributions while we do not (that is, we do not abstract from the transitional paths). Second, we characterize competitive equilibrium rather than a search equilibrium. Third, we evaluate the optimal policy and we allow the planner to tie its policy to the distribution of money.3

Our paper also relates to Cavalcanti and Erosa (2008) which shows that in a search model with pairwise meetings and aggregate shocks, the return on money should be state dependent. To reach to this conclusion the paper characterizes the allocation under a constant return on money and shows that there is a welfare enhancing deviation from a constant return. Our paper differs from this one in several aspects. First, we evaluate a competitive equilibrium rather than a search equilibrium. Second, they restrict attention to equilibria where agents hold either one or zero units of money while we do not impose any restriction of this kind. Third, we characterize the return on money as a function of the aggregate state of the economy. And fourth, we explore optimal policy while they do not.

Our setup shares several features with the quantitative general equilibrium models of Brunnermeier and Sannikov (2010, 2011), such as the assumption of incomplete markets /

3Our paper also relates to Chiu and Molico (2010), which evaluates the welfare costs of inflation in a search model of money as in Lagos and Wright (2005) but with endogenous participation in the centralized market (fixed cost of participation). Participation depends on the current amount of money holdings and realization of participation cost, therefore making the distribution of money an endogenous object. Precautionary motives are not present in this model and therefore inflation is not optimal. Chiu and Molico (2010) deals with a stationary environment and do not evaluate the optimality of state dependent policies.
financial frictions, the random productive abilities of agents, and the equilibrium interaction between the wealth distribution, the value of liquidity and total production. Compared to our model these papers add the following elements to be able to directly analyze the role of financial intermediation: specialized financial intermediaries, money and physical capital. We see these approaches as complementary. The simple structure of our economy allows us to derive a sharper characterization of the equilibrium and of the (ex-ante) socially efficient policy. We think our simple framework is useful to interpret the workings of the more realistic, but more involved, quantitative results of larger general equilibrium models. For instance, our characterization of the nature of the ex-ante optimal policy provides a simple interpretation of the result, discussed in Section 6 of Brunnermeier and Sannikov (2010), that monetary expansions may (ex-post) alleviate the effects of recession by redistributing wealth towards poor but productive agents.

A key feature of our model is that business cycles, and the magnitude of fluctuations, depend on the tightness of the borrowing constraint. This relates to the results of Guerrieri and Lorenzoni (2009) and Guerrieri and Lorenzoni (2011), which explore the effects of borrowing constraints on business cycles in a model with liquid assets. The first paper shows that in a model where there is a complementarity between consumption and production decisions a tightening of the borrowing constraint magnifies aggregate shocks. The second paper evaluates the effects of a credit crunch, an exogenous tightening of the borrowing constraint, to show that the interest rate decreases and output drops. In our model, as the borrowing constraint becomes tighter, economic fluctuations become more severe. The relative simplicity of our setup allows us to investigate the optimal (state dependent and independent) provision of liquidity.

Finally, our paper is also related to Hauser and Hopenhayn (2004), which explores a risk neutral environment where two agents have random opportunities to give a favor to the other party. The paper solves for the optimal mechanism and shows that money, albeit not essential, attains the optimal allocation; monetary expansions are necessary as there are histories where an agent runs out of money precluding her from acquiring another favor.

2 The model

This section describes the model economy: agents’ preferences, production possibilities, and markets. Two useful benchmarks are presented: the (efficient) allocation with complete markets and the optimal monetary policy with no uncertainty. To finalize, we also argue that

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4 A related mechanism is explored by Chamley (2010), which shows that a liquidity shock to a fraction of the population can propagate and generate a liquidity trap.
value functions and allocations are homogeneous in the exogenous parameters of interest; this is useful as it allow us to reduce the dimensionality of the problem by doing an appropriate normalization.

We consider two types of infinitely lived agents (with a large mass of agents of each type), indexed by \( i = 1, 2 \), and assume that at each point in time only one type of agent can produce. We further restrict attention to the case where agents of the same type play the same action at every point in time so that we can discuss the model in terms of two representative agents, one of each type. The productive agent transforms labor into consumption one for one, the unproductive agent cannot produce. The productivity of labor is state dependent: the duration of productivity spells is random, exponentially distributed, with mean duration \( 1/\lambda > 0 \). Money is distributed at each time \( t \) between the two agents so that \( m^1_t + m^2_t = m_t \). The growth rate of the money supply at time \( t \) is \( \mu_t \), so that the money supply follows \( m_t = m_0 e^{\mu_t} \) with \( m_0 \) given. As in Scheinkman and Weiss (1986) and Levine (1991), we let the individual state of an agent to be private information, precluding agents from issuing private debt.\(^5\) A key assumption is that agents face a borrowing constraint restricting their unique savings instrument, money, to be non-negative. Because of the assumption of anonymity fiscal policy has limited powers in this setup. Appendix A discusses what allocations can be achieved using tax policy under various assumptions about government powers (commitment vs. no commitment), types of available taxes (lump-sum vs. distortionary), and government knowledge about the state (agent’s type observable vs. not observable).

Let \( \rho > 0 \) denote the time discount rate, \( \omega \) denote a history of shocks and money supply levels, and \( s(t, \omega) = \{1, 2\} \) be an indicator function denoting which agent is productive for a given history \( \omega \) and current time \( t \). Agent \( i \) chooses consumption \( c^i \), labor supply \( l^i \), and depletion of money balances \( \dot{m}^i \), in order to maximize her (time-separable) expected discounted utility,

\[
\max_{\{c^i(t, \omega), l^i(t, \omega), \dot{m}^i(t, \omega)\}} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[ \ln \left( c^i(t, \omega) \right) - l^i(t, \omega) \right] dt \right\} \tag{1}
\]

\(^5\)Having a large mass of agents of each type is important for the argument as it implies that a single agent cannot infer the productive state of a different agent given his own state. Note that if there were only 2 agents, both agents know the state of the other agent, therefore individual states are not any more private information implying that debt contracts are easy to write and fulfill.
subject to the constraints

\[
\dot{m}_i(t, \omega) \leq \left[ l_i(t, \omega) + \tau(t, \omega) - c_i(t, \omega) \right] / \tilde{q}(t, \omega) \quad \text{if } s(t, \omega) = i \\
\dot{m}_i(t, \omega) \leq \left[ \tau(t, \omega) - c_i(t, \omega) \right] / \tilde{q}(t, \omega) \quad \text{and } l_i(t, \omega) = 0 \quad \text{if } s(t, \omega) \neq i \\
m_i(t, \omega) \geq 0 \quad l_i(t, \omega) \geq 0 \quad c_i(t, \omega) \geq 0 \quad m_0^i \text{ given}
\]

where \( \tilde{q}(t, \omega) \) denotes the price of money, i.e. the inverse of the consumption price level, \( \tau(t, \omega) \) denotes a government lump-sum transfer to each agent, and expectations are taken with respect to the processes \( s \) and \( m \) conditional on time \( t = 0 \).

A monetary policy with \( \mu_t > 0 \) is called expansionary, a policy with \( \mu_t < 0 \) is called contractionary. It is immediate that when the money supply is constant for all \( t \) (i.e. \( \mu_t = 0 \) \( \forall t \)) the economy is the one analyzed by Scheinkman and Weiss (1986). For any history \( \omega \) the monetary policy \( \mu_t \) determines the transfers to the agents \( \tau_t \) through the government budget constraint,

\[
\tilde{q}_t \mu_t m_t = 2 \tau_t
\]

The government transfer scheme implies that in the case of a contractionary policy agents must use their money holdings to pay taxes (i.e. \( \tau < 0 \)). The “tax solvency” constraint, \( m_i(t, \omega) \geq 0 \), imposes this restriction. Notice that in the continuous time characterization of the model the tax solvency constraint coincides with the borrowing constraint.\(^6\)

Notice that the government cannot differentiate transfers across agent-types. This follows from the assumption that the identity of the productive type is not known to the government. Levine (1991) shows in a similar setup that, because of anonymity, the best mechanism is linear and resembles monetary policy.

Next we state two important remarks. The first one characterizes a symmetric efficient allocation with complete markets (the proof is standard so we omit it):

**Remark 1** Assuming complete markets and an ex-ante equal probability of each state \( (s = 1, 2) \), the symmetric efficient allocation prescribes the same constant level of consumption, \( \bar{c} \), for both agents, where \( \bar{c} = 1 \).

Thus without borrowing constraints the efficient allocation solves a static problem, and it encodes full insurance: agents consume a constant amount (equal since we assume ex-ante

\(^6\) To see this let \( \Delta \) denote the period length and \( m_i^t \) the money holdings of agent \( i \) at period \( t \). The borrowing constraint is \( m_i^t \geq 0 \), \( i \in \{1, 2\} \), while the tax solvency constraint is \( m_i^t \geq -\frac{\mu}{2} \Delta m_t - \Delta \frac{c_i(t, \omega)}{\tilde{q}(t, \omega)} \) for a productive agent and \( m_i^t \geq -\frac{\mu}{2} \Delta m_t + \Delta \frac{c_i(t, \omega)}{\tilde{q}(t, \omega)} \) for an unproductive one. It is immediate to see that (i) the restrictions imposed by the two types of constraints differ for a given \( \Delta > 0 \), and (ii) the restrictions imposed by the tax solvency constraint approaches the restriction imposed by the borrowing constraint as \( \Delta \downarrow 0 \). See Appendix I, where we write the model in discrete time for period length \( \Delta \), for more details.
identity) and the aggregate output is constant.

The second remark characterizes the optimal monetary policy in the case of no uncertainty. This helps highlighting the essential role of uncertainty in our problem. In particular, consider the case where each agent is productive for $T$ time, and then becomes unproductive for the next $T$ time. Without loss of generality, for the characterization of the stationary equilibria, let us assume that the economy starts in period $t = 0$ with agent 1 being productive and agent 2 owning all the money, so that $m_1^1 = 0$. We have

**Remark 2** Consider deterministic production cycles of length $T$. The symmetric efficient allocation, $c_i^t = \bar{c}$ for $i = 1, 2$ and all $t$ is attained by deflating at the “Friedman rule”, $\mu_t = -\rho$ for all $t$.

This remark, together with the efficient allocation described in Remark 1, shows that without uncertainty this economy replicates Townsend (1980), Bewley (1980) result on the optimality of the “Friedman rule” (see the Online appendix for the proof which is standard).

To conclude notice that, by inspection of the agent problem presented in equations (1) to (4) and the evolution of money supply ($m_t = m_0 e^{\mu t}$), it is easily seen that the problem is homogeneous on $\{\lambda, \rho, \mu_t\}$: allocations (the flows) are homogeneous of degree 0 while prices and values (the stocks) are homogeneous of degree minus 1. This result is a natural consequence of the Poisson rate of changing states $\lambda$, the discount rate $\rho$, and the monetary expansion rate $\mu_t$ all being measured with respect to calendar time.\(^7\) This result is useful as it shows that, after normalizing by $\lambda$, the model has only two parameters: the normalized discount rate, $\rho/\lambda$, and the normalized money growth rate, $\mu_t/\lambda$. Later on, when we treat $\mu_t$ as a policy instrument, this implies that the model has a unique exogenous parameter given by the normalized discount rate.

### 3 Characterization of monetary equilibrium

This section defines a monetary equilibrium and derives the system equations (and boundary conditions) that characterize an equilibrium. We will show that the wealth distribution summarizes the history of shocks and derive expressions for the consumption of agents and the expected return of money as a function of the wealth distribution. The invariant distribution of wealth is also analytically characterized.

\(^7\)See Appendix I where we write the model in discrete time to see that the homogeneity result holds by looking at the problem under $\{\lambda, \rho, \mu_t\}$ with period length $\Delta$ and under $\{k \lambda, k \rho, k \mu\}$ with period length $\tilde{\Delta} \equiv \Delta/k$, for $K > 0$. 

8
We look for an equilibrium where the price of money may depend on the whole history of shocks, as encoded in the current values of the money supply, the distribution of money holdings, and the current state of productivity. Formally we let \( \tilde{q}(t, \omega) = \tilde{q}(m(t, \omega), m^i(t, \omega), s(t, \omega)) \). With a slight abuse of notation this implies \( c^i(t, \omega) = c^i(m(t, \omega), m^i(t, \omega), s(t, \omega)) \), and \( l^i(t, \omega) = l^i(m(t, \omega), m^i(t, \omega), s(t, \omega)) \), and \( \dot{m}^i(t, \omega) = \dot{m}^i(m(t, \omega), m^i(t, \omega), s(t, \omega)) \). In other words, we are looking for an equilibrium that depends solely on three states: the level of the money supply, the distribution of money holdings, and the current state.

By a simple quantity theory argument it is easily seen that the nominal variables of this economy are homogenous of degree one in the level of money, which acts as a numeraire. Hence we simplify the state space using that the nominal price of money is homogenous of degree minus one in the level of the money supply, i.e. \( \tilde{q}(m_t, m^i_t, s_t) = \frac{1}{m_t} q(x^i_t, s_t) \), where \( x^i_t \equiv \frac{m^i_t}{m_t} \) with \( i = 1, 2 \) and \( s_t = 1, 2 \). The variable \( x^i_t \in [0, 1] \) is the share of total money balances in the hands of agent \( i \), i.e. a measure of the wealth distribution. Likewise the consumption rule is homogeneous of degree zero in the level of the money supply \( c^i(m_t, m^i_t, s_t) = c^i(x^i_t, s_t) \).

Note that the state of the economy can then be summarized by the wealth share in the hands of the unproductive agent, which we will denote using the variable \( x_t \) (with no superscript). Note that this new state variable will record discrete jumps every time the identity of the productive type will change: for instance if type 1 switches from being unproductive to being productive the state jumps from \( x_t = x^1_t \) to \( x_t = 1 - x^1_t \). We focus on equilibria where the price of money depends on the share of money in the hands of the unproductive agent (but not on the identity of this agent). We will simply denote this price by the function \( q(x_t) \), i.e. a function of the wealth share of the unproductive agent. Likewise, we let the planner choose a monetary policy \( \mu_t = \mu[x_t] \). Next we define a monetary equilibrium.

**Definition 1** Let \( x_t \) be the wealth share of the unproductive agent. For a given policy rule \( \mu[x_t] \) and an initial distribution of money holdings \( m^i_0 \) for \( i = 1, 2 \), a monetary equilibrium is a price function \( \tilde{q}(t, \omega) = \frac{1}{m_t} q(x_t) \), with \( q : [0, 1] \rightarrow \mathbb{R}^+ \) and a stochastic process \( x_t \) with values in \( [0, 1] \) such that, for all \( t \), all consumers maximize expected discounted utility (equation (1)) subject to (2), (3) and (4), and the market clearing constraint \( c^1(1 - x_t, 1) + c^1(x_t, 2) = l^1(1 - x_t, 1) \) and the government budget constraint are satisfied.

From now on we omit the time index \( t \) to simplify the notation. A straightforward, but important, result is that permanent deflations cannot be attained in equilibrium.\(^8\) We state this result in the next proposition.

\(^8\)This result relates to Bewley (1983) that shows that in a neoclassical growth model with incomplete markets there is no monetary equilibrium if the interest rate is lower than the discount rate.
Proposition 1 There is no monetary equilibrium where $\mu[x] < 0 \forall x$. Moreover, all monetary equilibria must satisfy $\mu[0] \geq 0$.

See Appendix B for a proof. The economics of this result is simple. As the length of the unproductive spell cannot be bounded above, there is a nonzero probability that a poor unproductive agent fails to cover her tax needs. The only way she can fulfill her tax obligations is by keeping half of the money stock and not trading for goods. Because of no trade, money has no value (i.e. $q = 0$), there is no monetary equilibrium, and the allocation is autarkic. Notice that Proposition 1 implies that there is no monetary equilibrium under the Friedman rule. Moreover, the remark shows that, for all rules, possibly those that include monetary contractions, the money growth rate cannot be negative at $x = 0$. The result follows from noticing that when $x = 0$ unproductive agents hold no money and therefore are not able to cover their tax obligations.

Solving the model requires characterizing the marginal value of money given by the lagrange multipliers for $m^i$ in the problem defined in (1): $\tilde{\gamma}(m, x^i, s)$. Without loss of generality, let us look at this problem from the perspective of agent 1. Let $\tilde{\gamma}(m, x^1, 1)$ and $\tilde{\gamma}(m, x^1, 2)$ denote the (un-discounted) multipliers associated to the constraints in equation (2) and (3), respectively, so that e.g. $\tilde{\gamma}(m, x^1, 2)$ measures the marginal value of money when the money supply is $m$, agent 1 holds a share $x^1$ of it and she is unproductive. Likewise, the multiplier $\tilde{\gamma}(m, x^1, 1)$ measures the marginal value of money when the money supply is $m$, agent 1 holds a share $x^1$ of it and she is productive. The first order conditions with respect to $l(t, \omega)$ and $c(t, \omega)$ give

$$\tilde{\gamma}(m, x^1, 1) = \frac{1}{m} q(x^1, 1) \quad \text{and} \quad \tilde{\gamma}(m, x^1, 2) = \frac{1}{m} \frac{q(x^1, 2)}{c^1(x^1, 2)}$$

where we used the homogeneity with respect to $m$ of the price $\tilde{q}$ discussed above. As usual, these conditions equate marginal costs and benefits of an additional unit of money. For a productive agent the marginal benefit, $\tilde{\gamma}(m, x^1, 1)$, equals the cost of obtaining that unit, i.e. the disutility of work to produce and sell a consumption amount $q(x^1, 1)/m$, in nominal terms. For an unproductive agent the marginal cost is the value of the forgone unit of money $\tilde{\gamma}(m, x^1, 2)$, while the benefit is the additional units of consumption that can be bought with it, given by the product of the price $\tilde{q}$ (consumption per unit of money) times the marginal utility of consumption. The homogeneity of $\tilde{q}(m, x^1, 1)$ with respect to $m$ implies that the lagrange multiplier $\tilde{\gamma}$ is also homogenous, i.e. $\tilde{\gamma}(m, x^1, 1) = \gamma(x^1, 1)/m$. We can then rewrite the first order conditions in real terms as

$$\gamma(x^1, 1) = q(x^1, 1) \quad \text{and} \quad \gamma(x^1, 2) = \frac{q(x^1, 2)}{c^1(x^1, 2)} \quad (5)$$
It is shown in Appendix C that for \( x \in (0, 1) \) the lagrange multipliers \( \gamma(x^1, s) \) solve a system of differential equations, which is the continuous time counterpart of the system of Euler equations in discrete time.\(^9\) We present the system following the Hamilton-Jacobi-Bellman representation,

\[
\begin{align*}
(p + \mu[x])\gamma(x^1, 1) &= \gamma_x(x^1, 1)\dot{x}^1(x^1, 1) + \lambda(\gamma(x^1, 2) - \gamma(x^1, 1)) \\
(p + \mu[x])\gamma(x^1, 2) &= \gamma_x(x^1, 2)\dot{x}^1(x^1, 2) + \lambda(\gamma(x^1, 1) - \gamma(x^1, 2))
\end{align*}
\]

(6) (7)

To provide some intuition consider the first equation: when the agent is productive and holds a share of money \( x^1 \) the value flow (discounted by the nominal rate \( (p + \mu[x]) \)) is equal to the change in the marginal value due to the evolution of her money holdings, \( \gamma_x(x^1, 1)\dot{x}^1(x^1, 1) \), and to the expectations of the change in value in case the state switches and the agent becomes unproductive: \( \lambda(\gamma(x^1, 2) - \gamma(x^1, 1)) \).

The next proposition characterizes the functions \( \gamma(x^1, 1) \) and \( \gamma(x^1, 2) \) that solve the system of differential equations:

**Proposition 2** Properties of the marginal value of money in equilibrium.

1. Consider \( \mu[x] = \mu \geq 0 \). Then, \( \gamma(x^1, 1) < \gamma(x^1, 2) \), \( \gamma_x(x^1, 2) < 0 \), \( \gamma_x(x^1, 1) < 0 \), \( \dot{x}^1(x^1, 1) < 0 \), and \( \lim_{x^1 \to 0} \dot{x}^1(x^1, 2) = 0 \) for \( x^1 \in (0, 1) \). Moreover \( \frac{\partial^2 \gamma^1(x^1, 2)}{\partial x^1} > 0 \).

2. If the state dependent policy rule \( \mu[x] \) satisfies

\[
\frac{\gamma(x^1, 1)}{\gamma(x^1, 2)} \leq \frac{1 + \frac{\rho}{1}}{\frac{\mu[x]}{\lambda}} \leq \frac{\gamma(1 - x^1, 2)}{\gamma(1 - x^1, 1)} \quad \text{for} \quad x \in [0, 1],
\]

(8)

then there is an equilibrium where the properties listed in 1) are satisfied.

See Appendix D for the proof. The first part of Proposition 2 characterizes the functions \( \gamma(x^1, 1), \gamma(x^1, 2) \) in a monetary equilibrium for \( \mu[x] = \mu \geq 0 \). It is shown that for any wealth share \( x^1 \) money is more valuable to unproductive agents than to productive agents, i.e. \( \gamma(x^1, 1) < \gamma(x^1, 2) \). The fact that \( \gamma(x^1, 1) \) and \( \gamma(x^1, 2) \) are decreasing implies that the value of money for an agent is decreasing in her wealth. Moreover, it is shown that an unproductive agent will be depleting her wealth share at any level of wealth \( x^1 \), i.e. \( \dot{x}^1(x^1, 2) < 0 \). The limit \( \lim_{x^1 \to 0} \dot{x}^1(x^1, 2) = 0 \) shows that as the wealth of the unproductive agent approaches zero it is optimal for her to spend all the money transfer. Notice also that the consumption of an

\(^9\)In Appendix I we write the model in discrete time and derive the system of Euler equations as a function of the period length parameter \( \Delta \). It can be seen that equations (6) and (7) are the limit of the Euler equations in discrete time as \( \Delta \) approaches 0. See Dixit (1993) and Stokey (2008) for a rigorous analysis of the relation between discrete and continuous time models.
unproductive agent is increasing in her wealth; as the agent gets poorer she hedges against running out of resources by reducing her consumption. The second part of the lemma extends the result to the case where the money growth rate is state dependent. A sufficient condition for the characterization is that the solution to the system of differential equations for $\gamma(x^1,1)$ and $\gamma(x^1,2)$ is stable which is guaranteed if the monetary policy rule $\mu[x]$ satisfy the bounds in equation (8). Later on these bounds will be useful in the characterization of the state dependent policy.

The fact that the function $\gamma(x^1,1)$ is decreasing implies that the price of money is increasing in the wealth of the unproductive agent. To see this, define $x$ to be the wealth share of the unproductive agent (as in the equilibrium definition given above), assumed to be type 2, so that the wealth share of type 1 is $1-x$. Then, using equation (5), the price of money is $q(x) \equiv q(1-x,1) = q(x,2)$, which will be used throughout the paper. Notice then that $\partial q(x)/\partial x = -\gamma_x(1-x,1) > 0$, which shows that the price of money $q(x)$ is increasing in $x$. Intuitively, as the unproductive agent uses money to consume her wealth decreases and the price of money goes down, i.e. less consumption is bought by 1 unit of money. Such dynamics of the terms of trade reflects the fact that as the productive agent accumulates wealth her interest in the exchange becomes small. This result shows that the terms of trade are history dependent, which in turn generate history dependence in aggregate fluctuations in production.

To complete the description of the equilibrium we provide the evolution of money holdings $\dot{x}^1(x^1,s)$ and the boundary conditions for the marginal value of money $\gamma(x^1,s)$. Consider the law of motion for the share of money held by type 1 when unproductive:

$$\dot{x}^1(x^1,2) = \frac{\dot{m}^1}{m} - \frac{m^1}{(m)^2} \dot{m} = \mu[x] \left( \frac{1}{2} - x^1 \right) - \frac{c^1(x^1,2)}{q(x^1,2)} = \mu[x] \left( \frac{1}{2} - x^1 \right) - \frac{1}{\gamma(x^1,2)} \tag{9}$$

where we used the budget constraint of the unproductive agent, equation (3), the government budget constraint, and the first order condition in equation (5). Because there are two types of agents and shocks are perfectly negatively correlated if unproductive agents money holdings are $x$ productive agents are holding $1-x$. Therefore, $\dot{x}^1(1-x,1) + \dot{x}^1(x,2) = 0$ so that the accumulation rate of money holdings is perfectly negatively correlated across agent’s types. This condition, together with equation (9) can be used to produce the evolution of money holdings for productive agents.

The equation describing the evolution of money holdings for unproductive agents, i.e. equation (9), shows that the optimal change of real money holdings, on top of the government transfer, depends on the value of money for the unproductive agent relative to the value of money for the productive agent: $\gamma(x^1,2)/q(x^1,2) = \gamma(x^1,2)/q(1-x^1,1)$. Notice from the
first order conditions that the smaller is the consumption level of the unproductive agent, the higher is the value of $\gamma(x^1, 2)/q(x^1, 2)$, hence the smaller is be the (absolute) real value of money transfers.

For expositional purposes we rewrite the system of ODEs as a function of the aggregate state $x$ and we substitute in for $\dot{x}^1(x^1, s)$ the corresponding expressions from equation (6) and (7),

$$\frac{\gamma_x(1-x^1, 1)}{\mu[0]} \left[ \frac{1}{\gamma(x^1, 2)} \right] = \frac{(\rho + \lambda + \mu[x^1])}{\gamma(1-x^1, 1)} - \lambda \gamma(1-x^1, 2)$$  

$$\frac{\gamma_x(x^1, 2)}{\mu[0]} \left[ \frac{1}{\gamma(x^1, 2)} \right] = \frac{(\rho + \lambda + \mu[x^1])}{\gamma(x^1, 2)} - \lambda \gamma(x^1, 2)$$

where for each unproductive agent with money holdings $x^1$ there is a productive agent with money holdings $1-x^1$. Note that this is a system of delay differential equations.\footnote{Notice in particular that the delay is non-constant, which prevents an analytical solution in closed form. Notice however that the functions $\gamma(1-x^1, 1), \gamma(x^1, 2)$ are analytical, and that the above system allows us to completely characterize these functions given initial values for $\gamma(\frac{1}{2}, 1), \gamma(\frac{1}{2}, 2)$.}

Now we derive the boundary conditions for this problem. The boundaries concern the state in which the unproductive agent has no money. In this case an unproductive agent spends the whole money transfer to finance her consumption. Appendix D gives a formal proof of this statement. The budget constraint gives $c^1_t = \tau_t = q(0, 2)\mu[0]/2$. Using equation (5),

$$\gamma(0, 2) = \frac{2}{\mu[0]} \quad \text{with} \quad \lim_{\mu[0] \to 0} \gamma(0, 2) = \infty$$

where the limit obtains because of Inada conditions. This is an important result in our analysis. An expansionary policy provides an upper bound to the marginal utility of money holdings because the price of money is always finite in an equilibrium and therefore the agent enjoys positive consumption even when she is very poor. If there is no money growth when the unproductive agent is poor (i.e. as $\mu[0] \to 0$), the agent is not able to consume in poverty and therefore Inada conditions imply that the marginal utility of money approaches infinity.

The second boundary condition is also associated to the state in which the unproductive agent has no money and the productive agent has it all. Evaluating equation (10) at $x^1 = 0$ gives

$$\gamma(1, 2) = \left( 1 + \frac{\rho}{\lambda} + \frac{\mu[0]}{\lambda} \right) \gamma(1, 1)$$

as $\dot{x}^1(1, 1) = 0$ (see Proposition 2). The reason that the share of money remains constant in this state is the following. The unproductive agent, who has a zero share of money,
will spend all the money she receives from the government transfer in consumption goods. Intuitively, as the unproductive agent is impatient and there is a positive probability of becoming productive, saving a part of the transfer when $x^1 = 0$ would be optimal if she expected to remain unproductive and that the value of money will go up in the future. In a rational expectations equilibrium, however, conditional on remaining unproductive this agent expects the value of money to go down. Recall that $x$ measures the fraction of money in hands of unproductive agents. The amount of consumption goods bought by a unit if money, measured by $q(x)$, is increasing in $x$. Hence as $x \downarrow 0$, i.e. as the unproductive agent runs out of money, the real value of money falls.

Notice that $\mu[0]$, the money growth rate when the unproductive agent has zero wealth, appears in both boundaries. It appears in equation (12) because it bounds above the marginal value of money; this constitutes a force for having high $\mu[0]$. It appears in equation (13) because positive money growth erodes the return on money and therefore reduces the production incentives; this pushes $\mu[0]$ downwards. This insurance-production incentives tradeoff when the unproductive agent has zero money shows that policy is not able to completely decouple the two opposing forces, not even with a state dependent policy. This also happens for $x > 0$ as it will be clear when we explore the supply of liquidity.

We can explore a few properties of $\mu[x]$ by connecting the boundary conditions with the bounds described in equation (8). First, it can be seen that any $\mu[0] \geq 0$ satisfies the bounds. Second, after evaluating the bounds at $x = 1$ we get a relation between $\mu[0]$ and $\mu[1]$:

$$1 + \frac{\rho}{\lambda} + \frac{\mu[1]}{\lambda} \geq \left(1 + \frac{\rho}{\lambda} + \frac{\mu[0]}{\lambda}\right)^{-1}.$$  

This inequality is interesting as it bounds the choice of the money growth rate when the unproductive is very rich, i.e. $\mu[1]$, with the choice of money growth rate when she is very poor, i.e. $\mu[0]$: A higher $\mu[0]$ implies that $\mu[1]$ can take lower values. This is interesting because it shows a connection between the policy choices at $x = 0$, where the insurance problem is the highest, and at $x = 1$ when it is low: the provision of insurance when mostly needed allows for policies that aim to maximize production when the insurance problem is low.

### 3.1 The second best nature of policy

In this section we show that there is no monetary policy $\mu$ where both types of agents consume the allocation of the complete markets economy. We state this in the next proposition.
Proposition 3  If $\rho/\lambda > 0$ and finite there is no policy rule, i.e. no function $\mu[x]$, such that $c^1(x^1, 1) = c^1(x^1, 2) = \bar{c}$.

See Appendix E for a proof. In order to prove the proposition we start by noting that the complete markets allocation sustains constant price and marginal value of money. We then use this result to show that no monetary equilibrium in the economy with uncertainty can be supported. When no monetary equilibrium exists, agents do not trade and therefore are not able to insure at all. This implies that they cannot attain constant consumption as prescribed in the complete markets allocation which completes the proof.

To prove that no monetary equilibrium can be supported with a constant marginal value of money we note that this requires either a zero marginal value of money, and a zero price of money, or a contractionary policy at the discount rate $-\rho$. The first trivially cannot constitute a monetary equilibrium as one of the conditions for a monetary equilibrium is positive price level and marginal value of money (see Definition 1 and Proposition 2). When $\mu[x] = -\rho$ we know from Proposition 1 that this cannot constitute a monetary equilibrium as we restricted the normalized discount rate $\rho/\lambda$ to be positive. This last result contrasts with the optimal policy under deterministic fluctuations (see Remark 2) where the complete markets allocation can be attained by deflating at the Friedman rate. The difference is that the length of unproductive spells is random here and therefore agents cannot trade in order to comply with their tax obligations. Both cases imply a collapse of the monetary economy resulting in no trade, money is valueless, and as a result there is no monetary equilibrium.

The state reversal parameter $\lambda$ being finite is important for the result to hold. First, as $\lambda$ approaches infinity the production cycles become deterministic so that insurance motives vanish. Second, as $\lambda$ diverges the normalized discount rate approaches zero, which means that the the discount rate per unit of time becomes arbitrarily small, making Friedman’s rule attainable. Therefore, $\mu = 0$, which is Friedman’s rule when $\lambda \to \infty$, attains the complete markets allocation as described in Remark 1.

3.2 Consumption when unproductive and the return on money

We now discuss the optimal consumption when unproductive. This is an important object because unproductive agents need to spend money to consume and therefore monetary policy will have a strong impact on their consumption behavior. Notice that

$$c^1(x^1, 2) = \frac{\gamma(1 - x^1, 1)}{\gamma(x^1, 2)} \text{ with } \frac{\partial c^1(x^1, 2)}{\partial x^1} > 0$$

(14)
which follows from Proposition 2. The fact that consumption when unproductive is increasing in money holdings, implies that for any given money growth rate consumption is smallest when the agent money holdings are zero, in particular at $x^1 = 0$ we have

$$c^1(0, 2) = \frac{\mu[0]}{2} \gamma(1, 1) \quad , \quad \text{with} \quad \lim_{\mu[0] \to 0} \frac{\mu[0]}{2} \gamma(1, 1) = 0$$

(15)

where the limit obtains from the agent’s budget constraint. This is important for the welfare analysis that will follow because it shows that monetary transfers provide the unproductive agent with a lower bound to her consumption level. Without transfers, an agent with no money cannot consume.

A plot of the consumption function is given in the left panel of Figure 1 for the case where monetary policy is state independent, i.e. $\mu[x] = \mu$. Our baseline parameterization assumes $\rho/\lambda = 1/2$ so that, given the standard value for the discount rate $\rho = 0.05$, $\lambda = 0.1$ which implies that the average length of a productive spell is 10 years. For $\mu = 0$, it can be seen that consumption when unproductive goes to zero as $x^1 \downarrow 0$. Note also that the consumption when unproductive may be above the consumption when productive (i.e. $\bar{c} = 1$) for high enough wealth. To see why note that as $x \to 1$ the productive agent holds almost zero money, and therefore is eager to accumulate money to insure against the possibility of a switch of the state. So the price of money is high (i.e. the price of consumption is low), and the productive agent is willing to produce a lot to refill his low money holdings. A comparison of the consumption function at zero and positive money growth (the two curves in the picture) illustrates the tradeoff of inflation: increasing the money growth rate (from 0 to 5 percent in the figure) provides higher consumption to relatively poor unproductive agents but decreases the consumption of unproductive agents with “high” wealth. An optimal choice of the money growth rate trades off these effects. The next lemma characterizes the consumption behavior as the unproductive agent approaches extreme poverty.

**Proposition 4** For any $\mu[0] > 0$ we have: $\lim_{x^1 \to 0} \frac{c^1(x^1, 2)x^1}{c^1(x^1, 2)} = Q_1$, where $Q_1 = 1 + \frac{\rho}{\mu[0]} + \frac{\lambda}{\mu[0]} \left( 1 - \frac{\gamma(0, 1)}{\gamma(0, 2)} \right) > 1$ and $Q_1$ is finite.

See Appendix F for the proof. Proposition 4 states that when money holdings, $x$, are “low”, the unproductive agent’s elasticity of consumption with respect to money holdings approaches a value that is larger than one. This implies that, around low values for money holdings, i.e. $x \approx 0$, the consumption of a poor unproductive agent decreases at a rate that is higher than the rate at which $x$ falls, so that the boundary $x = 0$ is never hit.

Another interesting variable is the expected return on money or, equivalently, the expected interest rate. This object is useful in understanding the workings of the model and illustrates
how the market incompleteness affects the economy by generating a risk premium. Let \( r(x) \) denote the expected (net) return on money,

\[
r(x) \equiv \mathbb{E} \left[ \frac{\tilde{q}(x)}{\tilde{q}(x)} \bigg| x_t = x \right]
\]

which is just the expected growth rate of the price of money. Using that \( \tilde{q}(x) = \gamma(1 - x, 1)/m \), the expected return on money can be written as (see Appendix J)

\[
r(x) = \rho + \lambda \frac{\gamma(x, 1)}{\gamma(1 - x, 1)} \left( 1 - \frac{1}{c^1(1 - x, 2)} \right).
\]

In a complete markets setting, such as the one described in Remark 2, consumption is constant (at \( \bar{c} = 1 \)) and the expected return on money equals the time discount, \( \rho \). With incomplete markets the return on money is history dependent. There are two channels through which history affects the interest rate. First, through changes in \( x \) when the identity of the productive agent does not change. These effects are “small”, in the sense they are continuous in time and proportional to \( \dot{x} \). The second channel operates through jumps in the value of \( x \), which occur when the identity of the productive agent changes. That is why the probability of a change in identity, \( \lambda \), appears in the equation. The latter effect can be quantitatively large and dominate all other variations. When the current unproductive agents is rich (a high value of \( x \)), a state switch implies that the new unproductive agents is poor. This
implies that the price of consumption might increase if the state switches (i.e. the price of the asset falls), so that the expected return on money is low.\footnote{Recall that the asset price is \( q(x) = \gamma(1 - x, 1) \), which is increasing in \( x \).} Alternatively, if the current unproductive agent is poor (small value of \( x \)), a state switch implies the new unproductive agents is rich. This yields a high expected return on money. This intuitive interpretation of the dynamics of the expected return on money abstracts from the change in consumption profiles that occur if there is no state switch, but much of the action in the return happens in the corners as state reversals imply large swings in the distribution of wealth and therefore in consumption. The expected return on money is plotted in the right panel of Figure 1. It is shown that the expected return is high when unproductive agents are poor. Moreover, a more expansionary policy implies a flatter profile for the expected return, as the changes in consumption after a state switch are smaller.

### 3.3 The stationary density of wealth distribution

This section completes the characterization of the equilibrium by computing the stationary density of wealth (money holdings). This density is useful because it shows that, independently of the monetary policy followed, there is a “non-negligible” amount of histories for which unproductive agents approach extreme poverty. This helps explaining why an expansionary policy, providing a floor to consumption (and utility) of unproductive agents who incur in these histories, might be desirable ex-ante.

Let \( F(x, s) \) denote the CDF for the share of money holdings in state \( s \) with density \( f(x, s) = \frac{\partial F(x, s)}{\partial x} \). The density function of the invariant distribution is derived from the usual Kolmogorov Forward Equation (KFE) after imposing for stationarity. Appendix K derives the KFE for our model with Poisson jumps in the state, which gives

\[
0 = f(x^1, s_i) \dot{x}^1(x^1, s_i) + f(x^1, s_i) \frac{\partial \dot{x}^1(x^1, s_i)}{\partial x} + \lambda \left[ f(x^1, s_i) - f(x^1, s_{-i}) \right]
\]

where \( s_i = 1, 2 \) denotes the current state and \( s_{-i} \) the other state. It is immediate that the densities satisfy \( f(x^1, 2) = f(1 - x^1, 1) \) since, given the assumed symmetry of the shocks, for each agent with money \( x = x^1 \) and \( s = 2 \) there is another agent with money \( 1 - x^1 \) and \( s = 1 \). This allows us to concentrate the analysis on only one density: \( f(x^1, 2) \). Using the
expression for \( \dot{x}^1(x^1, 2) \) derived in equation (9) and \( \dot{x}^1(x^1, 2) + \dot{x}^1(1 - x^1, 1) = 0 \) gives

\[
\frac{f_x(x^1, 2)}{f(x^1, 2)} = \frac{\gamma(x^1, 2) \left( \lambda - \mu[x^1] - \frac{\partial \mu[x^1]}{\partial x^1}(x^1 - \frac{1}{2}) \right)}{1 + \mu[x^1] \left( x^1 - \frac{1}{2} \right) \gamma(x^1, 2)} - \frac{\lambda}{\mu[1 - x^1] \left( \frac{1}{2} - x^1 \right) + \frac{1}{\gamma(1-x^1,2)}} \equiv \Omega(x^1)
\]

It follows that

\[
f(x^1, 2) = Ce^{\int_{x/2}^{x} \Omega(z) \, dz} \quad \text{where} \quad C = \left[ \int_0^1 e^{\int_{x/2}^{x} \Omega(z) \, dz} \, dx^1 \right]^{-1}
\]

where the constant \( C \) ensures that \( \int_0^1 f(x^1, 2) \, dx^1 = 1 \). The next lemma establishes useful properties of the invariant density function for money holdings:

**Proposition 5** For any given \( \mu[0] \geq 0 \) and \( x^1 \in (0, 1) \) the invariant density of money holdings \( f(x^1, 2) \) has the following properties: (i) continuous and differentiable in \( x^1 \), (ii) \( \lim_{x \to 0} f(x^1, 2) = +\infty \), and (iii) \( \lim_{x \to 0} f_x(x^1, 2) = -\infty \).

The details of the proof can be found in Appendix G. The lemma exhibits an important property of the model. That is, for any \( \mu[0] \geq 0 \) the density of unproductive agents diverges to infinity as \( x \) converges to zero. This asymptote plays an important role in providing the grounds for the optimality of an expansionary monetary policy. Note that the asymptote at zero shows that unproductive agents, unless being hit by a shock that reverses their productive state, deplete their money holdings to the point of almost exhaustion. As positive money growth rates provide a floor to consumption (see equation (15)), the asymptote at zero provides the basis for the optimal money growth rate being positive.

The left panel of Figure 2 plots the density function \( f(x^1, 2) \) for two parameterizations, one with zero money growth rate and one with positive money growth rate, under the maintained assumption that \( \rho/\lambda = 1/2 \). In both cases it can be seen that, as shown in Proposition 5, the density function has an asymptote as the share of wealth in hands of an unproductive agent approaches zero. Moreover, the plot shows what happens with the mass “near” zero as we change the money growth rate. As expected, increasing the money growth rate \( \mu \) increases the amount of histories where unproductive agents have little money. This follows because a higher \( \mu \) implies higher insurance, and therefore a lower cost of running out of money. In the right panel of Figure 2 we explore the effect of \( \rho/\lambda \) on shaping the density function. We do so by decreasing \( \rho/\lambda \) from 1/2 to 1/10 in the case where there is no money growth (i.e. \( \mu = 0 \)).

The same qualitative results hold for positive money growth rates.
the mass at average values of $x$ larger. The intuition is as follows. $\rho/\lambda$ falls either because $\rho$ decreases or because $\lambda$ increases. When $\rho$ decreases agents care more about the future, save more, and therefore deplete their money holdings slower. When $\lambda$ increases the shorter the unproductive state. Both cases imply that there is a lower probability that money holdings of unproductive agents gets depleted. This will be important later on in understanding the effect of $\rho/\lambda$ on the optimal monetary policy.

Figure 2: Invariant distribution of wealth $f(x^1, 2)$

Parameters: $\rho/\lambda = 1/2$ (left panel), and $\mu = 0$ (right panel).

4 On the supply of liquidity

In this section we define a welfare criterion and explore the properties of different rules for the supply of liquidity. Our objective is to evaluate a state dependent policy which can be made contingent on some aggregate variables that summarize the state of the economy, such as total output. However, we start by analyzing a state independent policy as it proves useful in understanding how the policy choice strikes a balance between insurance motives and production incentives.

Let $V(x; \mu)$ denote the discounted present value of the sum of utilities of both types of agents, where agents are given the same Pareto weight. This is a function of the money share of unproductive agents, $x$. Monetary policy $\mu = \mu[x]$ affects the value only through changes in the evolution of the state $x$. Still, we make explicit the dependence of the value function on the policy function $\mu$ as it will prove to be useful for the comparative statics analysis.
The continuous time Bellman equation is

\[
\rho V(x; \mu) = \ln c^1(x, 2; \mu) - 1 - c^1(x, 2; \mu) + V_x(x; \mu) \dot{x}(x, 2; \mu) + \lambda [V(1 - x; \mu) - V(x; \mu)]
\] (16)

where \(\dot{x}(x, 2; \mu)\) denotes the evolution of money holdings of an unproductive agent, and where we made explicit the dependence of the value function, consumption, and evolution of money holdings on the money growth rate \(\mu = \mu[x]\).\(^{13}\) The flow value \(\rho V\) is given by the sum of the period utility for both agents plus the expected change in the value function. The latter occurs because of the evolution of assets (the change in \(x\)) as well as of the possibility that identity of the productive agent will change. Notice that in this case the state, i.e. the wealth of the unproductive agent, switches from \(x\) to \(1 - x\).

We consider the problem from an ex-ante perspective, i.e. assuming that at the beginning of time nature assigns the initial productive states and the planner can choose the initial wealth distribution and a policy rule for money growth. We assume that the planner, once it chooses the policy, commits to it. Note that because individual types are not observable, and given the symmetry of the environment (and identical Pareto weights), the planner will give the same amount of liquidity to every agent and therefore at the beginning of time \(x = \frac{1}{2}\).

To compare different policies it is useful to define the welfare of a given policy using a certainty equivalent compensating variation. Let \(\alpha\) denote the consumption equivalent cost of market incompleteness associated with a given policy. That is, \(\alpha\) solves the following equation

\[
2 \ln (1 - \alpha) - 2 = \rho V\left(\frac{1}{2}; \mu\right)
\] (17)

so that \(\alpha\) measures the fraction of the consumption under complete markets that agents would be willing to forego to eliminate the volatility of consumption due to market incompleteness for a given policy rule \(\mu\).\(^{14}\)

\(^{13}\)This Bellman equation follows from writing the problem in discrete time. With a slight abuse of notation let \(V(x; \mu)\) denote the value function in discrete time,

\[
V(x; \mu) = \Delta \left( \ln c^1(x, 2; \mu) - 1 - c^1(x, 2; \mu) + (1 - \Delta \rho) ((1 - \Delta \lambda) V(\tilde{x}; \mu) + \Delta \lambda V(1 - \tilde{x}; \mu)) \right)
\]

where \(\Delta\) is the length of the time period, and \(\tilde{x}\) is the share of money in the hands of unproductive agents the next date. That is, \(\tilde{x} = x + \dot{x}^1(x, 2; \mu) \Delta\). Taking the limit as \(\Delta \downarrow 0\) gives the equation in the text.

\(^{14}\)Recall that under complete markets \(c^1(x^1, 1) = c^1(x^1, 2) = 1 \forall x^1\) and \(l^1(x^1, 1) = 2 \forall x^1\).
4.1 A constant anticipated policy

When the government does not observe individual states or economic outcomes (such as output or prices) the policy consists of a constant growth rate: \( \mu[x] = \mu \forall x \). Once the initial distribution of money \( x = 1/2 \) is fixed, the planner chooses the systematic (constant) money growth rate \( \mu \) to maximize \( V(\frac{1}{2}; \mu) \). This section is useful in understanding the underlying forces that shape up the monetary policy choice: the tradeoff between insurance motives and production incentives.

Notice that the planner would not choose to implement either an “hyperinflation”, i.e. \( \mu \to \infty \), or a systematic deflation, i.e. \( \mu < 0 \), as both cases imply no trade and autarkic allocations. In the first case, as the money growth rate becomes very large, the marginal value of a unit of money becomes small because inflation erases the value of accumulated money holdings. In the limit, as an “hyperinflation” is implemented the value of money becomes nil, so that a productive agent has no incentives to accept money in exchange for goods, and the consumption of an unproductive agent is zero. Likewise, for any contractionary policy, i.e. \( \mu < 0 \), there is no monetary equilibrium and hence no trade, as was shown in Proposition 1. With either type of policy the consumption allocations coincide with those under autarky, so that unproductive agents do not consume. Since agents spend, on average, half of their lives in the unproductive state, their expected utility under this policy diverges to \(-\infty\). In Appendix L we present a formal proof of the statement.

Figure 3: On the best constant monetary policy \( \mu \)

For finite non-negative values of money growth rate \( \mu \) the welfare \( V(\frac{1}{2}; \mu) \) is finite.\(^{15}\) Notice

\(^{15}\)That \( V(\frac{1}{2}; 0) \) is finite follows immediately from Scheinkman and Weiss (1986). For \( \mu > 0 \) note that
that there is only one parameter affecting the value of the policy: the normalized discount rate $\rho/\lambda$. We study the optimal choice of $\mu$ as we vary $\rho/\lambda$ in Figure 3. The figure shows that the optimal $\mu$ increases with $\rho/\lambda$. The reason is that agents with a higher normalized discount rate become poor faster, since their savings motive is weaker. This translates into a higher need for insurance, i.e. a higher growth rate of money. Conversely, when $\rho/\lambda$ is low enough the production incentives outweighs the insurance motives and therefore the optimal choice of $\mu$ is zero. Moreover, when $\rho/\lambda$ approaches zero the economy can actually sustain first best allocations as it attains in this limiting case Friedman rule; this happens because absent the time discount, or when productive states switch instantaneously, the insurance motive vanishes so that production cycles become deterministic as in Remark 2. Overall, what happens is that in an economy with borrowing constraints agents occasionally incur into histories in which they “almost” run out of money. The amount of histories, as discussed in Section 3.3, depends on $\rho/\lambda$: a higher value implies that agents deplete their money holdings faster. Monetary expansions provide a floor to how bad consumption looks in these states. But this insurance provision comes at a cost, as productive agents are less willing to accept money balances in exchange for goods as monetary expansions decrease the return of money. The optimal finite money growth rate strikes a balance between these opposing forces.

### 4.2 A state dependent policy

The analysis of the policy in Section 4.1 assumed that money growth $\mu$ was constant and provided the necessary intuition to understand that the choice of monetary policy strikes a balance between insurance motives and production incentives. Because the terms of trade depend on the aggregate state of the economy, i.e. the price of money changes with $x$, the tradeoff also inherits this property. It follows that a policy that accounts for this dependence should be welfare enhancing. With this in mind we allow the government to observe some economy aggregates, such as the price level or aggregate production, that can be used to condition the policy, under the maintained assumption that the individual state of each agent is not observable. As the aggregate production and the price level are monotone functions of $x$, the wealth share in the hands of the unproductive agent, we assume that the government can tie the policy to $x$. That is, $\mu_t = \mu[x_t]$. This policy is such that the government observes the share of wealth held by unproductive agents, but cannot identify individual agents nor observe transactions. This implies that the complete markets allocation, where both types of agents consume $\bar{c}$ for every value of wealth and state of nature, cannot be attained (see

\[
\int \frac{\ln [\bar{c}(0)+2]}{\rho} - \rho > -\infty \quad \forall \mu > 0, \text{ where } \bar{c}(0) = c^1(0,2).
\]

23
Recall that Proposition 1 showed that \( \mu(x) \) could not be negative for all \( x \) as all trade would break down. However, a state dependent policy implies that \( \mu(x) < 0 \) at least for some values of \( x \). In fact, as long as \( \mu(0) \geq 0 \), negative values of \( \mu(x) \) are feasible for all \( x > 0 \).

We develop the analysis of the state dependent policy by restricting the policy rule \( \mu(x) \) to be piecewise linear. Letting \( \tilde{x} \) denote the value at which the two linear functions cross and \( \tilde{\mu} = \mu(\tilde{x}) \), this functional assumption allows us to explore the role of state dependent policy by a parsimonious parameterization of the policy choices. Obviously this simple functional form nests the constant policy analyzed above, as well as monotone (increasing or decreasing) or \( V \)-shaped and tent-shaped functions. We discuss in the concluding remarks the challenges involved in solving the unconstrained Ramsey plan without imposing a functional form on the policy rule. Section 4.3 discusses alternative functional forms for the policy rule. Given the piecewise linear form assumed for \( \mu(x) \), now the planner chooses 4 parameters \( \{\mu_0, \mu_1, \tilde{\mu}, \tilde{x}\} \), where \( \mu_0 \equiv \mu(0) \) and \( \mu_1 \equiv \mu(1) \).

We compute the best policy that maximizes ex-ante expected welfare, defined in equation (16), by searching on a 4-dimensional grid over \( \mu_0, \mu_1, \tilde{\mu} \) and \( \tilde{x} \), where the set of parameters is constrained in order to satisfy the bounds on \( \mu(x) \) described in equation (8). We compare the best constant – and therefore state independent – policy, \( \tilde{\mu} \), with the best state dependent policy: \( \hat{\mu}(x) \). Figure 4 plots the best state independent and state dependent policies obtained for the baseline parametrization of the model where the normalized discount rate \( \rho/\lambda \) equals 1/2. Given this parameter choice, by setting the discount rate to the standard value of 0.05, implies that \( \lambda \) equals 1/10, so that the average length of a productive state is 10 years. Qualitatively similar results are obtained for other parameterizations for \( \rho/\lambda \).

The best state independent policy \( \tilde{\mu} \) consists of a constant expansion of the monetary base of 0.11% per year. As was discussed in the previous subsection this result is due to the fact that the expected duration of the cycle is sufficiently long so that the corner solution of \( \mu = 0 \) is dominated by a strictly positive (albeit small) money growth rate. The best state dependent policy \( \hat{\mu}(x) \) is very different: its general shape is decreasing in \( x \), and the rule prescribes liquidity injections when unproductive agents are very poor. The monetary growth rate is expansionary when \( x \) is below the “poverty threshold” \( \tilde{x} = 0.02 \); when the unproductive agent’s money holdings get close to the borrowing constraint, \( x \approx 0 \), the policy prescribes an expansion of the money supply at a rate around 5.5 per cent. On the other hand, the policy rule prescribes liquidity contractions for \( x > \tilde{x} \), a contraction rate greater than the rate of time discount is prescribed for \( x > 0.4 \), i.e. \( \mu(x) < -\rho \) for all \( x > 0.4 \). The state dependent pattern of the policy rule can be equivalently interpreted in terms of
Figure 4: Best state dependent and state independent policy rules ($\rho/\lambda = 1/2$)

the business cycle. Since aggregate production is $c^1(1 - x, 1) + c^1(x, 2) = 1 + c^1(x, 2)$ and from Proposition 2 we have that $c^1(x, 2)$ is increasing in $x$, then the policy $\mu[x]$ is such that monetary expansions happen when aggregate production is low and monetary contractions occur when it is high. In other words, the policy is expansionary during recessions and contractionary during expansions. A similar argument for the (ex-post) beneficial effect of liquidity expansions during recessions is developed by Brunnermeier and Sannikov (2010) in the context of a larger quantitative model of a monetary economy.

The comparison of the state independent rule $\bar{\mu}$ with the state dependent policy $\hat{\mu}[x]$ shows that the latter allows the planner to provide more insurance when needed. This can be seen by noting that when $x$ is small, the optimal state dependent policy $\hat{\mu}[x]$ expands the money supply at a faster rate than the optimal state independent policy $\bar{\mu}$. Moreover, the state dependent policy $\hat{\mu}[x]$ provides high production incentives, as shown by the higher expected return on assets in Figure 5 (red dashed line). This characterization shows that the constant policy is some kind of average of the values prescribed by the state dependent one. This follows since the state dependent rule allows the planner to partially decouple insurance motives and production incentives.

Figure 5 illustrates the profiles of the consumption function and the expected return on money under the two policy rules: $\bar{\mu}$ and $\hat{\mu}[x]$. The figure shows that under the best state dependent policy $\hat{\mu}[x]$ the consumption rule for unproductive agents provides a smoother consumption profile as a function of $x$ than is produced under the constant policy $\bar{\mu}$. The smoother consumption profiles of the state dependent rule also yields a flatter profile for the expected return on money $r(x)$, as shown in the right panel of the figure. This flatter profile
Figure 5: Consumption and the return on money under different policy rules ($\rho/\lambda = 1/2$)

Consumption: $c^1(x^1, 2)$

Expected return: $r(x)$

reflects the fact that the consumption of an unproductive agent is less extreme under $\hat{\mu}[x]$: the smaller expected changes in consumption (hence marginal utilities) associated to a state switch dampen the risk premia and lead to a smoother expected return of the asset.

Recall that, as mentioned, for $x > 0.4$ the best state dependent policy prescribes liquidity contractions at a faster rate than $\rho$, the rate prescribed by the Friedman rule. We interpret this feature of the policy rule by the forward looking nature of agents’ decisions: the agents’ expectations of the asset return are a weighted average of future expected periods of monetary expansions and other periods of monetary contractions. The contractions in excess of the rate of time preference partially compensate for the fall in the asset’s expected return due to the anticipated monetary expansions. This raises the mean expected asset return (see Figure 5), and increases the incentives to produce. This is shown in the right panel of the figure which also plots the expected return implied by a state dependent rule which replaces the value of $\mu_1$ prescribed by the best rule with the (negative of the) time discount rate $-\rho$ (green line). It appears that returns are uniformly lower under such a policy.

A summary of some key features of each policy rule is given in Table 1, which reports the welfare costs of market incompleteness and some time-series statistics implied by the two policy rules under our benchmark parametrization with $\lambda = 0.10$ and $\rho = 0.05$. Also, to aid in the analysis, we include the case with constant money in the first column of the table. The first row of Table 1 reports the welfare cost of market incompleteness $\alpha$ using the compensating variation defined in equation (17). Recall that $\alpha$ measures how much of the
Table 1: Economic outcomes under alternative policy rules: summary statistics

<table>
<thead>
<tr>
<th></th>
<th>Constant money $\mu = 0$</th>
<th>State independent $\tilde{\mu} = 0.0011$</th>
<th>State dependent $\tilde{\mu}[x]$ $\rho, \mu_1, \tilde{\mu}, \tilde{\mu}[x] = (0.0545, -0.083, -0.03, 0.028)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Welfare cost $\alpha$</td>
<td>31.2%</td>
<td>31.1%</td>
<td>13.9%</td>
</tr>
<tr>
<td>$x$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mathbb{E} \left[ \ln(\xi(x, 2)) \right]$</td>
<td>-1.7</td>
<td>-1.7</td>
<td>-0.6</td>
</tr>
<tr>
<td>std. dev. $\ln(\xi(x, 2))$</td>
<td>1.5</td>
<td>1.4</td>
<td>0.8</td>
</tr>
<tr>
<td>$\mathbb{E}[x]$</td>
<td>0.32</td>
<td>0.31</td>
<td>0.38</td>
</tr>
<tr>
<td>fraction of time $x &lt; 0.02$</td>
<td>0.07</td>
<td>0.11</td>
<td>0.36</td>
</tr>
<tr>
<td>years to (first) hit $x = 0.02$</td>
<td>133.2</td>
<td>85.1</td>
<td>21.5</td>
</tr>
<tr>
<td>$\mathbb{E}[r(x)]$</td>
<td>0.046</td>
<td>0.04</td>
<td>0.1</td>
</tr>
<tr>
<td>$\mathbb{E}[\mu]$</td>
<td>0</td>
<td>0.0011</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.001)</td>
</tr>
</tbody>
</table>

* standard errors in parenthesis.  

Percentage loss (in consumption equivalent units) relative to a complete markets equilibrium. Computation does not require simulation. Parameter: $\rho/\lambda = 1/2$. The Monte Carlo simulation uses a discretized time grid with a period length of 12 hours.

consumption under the complete markets allocation agents are willing to forego to smooth all the consumption fluctuations that occur under incomplete markets. It can be seen that the state dependent policy $\tilde{\mu}[x]$ is able to provide a much higher welfare to agents: the cost $\alpha$ associated to the constant policy is more than twice the cost associated to the best state dependent policy.

The other lines of the table report some unconditional moments of the model statistics, computed through computer simulations. These statistics are related to the distribution of money holdings $x$, discussed in Section 3.3. Those moments thus combine the state contingent behavior of e.g. the consumption and the expected return on money, which depend on the value of $x$, with the “local time” that the model spends around each value of $x$. The mean and
standard deviation of the unproductive agent instantaneous utility derived from consumption rate indicate why the state dependent policy $\hat{\mu}[x]$ provides higher utility than the constant policy: the mean consumption utility is higher, and less volatile.

Another interesting aspect of $\hat{\mu}[x]$ is that the greater insurance that it provides makes the unproductive agents less fearful of being poor. The fact that more insurance is provided by the state contingent policy makes the agent less afraid of getting close to the borrowing constraint. This can also be seen by analyzing the invariant density of money holdings discussed in Section 3.3, where we showed that there are more histories for which money balances get close to zero as the provision of insurance increases. Starting from an egalitarian distribution of money holdings (i.e. $x = 1/2$) an unproductive agent depletes most of her money holdings in 21 years under $\hat{\mu}[x]$, while it takes 85 years under $\bar{\mu}$ and 133 years with constant money. Likewise, using the poverty threshold $x = 0.02$ defined above, it appears that unproductive agents spend 36% of their time in poverty (i.e. with $x < \bar{x}$) under $\hat{\mu}[x]$, while only 11% of their time under $\bar{\mu}$.

The last row of Table 1 shows that the average money growth rate under $\hat{\mu}[x]$ is negative, but above the rate prescribed in Friedman’s rule (i.e. $-\rho$). Thus, in spite of the fact that the money growth rate is greater than the rate of time preference in many states of the world, namely those where $x > 0.4$, we find that the average money growth rate of the economy is above $-\rho$, as the economy spend a significant amount of time at low values of $x$, where the policy rules prescribes a positive money growth rate.

Finally, although the quantitative features of the optimal policy, and the implied behavior of consumption, naturally depend on the benchmark value of $\rho/\lambda = 1/2$, the qualitative features of the state dependent optimal policy, such as its decreasing profile, and the implied smoothness of consumption and the expected return on money are general features, are obtained for any value for the normalized discount rate $\rho/\lambda$.

### 4.3 Alternative specifications for the policy rule

We conclude by discussing some alternative functional forms for the policy rule that we have explored. The first one involves a minimal deviation from the flat policy of Section 4.1, i.e. a constant money growth rule that allows for a discontinuity at $x = 0$. Since the essence of the borrowing constraint is to prevent the agents to incur into negative $x$, one might conjecture that a different behavior at this point would be welfare improving. While this conjecture holds true, we find that a continuous form of the policy rule yields better outcomes since, given the forward looking behavior of agents in this economy, a policy that is continuous in $x$ allows agents a better smoothing of consumption across states. The Online Appendix reports more details on the solution method and the behavior of economic outcomes under
such a rule. Moreover a comparison with the outcomes produced by the piecewise linear rule is discussed.

Another specification involves allowing for the piecewise linear rule to be “U” shaped. We have that the only states where an expansionary policy is desirable are states where \( x \) is close to zero, i.e. recessions, and that elsewhere the profile of the policy rule was strictly decreasing. We interpreted the decreasing profile of the curve using the idea that the policy uses the states where \( x \) is high to compensate for the low return that will eventually occur when \( x \) is low. One might conjecture, however, that at very high values of \( x \) there might be a benefit from expanding the money supply in order to push \( x \) towards a smaller value so that after a state reversal the new unproductive agent does not hold very low money balances. This logic however ignores the effect that a different rule induces on the behavior of agents, which ultimately determines the speed with which \( x \) moves. To evaluate the possibility of a “U” shaped policy we explored whether deviations from the optimal piecewise rule described above were beneficial. To this end we allow the numerical optimization routine to choose a point \( \bar{x} > \tilde{x} \) where the policy rule could take a different shape, possibly increasing. Our results (available upon request) ruled the out the optimality of such policies. We were not able to found piecewise deviations that improved the ex ante welfare compared to the simpler piecewise rule discussed above.\(^{16}\)

5 Concluding remarks

Friedman’s rule prescribes a liquidity contraction at the rate of time discount. When liquidity is essential for trading, which stems from the private nature of individual histories, an expansionary policy can be desirable due to insurance needs. A natural trade-off arises as expanding the liquidity base dampens the return on the asset, therefore reducing the production incentives. The regulation of liquidity strikes a balance between these two forces.

As we discussed in the paper, insurance needs and production incentives depend on the wealth distribution. The novelty of our paper is that we acknowledge this dependence and we explore how it shapes up the regulation of liquidity. We find that the best policy expands the supply of liquidity when the unproductive agents are poor (when the insurance needs are large), and that it contracts the liquidity base otherwise to maximize production incentives. Because aggregate production is low when the unproductive group is poor and high when they are rich, the best policy can be interpreted as counter-cyclical.

\(^{16}\)We also explored non-linear monotone specifications for the policy rule, parsimoniously parametrized. The results, reported in Lippi and Trachter (2012), were not able to deliver any noticeable improvement in welfare compared to the ones reported in this paper. In fact, the piecewise linear policy dominates this non-linear specifications in terms of welfare.
Some interesting extensions are left for future work. First, it would be interesting to provide a more general characterization of the socially optimal policy. Our analysis in Section 4.1 assumed a parametric functional form for the policy rule (monotone and differentiable). More generally, one might want to solve for the state-dependent policy using a Ramsey plan as in Chang (1998), where the planner chooses the policy by picking the best competitive equilibria implied by it. One key difference is that in solving this problem we would not be imposing a functional form on the policy rule. This application is involved because in our problem the set of continuation values will depend on the aggregate state, due to the presence of a physical state variable (wealth). We conjecture that this might be dealt with by applying the ideas developed in Fernandes and Phelan (2000) and Phelan and Stacchetti (2001).

Another extension is to improve the quantitative properties of the model. The assumption that the shocks are perfectly correlated makes the model tractable, but perhaps unrealistic. In Appendix O we explore a “Bewley” model with a continuum of agents whose shocks are uncorrelated. In this model aggregate production is constant so that there is no room for studying how to control liquidity over the business cycle. It is shown that in this model a socially efficient policy also involves ex-ante expansions if the normalized discount rate $\rho/\lambda$ is sufficiently high, for the same reasons discussed in Section 4.1. A more realistic model might combine these two extreme cases, by exploring intermediate cases for the shock’s correlations across agents types. Another model that would fit the data better would be to consider a “Bewley” type economy with aggregate shocks to $\rho/\lambda$. These interesting assumptions would come at the cost of losing the model’s tractability, but might be explored by means of numerical simulations, as in the recent related work by Brunnermeier and Sannikov (2011) and Guerrieri and Lorenzoni (2011).

References


Appendix

A Fiscal policy under alternative government powers

A central assumption in our analysis is that the government does not know $s(t, \omega)$, i.e. the identity of the productive type. It is useful to explore the consequences of relaxing this assumption to better understand the nature of the monetary policy problem. Without loss of generality, given the symmetry of the states, let us assume that type 1 is productive and type 2 is not productive. Also, for simplicity, let us set the money supply equal to zero in what follows.

We begin by assuming that the government observes the identity of the productive type and is able to tax productive agents and transfer resources to unproductive agents. We consider two taxing technologies. The first one is lump sum taxes: in this case the productive agent pays a flat tax $\tilde{\phi} = \tilde{c}$, and the government uses the proceedings to finance the consumption of the unproductive agent. It is immediate that under these assumptions the complete markets allocation can be replicated. Alternatively, consider a setup where the only available taxes are distortionary, say proportional to production: then the transfer to the unproductive agent is $\tau = \phi l$. For a generic tax rate $\phi \in (0, 1)$ the consumption of the two agents solves $u'(c_1) = \frac{1}{1-\phi}$, $u'(c_2) = \frac{1}{\phi}$. An ex-ante optimal policy, maximizing the expected utility of the two types with equal weights, gives $\phi = 1/2$. Under this setting the government fiscal policy provides insurance, consumption is constant through time, though the level of consumption is smaller than under complete markets.

<table>
<thead>
<tr>
<th>type known</th>
<th>type not known</th>
</tr>
</thead>
<tbody>
<tr>
<td>lump-sum-tax</td>
<td>distortionary tax</td>
</tr>
<tr>
<td>$u'(\tilde{c}_1) = u'(\tilde{c}_2) = 1$</td>
<td>$u'(\tilde{c}_1) = u'(\tilde{c}_2) = 2$</td>
</tr>
<tr>
<td>Gvt. commitment</td>
<td>No commitment</td>
</tr>
<tr>
<td>$u'(\tilde{c}_1) = u'(\tilde{c}_2) = 1$</td>
<td>$u'(\tilde{c}_1) = 1$, $u'(\tilde{c}_2) = \infty$</td>
</tr>
</tbody>
</table>

Let us next consider a government who does not know the type’s identities. In this case the efficient stationary allocation with $c = \tilde{c}$ at all times for all types can be sustained if the government has the ability to commit to a trigger policy. Suppose the government credibly announces: “productive types must pay a tax $\tilde{c}$ to the government, who will then transfer it to the other types. If at any point in time the tax is not enough to pay for the transfer, the scheme will be shut down and the economy will be left in autarky forever”. Assuming the threat is credible (and discounting is finite) then it is in the interest of every individual agent to comply, because deviating implies that the agent consumption is zero when unproductive
which, due to Inada conditions, delivers an expected utility of $-\infty$ (since, on average, agents are unproductive for half of the times). The various outcomes sustainable under alternative fiscal policy assumptions are summarized in Table 2.

In what follows, we consider a less powerful government than the one depicted above. We assume the government does not know the identity of productive types, and that it cannot commit to trigger policies. In such a situation fiscal policy, i.e. direct taxation, is powerless. Absent a liquid asset, the resource allocation is autarkic, and individuals experience inefficient fluctuations in utility. We next study the powers of monetary policy, under the maintained assumptions of type-ignorance and no-commitment.

**B Proof of Proposition 1**

We first present a proof of the first part of the statement. That is, that there is no monetary equilibrium where $\mu[x] < 0 \ \forall \ x$. We then present a proof of the second part of Proposition 1.

A contractionary policy $\mu[x] < 0 \ \forall \ x$ requires agents to pay lump sum taxes ($\tau[x] < 0 \ \forall \ x$). Consider the case where agent 1 has fraction of money balances $x_1^t$, and the current state of the economy is $s(t, \omega) = 2$, which means that agent 1 is unproductive. If $x_1^t$ is low enough, given that $\lambda > 0$ and finite, the agent will fail to comply with the monetary authorities with non-zero probability. On the other hand, consider the case where $x_1^t = 1$. In this case the agent is able to comply with her tax obligations with certainty, as she can make her consumption profile to be arbitrarily low. This implies that there exists a threshold $\zeta \in (0, 1)$ such that for $x_i^t \geq \zeta$ the agent is able to cover her lifetime tax needs with probability one. Note that the threshold must be independent of the current state as with positive probability the states are reversed. In the next claim we characterize this threshold.

**Claim 1** If $\mu_t < 0 \ \forall \ t$, for any state of the world $s(t, \omega)$, there is a unique threshold: $\zeta = 1/2$, and a unique ergodic set where $x_1^t = x_2^t = 1/2$, $\ \forall \ t$, that ensures tax solvency.

**Proof.** We will first prove by contradiction that $\zeta \notin [0, 1/2)$. Then we will show that $\zeta = 1/2$ is enough to cover the lifetime tax obligations. Suppose that $\zeta < 1/2$. Without loss of generality assume that $x_1^t \in (\zeta, 1/2)$ and agent 1 is unproductive. Conditional on no reversal of the state, it follows that $x_1^{t+dt} < x_1^t$. Then for a given $\Delta \in \mathbb{R}^+$, $\Pr[x_1^{t+\Delta} < \zeta] > 0$ and therefore the agent will fail to comply with her tax obligations with positive probability. Then, $\zeta \notin [0, 1/2)$. Consider now the case where $x_1^t = \zeta = 1/2$. As the agent can decide not to trade she can always keep her share of outstanding money balances $x^1$ above $1/2$ and therefore for any $\mu \in (0, 1)$ she will be able to cover her tax needs. That $x^1 = 1/2 \ \forall \ s$ is the ergodic set is trivial. If $x_0^1 < 1/2$ there is a positive probability that an agent fails to pay for
her lifetime taxes. An unproductive agent with money holdings \(x^1 > 1/2\) is willing to buy goods (and the productive one with \(x^1 < 1/2\) willing to take the money) until \(x^1\) reaches \(1/2\).

Intuitively, given the uncertain duration of the productivity spell, the only value of money holdings that ensures compliance with tax obligations for both types of agents is \(x^1 = x^2 = 1/2\). At this point, for any history of shocks, the identical lump-sum (negative) transfers reduce the money holdings of both agents proportionally, leaving the wealth distribution unaffected. This leads us to

**Remark 3** Let \(\mu < 0\): In the ergodic set there is no stationary monetary equilibrium and consumption allocations are autarkic.

The proof of Remark 3 follows from noting that Claim 1 implies no trade in the ergodic set. Productive agents have an unsatisfied demand for money and unproductive ones have an unsatisfied demand for consumption goods.

Now we turn to prove that \(\mu[0] \geq 0\). Consider the law of motion for the share of money held by type 1 when unproductive,

\[
\dot{x}^1(x^1, 2) = \frac{\dot{m}^1}{m} - \frac{m^1}{(m^2)} \dot{m} = \mu[x] \left( \frac{1}{2} - x^1 \right) - \frac{c^1(x^1, 2)}{q(x^1, 2)}
\]

because \(c^1(x^1, 2) \geq 0\) and \(q(x^1, 2) > 0\) (i.e. prices are positive in a monetary equilibrium) we can bound above the law of motion \(\dot{x}^1\),

\[
\dot{x}^1(x^1, 2) \leq \mu[x] \left( \frac{1}{2} - x^1 \right)
\]

Notice that because the agent is unproductive \(x^1 = x\). We apply this result into the previous equation and evaluate it at \(x = 0\) to get

\[
\dot{x}^1(0, 2) \leq \frac{\mu[0]}{2}
\]

from where it can be seen that \(\dot{x}^1(0, 2) < 0\) if \(\mu[0] < 0\) which is inconsistent with the borrowing and tax-solvency constraints. Therefore, \(\mu[0] \geq 0\).
C The Euler equation for the marginal utility of money

The Hamilton-Jacobi-Bellman equation implies that the Lagrange multiplier $\hat{\gamma}$ follows

$$ E \{ e^{-\rho(t+dt)} \hat{\gamma}(m(t+dt), x^1(t+dt), s(t+dt)) \} \] m(t, \omega) = m, x^1(t, \omega) = x^1, s(t, \omega) = 1 \}
\approx e^{-\rho t} \left[ -\rho \hat{\gamma}(m_t, x^1_t, 1) dt + \hat{\gamma}_x(m_t, x^1_t, 1) \dot{x}^1_t dt + \hat{\gamma}_m(m_t, x^1_t, 1) m_t dt + \hat{\gamma}(m_t, x^1_t, 1)(1 - \lambda dt) \right] 
+ e^{-\rho t} \hat{\gamma}(m_t, x^1_t, 2) \lambda dt 
= \frac{e^{-\rho t}}{m_t} \left[ -\rho \gamma(x^1_t, 1) dt + \gamma_x(x^1_t, 1) \dot{x}^1_t dt - \gamma(x^1_t, 1) \frac{\dot{m}_t}{m_t} dt + \gamma(x^1_t, 1)(1 - \lambda dt) + \gamma(x^1_t, 2) \lambda dt \right] 
= \frac{e^{-\rho t}}{m_t} \left[ \gamma(x^1_t, 1) + \gamma_x(x^1_t, 1) \dot{x}^1_t dt - \gamma(x^1_t, 1)(\rho + \lambda + \mu[x]) dt + \gamma(x^1_t, 2) \lambda dt \right] 

Subtracting $e^{-\rho t} \gamma(x^1_t, 1)/m_t$ from both sides, dividing by $dt$, taking the limit for $dt \downarrow 0$, gives equation (6). An identical logic gives equation (7).

D Proof of Proposition 2

We start by analyzing the case where $\mu$ is constant. Then, we analyze the case where $\mu = \mu[x]$.

Let $\mu$ be a non-negative constant i.e. $\mu[x] = \mu \geq 0, \forall x$. That $\gamma(x^1_s, s) > 0$ for $s = 1, 2$ is implied for all internal solutions from the Khun-Tucker theorem and increasing utility. Next we show that $\gamma_x(x^1_s, s) < 0$ for $s = 1, 2$, $\gamma(x^1, 1) < \gamma(x^1, 2)$, and $\dot{x}^1(x^1, 2) < 0$. The proof follows by conjecturing that $\dot{x}^1(x^1, 2) < 0$ in the equilibrium, i.e. that the wealth of the unproductive agent decreases as long as she stays unproductive. We then show that $\dot{x}^1(x^1, 2) < 0$ is consistent with the equilibrium, which completes the proof.

Conjecture that $\dot{x}^1(x^1, 2) < 0$. Note that here cannot be an equilibrium where $\gamma_x(x^1, 2) \geq 0$ since the system of differential equations in equation (6) and (7) implies that $\gamma(x^1, 1)$, and the price level $q(x)$, will diverge, which constitutes a violation of the equilibrium definition (see page 29, Figure 1, in Scheinkman and Weiss (1986)). Thus the stability of the system of ODEs requires $\gamma_x(x^1, 2) < 0$.

Now consider $\dot{x}^1(x^1, 2) < 0$ and $\gamma_x(x^1, 2) < 0$. The stability of the system in equation (6) and (7) gives that $\gamma_x(x^1, 1) < 0$: if $\gamma_x(x^1, 1) \geq 0$, eventually $\gamma_x(x^1, 2) > 0$, which constitutes a contradiction. The fact that $\gamma_x(x^1, 1) < 0$ and $\gamma_x(x^1, 2) < 0$ implies that in equilibrium $\gamma(x^1, 1)/\gamma(x^1, 2) < \frac{\lambda}{\rho + \lambda + \mu}$ (the locus $\gamma_x(x^1, 1) = 0$ provides $\gamma(x^1, 1) = \frac{\lambda}{\rho + \lambda + \mu}\gamma(x^1, 2)$). Because $\frac{\lambda}{\rho + \lambda + \mu} < 1$, it follows immediately that

$$ \gamma(x^1, 1) < \gamma(x^1, 2) \ \forall \ x^1 \in (0, 1) \quad (18) $$

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It remains to be shown that the conjecture $\dot{x}^1(x^1, 2) < 0$ is satisfied in equilibrium and that $\gamma_x(x^1, 2) < 0$ is consistent with it. We do both things next.

We show that if $\gamma(x^1, 1) < \gamma(x^1, 2)$ then $\dot{x}^1(x^1, 2) < 0$ and $\gamma_x(x^1, 2) < 0$ for $x^1 \in (0, 1)$, which will complete the proof. Note that the right hand side of equation (7) and the inequality in equation (18) imply that $\gamma_x(x^1, 2)$ and $\dot{x}^1(x^1, 2)$ have the same sign. We now argue that the sign must be negative, otherwise the optimality of the consumption plan is violated. Differentiating the unproductive agent first order condition (equation (5)) gives

$$\frac{\dot{c}^1(x^1, 2)}{\gamma(x^1, 2)} = \frac{\dot{q}(x^1, 2)}{q(x^1, 2)} + \frac{u''(c^1(x^1, 2))}{u'(c^1(x^1, 2))} \dot{c}^1(x^1, 2)$$

or, using that $q(x^1, 2) = \gamma(1 - x^1, 1)$ and $\dot{c}^1 = c_x^1 \dot{x}^1$,

$$c_x^1(x^1, 2) \dot{x}^1(x^1, 2) = \left[ \frac{\dot{c}^1(x^1, 2)}{\gamma(x^1, 2)} - \frac{\dot{c}^1(1 - x^1, 1)}{\gamma(1 - x^1, 1)} \right] \frac{u'(c^1(x^1, 2))}{u''(c^1(x^1, 2))}$$

Using equation (6) and equation (7) to replace the terms in the square parenthesis gives

$$c_x^1(x^1, 2) \dot{x}^1(x^1, 2) = \lambda \left[ \frac{\gamma(1 - x^1, 2)}{\gamma(1 - x^1, 1)} - \frac{\gamma(x^1, 1)}{\gamma(x^1, 2)} \right] \frac{u'(c^1(x^1, 2))}{u''(c^1(x^1, 2))} < 0$$

(19)

where the inequality follows since the term in the square parenthesis is positive, as implied by equation (18). The right hand side of equation (19) implies that consumption is decreasing, i.e. $\dot{c}^1(x^1, 2) < 0$. This could happen in one of two ways. First, with $\dot{x}^1(x^1, 2) > 0$ and $c_x^1(x^1, 2) < 0$. But this violates optimality: consumption is decreasing in the agent’s wealth share. The agent could deviate from this plan and increase her welfare. The other possibility, consistent with optimality, is that $\dot{x}^1(x^1, 2) < 0$ and $c_x^1(x^1, 2) > 0$. Then, in the equilibrium, $\gamma_x(x^1, 2) < 0$ and $\dot{x}^1(x^1, 2) < 0$, which confirms the conjecture. Finally we show that $\lim_{x^1 \to 0} \dot{x}^1(x^1, 2) = 0$. The right hand side of equation (19) is strictly negative at all $x^1$, including the boundary $x^1 = 0$. Since $x^1$ cannot be negative, this implies that $c_x^1(x^1, 2) \uparrow +\infty$ and $\dot{x}^1(x^1, 2) \uparrow 0$ as $x^1 \downarrow 0$. Equation (21) in Appendix F can be used to verify that $\lim_{x^1 \to 0} c_x^1(x^1, 2) = +\infty$.

Now we deal with the case where money growth rate is state dependent: $\mu = \mu[x]$, where $x$ is the aggregate state. Notice that the proof for constant $\mu > 0$ only required that the marginal value of money $\gamma(x^1, 1)$ and $\gamma(x^1, 2)$ lies in the region where $\gamma_x(x^1, 1) < 0$ and $\gamma_x(x^1, 2) < 0$, which ensures the stability of the system of ODEs in equation (6) and (7). When $\mu[x]$ is state dependent the same stability requires that two inequalities must be
simultaneously satisfied at all \( x \in (0, 1) \)

\[
\gamma(x, 1) \leq \frac{(\rho + \lambda + \mu[x])}{\lambda} \gamma(x, 2) \quad , \quad \gamma(1 - x, 1) \leq \frac{\lambda}{(\rho + \lambda + \mu[x])} \gamma(1 - x, 2)
\]

where the first one ensures that \( \gamma_x(x, 2) \leq 0 \) and the second one that \( \gamma_x(1 - x, 1) \leq 0 \).

Some algebra shows that to satisfy both conditions the function \( \mu[x] \) must satisfy the inequality in equation (8), which states that there is both a lower and an upper bound on the rates of money growth that ensure the stability of the system. Under these conditions the properties of the \( \gamma \) functions are the same ones that were stated above, so that the lower bound is smaller than 1 and the upper bound is higher than one. This implies, for instance, that \( \mu(x) > -(\rho + \lambda) \). At \( x = 0 \) simple algebra shows (using the boundary condition) that the right hand side of the inequality, is \( \mu[0] \leq \mu[0] \) which is always true, in other words at \( x = 0 \) there is no upper bound. At \( x = 1 \) the left hand side is \(-(\rho + \lambda) + \lambda \frac{\lambda}{\lambda + \rho + \mu[0]} \leq \mu[1] \) which gives a lower bound for \( \mu[1] \), as a function of \( \mu[0] \).

**E Proof of Proposition 3**

Conjecture that \( c^1(x^1, 1) = c^1(x^1, 2) = \bar{c} \). From the first order conditions in equation (5),

\[
\gamma(x^1, 2) = q(x^1, 2) = q(1 - x^1, 1) = \gamma(1 - x^1, 1) \quad \forall \ x^1
\]

as \( u'(\bar{c}) = 1 \). When \( x^1 = 0 \) we have that

\[
\gamma(0, 2) = \gamma(1, 1) \equiv \gamma_a
\]

and when \( x^1 = 1 \) we get

\[
\gamma(1, 2) = \gamma(0, 1) \equiv \gamma_b
\]

There are three possibilities: (i) \( \gamma_a < \gamma_b \), (ii) \( \gamma_a > \gamma_b \), and (iii) \( \gamma_a = \gamma_b \). In case (i) we have that \( \gamma(0, 1) > \gamma(0, 2) \) which contradicts Proposition 2. In case (ii) we have that \( \gamma(1, 1) > \gamma(1, 2) \) which again contradicts Proposition 2. The only remaining possibility is (iii) where the marginal value of money is constant for every pair \( \{ x^1, s \} \). Let \( \bar{\gamma} \) denote this value. From equation (6), after imposing the constant value \( \bar{\gamma} \) for the marginal value of money, we obtain

\[
(\rho + \mu)\bar{\gamma} = 0
\]

which is satisfied if \( \mu = -\rho \) or when \( \bar{\gamma} = 0 \). When \( \bar{\gamma} = 0 \) note that the price level has to be constant and equal to the marginal value of money \( \bar{\gamma} \) which follows from direct inspection of
the first order conditions presented in equation (5). Let \( \bar{q} \) denote this constant level. Because \( \bar{\gamma} = 0, \bar{q} = 0 \) which cannot constitute an equilibrium as one of the conditions for a monetary equilibrium is positive price \( \bar{q} \) (see Definition 1). When \( \mu = -\rho \) we know from Proposition 1 that it cannot constitute a monetary equilibrium. This implies that agents do not trade, money is valueless, and as a result there is no monetary equilibrium. This completes the proof of Proposition 3.

This result does not hinge on \( \mu \) being constant. Let \( \mu = \mu[x] \). Note that equation (20) has to hold for every value of \( x \), where \( x \) is the share of money in the hands of unproductive agents. This again implies that solution requires either \( \bar{\gamma} = 0 \) or \( \mu[x] = -\rho \) \( \forall \) \( x \), which we know does not constitute an equilibrium.

**F Proof of Proposition 4**

That \( Q_1 > 1 \) follows since \( \gamma(0, 2) > \gamma(0, 1) \). Using equation (14) we write

\[
\frac{c_1^x(x^1, 2)}{c_1^1(x^1, 2)} = \frac{\gamma_x(1 - x^1, 1)}{\gamma(1 - x^1, 1)} - \frac{\gamma_x(x^1, 2)}{\gamma(x^1, 2)}
\]

From equation (10) (evaluated at \( 1 - x^1 \)) and equation (11) (evaluated at \( x^1 \)) we get

\[
\frac{\gamma_x(1 - x^1, 1)}{\gamma(1 - x^1, 1)} = \frac{\rho + \lambda + \mu[x]}{\mu[x]} (1 - \frac{1}{2}) + \frac{1}{\gamma(x^1, 2)} \,, \\
\frac{\gamma_x(x^1, 2)}{\gamma(x^1, 2)} = \frac{\rho + \lambda + \mu[x]}{\mu[x]} (\frac{1}{2} - x^1) - \frac{1}{\gamma(x^1, 2)}
\]

Then, noting that \( \gamma(0, 2) = \frac{2}{\mu[0]} \), some algebra gives that

\[
\lim_{x \to 0} \frac{c_1^x(x^1, 2)x^1}{c_1^1(x^1, 2)} = 1 + \frac{\rho}{\mu[0]} + \frac{\lambda}{\mu[0]} \left(1 - \frac{\gamma(0, 1)}{\gamma(0, 2)}\right) \equiv Q_1
\]

which proves the lemma.

**G Proof of Proposition 5**

Note that \( f(x^1, 2) = Ce^{f_2/2 \Omega(x)dx} \), which follows from integrating the ODE for \( f(x^1, 2) \). Because \( \Omega(x^1) \) is continuous and differentiable for every \( x^1 \in (0, 1) \), it follows immediately that \( f(x^1, 2) \) is continuous and differentiable for every \( x^1 \in (0, 1) \).\(^{17}\)

We next show that \( \lim_{x^1 \to 0} f(x^1, 2) = +\infty \) and \( \lim_{x^1 \to 0} f_x(x^1, 2) = -\infty \). We will do this

\(^{17}\)Since \( \gamma(x^1, 1) \) and \( \gamma(x^1, 2) \) are continuous and differentiable functions of \( x^1 \) in \( (0, 1) \), inspection of the ODE shows that \( \Omega(x^1) \) is continuous and differentiable in \( x^1 \) for any \( x^1 \in (0, 1) \).
by looking at the behavior of a different function that is proportional to $f(x^1, 2)$ close to 0. Because $\Omega(x^1)$ is continuous as a function of $\gamma(x^1, 1) \text{ and } \gamma(x^1, 2)$ there exists a positive constant $k_x < \infty$ such that for any $x^1$

$$d \left( \Omega(x^1), -\frac{(\rho + \lambda + \mu[0])}{\mu[0]} - \lambda \gamma(0, 1) \frac{\lambda}{\mu[0]} \frac{1}{1 - x^1} + \frac{1}{1 - \gamma(1, 2)} \right) < k_x, \mu[0] > 0$$

where we used that $\gamma(0, 2) = \frac{2}{\mu[0]}$ and where $d(a, b) = |a - b|$. Let $k \equiv \max k_x$, and $\hat{\Omega}(x^1) \equiv -\frac{(\rho + \lambda + \mu[0])}{\mu[0]} - \lambda \gamma(0, 1) \frac{\lambda}{\mu[0]} \frac{1}{1 - x^1} + \frac{1}{1 - \gamma(1, 2)} = -\frac{Q_1}{2} - \frac{\lambda}{\mu[0]} (1/2 - x^1) + \frac{1}{1 - \gamma(1, 2)}$, with $Q_1 > 1$ as described in Proposition 4. This implies that

$$d \left( \Omega(x^1), \hat{\Omega}(x^1) \right) < k \ \forall \ x^1 \in (0, 1), \mu[0] > 0$$

in other words, the distance of these two functions is uniformly bounded.

Let $\hat{f}(x^1, 2) \equiv \hat{C} e^{\int_{x^1/2}^{x^1} \hat{\Omega}(z) dz}$. Note that

$$0 < \lim_{x^1 \to 0} \frac{\hat{f}(x^1, 2)}{\hat{f}(x^1, 2)} = \frac{C}{\hat{C}} e^{\lim_{x^1 \to 0} \int_{x^1/2}^{x^1} [\hat{\Omega}(z) - \hat{\Omega}(z)] dz} < K$$

where $K$ is a positive constant. This result follows because the distance of $\Omega(x^1) - \hat{\Omega}(x^1)$ is uniformly bounded. Then, the limiting behavior of $\hat{f}(x^1, 2)$ is the same as the limiting behavior of $f(x^1, 2)$.

We now explore the function $\hat{f}(x^1, 2)$. Simple integration yields

$$\hat{f}(x^1, 2) = \hat{C} \left( \frac{\mu[0]}{\gamma(1, 2)} \frac{1}{1 - \gamma(1, 2)} \right) e^{\frac{Q_1}{2} (\frac{1}{x^1} - \frac{1}{2})}$$

and

$$\hat{f}(x^1, 2) = -\hat{f}(x^1, 2) \left( \frac{\mu[0]}{\gamma(1, 2)} \frac{1}{1 - \gamma(1, 2)} \right) \left( \frac{1}{2} + \frac{Q_1}{2} (\frac{1}{x^1} - \frac{1}{2}) \right)$$

From these expressions it is clear that $\lim_{x^1 \to 0} \hat{f}(x^1, 2) = +\infty$ and $\lim_{x^1 \to 0} \hat{f}(x^1, 2) = -\infty$. Because we have already established that $0 < \lim_{x^1 \to 0} \frac{f(x^1, 2)}{f(x^1, 2)} < K < \infty$, it follows immediately that $\lim_{x^1 \to 0} f(x^1, 2) = +\infty$ and $\lim_{x^1 \to 0} f(x^1, 2) = -\infty$. 

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Additional documentation

On the supply of liquidity in an economy with borrowing constraints

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H Proof of Remark 2

Let \( i = 1 \) be the index for the unproductive agent, and consider her decision problem. The money supply growth is \( \mu = \frac{\dot{m}}{m_t} \) and let \( \bar{\xi}_t \) denote the lagrange multiplier of the money flow constraint in equation (3). The first order condition with respect to \( c_t^1 \) gives: \( \bar{\xi}_t^1 = q_t u'(c_t^1) \), where we used the homogeneity of degree -1 in the aggregate money supply \( m \).

Without loss of generality let’s consider the first cycle, starting at time 0, where the unproductive agent holds all the money supply \( m_0 \). Using that \( m_t = m_0 e^{\rho t} \), \( \mu = -\rho \), and \( \bar{q}_t = \frac{1}{m_t} q_t \), we have

\[
\int_0^T \frac{\bar{c}}{\bar{q}_t} dt + \int_0^T \rho m_t dt = m_0 , \quad \text{which gives} \quad q = \bar{c} \frac{1-e^{-\rho T}}{\rho e^{-\rho T}} \approx \bar{c} \frac{T}{1-\rho T},
\]

where the approximation is accurate for small \( T \). It is immediate to verify that this allocation also solves the Euler equation of the productive agent and that her money holdings are never negative.

I The model in discrete time

In this section we write the model in discrete time for period length \( \Delta \) and we characterize the system of Euler equations and its boundary conditions. Let \( \bar{V}(m_t^1, s_t, m_t) \) denote the value function for an agent type 1 with current money holdings \( m_t^1, s_t \in \{1, 2\} \) the her productive status (i.e. \( s_t = 1 \) if the agent is productive), and \( m_t \) the current money supply. If the agent is currently unproductive,

\[
\bar{V}(m_t^1, 2, m_t) = \max_{c_t, m_{t+\Delta}^1} \Delta \ln c_t^1 + \frac{1}{1 + \rho \Delta} \left( \lambda \Delta \bar{V}(m_{t+\Delta}^1, 1, m_{t+\Delta}) + (1-\lambda \Delta) \bar{V}(m_{t+\Delta}^1, 2, m_{t+\Delta}) \right) + \gamma_{t+\Delta}^2 \left( m_t^1 + \frac{\tau_t - \Delta c_t^1}{\bar{q}_t} - m_{t+\Delta}^1 \right) + \tilde{\theta}_t m_{t+\Delta}^1
\]

and if the agent is currently productive we have,

\[
\bar{V}(m_t^1, 1, m_t) = \max_{c_t, l_t, m_{t+\Delta}^1} \Delta (\ln c_t^1 - l_t) + \frac{1}{1 + \rho \Delta} \left( \lambda \Delta \bar{V}(m_{t+\Delta}^1, 2, m_{t+\Delta}) + (1-\lambda \Delta) \bar{V}(m_{t+\Delta}^1, 1, m_{t+\Delta}) \right) + \gamma_{t+\Delta}^1 \left( m_t^1 + \frac{\tau_t + \Delta (l_t - c_t^1)}{\bar{q}_t} - m_{t+\Delta}^1 \right)
\]
where $\hat{\gamma}_1^t$ and $\hat{\gamma}_2^t$ are the Lagrange multipliers of the budget constraints of productive and unproductive agents and $\hat{\theta}_t$ is the Lagrange multiplier of the borrowing constraint which can only potentially bind for the unproductive agent.

The intra-temporal first order conditions state that

\[
(c_1^t) : \quad c_1^t = \frac{\bar{q}_t}{\hat{\gamma}_2^t} \text{ if } s_t = 2 \text{ and } c_1^t = \frac{\bar{q}_t}{\hat{\gamma}_1^t} \text{ if } s_t = 1
\]

\[
(l_t) : \quad \frac{\hat{\gamma}_1^t}{\bar{q}_t} = 1
\]

The inter-temporal conditions (first order conditions with respect to $m_1^{t+\Delta}$) are

\[
(1 + \rho\Delta)\hat{\gamma}_1^t = \lambda \Delta \hat{\gamma}_2^{t+\Delta} + (1 - \lambda \Delta)\hat{\gamma}_1^{t+\Delta} \text{ if } m_1^{t+\Delta} > 0
\]

\[
(1 + \rho\Delta)(1 + \mu\Delta)\hat{\gamma}_1^t = \lambda \Delta \hat{\gamma}_2^{t+\Delta} + (1 - \lambda \Delta)\hat{\gamma}_1^{t+\Delta} \text{ if } m_1^{t+\Delta} > 0
\]

\[
(1 + \rho\Delta)(1 + \mu\Delta)\left(\hat{\gamma}_2^t + \hat{\theta}_t\right) = \lambda \Delta \hat{\gamma}_1^{t+\Delta} + (1 - \lambda \Delta) \frac{\partial \hat{V}(m_2^{t+\Delta}, 2, m_t^{t+\Delta})}{\partial m_2^{t+\Delta}} \bigg|_{m_1^{t+\Delta}=0}
\]

where we used the envelope theorem when possible.

Notice that prices are homogeneous of degree -1 in the money supply level, $\hat{\gamma}_1^t = \frac{\gamma_1^t}{m_t}$, $\hat{\gamma}_2^t = \frac{\gamma_2^t}{m_t}$, and $\bar{q}_t = \frac{q_t}{m_t}$, so that we can rewrite the intra-temporal first order conditions as

\[
c_1^t = \frac{q_t}{\gamma_2^t} \text{ if } s_t = 2 \text{ and } c_1^t = \frac{q_t}{\gamma_1^t} \text{ if } s_t = 1 , \quad \frac{\gamma_1^t}{q_t} = 1
\]

from where it can be obtained that the consumption of a productive agent is $c^1(m^1, 1) = 1$.

Also notice that

\[
\hat{V}(m_t^1, j, m_t^2) = \hat{V}(\frac{m_t^1}{m_t^j}, j, 1) \equiv V(x_t^i, j) , \quad x_t^i \equiv \frac{m_t^i}{m_t}
\]

The Euler equations become

\[
(1 + \rho\Delta)(1 + \mu\Delta)\hat{\gamma}_1^t = \lambda \Delta \hat{\gamma}_2^{t+\Delta} + (1 - \lambda \Delta)\hat{\gamma}_1^{t+\Delta} \text{ if } m_1^{t+\Delta} > 0
\]

\[
(1 + \rho\Delta)(1 + \mu\Delta)\hat{\gamma}_1^t = \lambda \Delta \hat{\gamma}_2^{t+\Delta} + (1 - \lambda \Delta)\hat{\gamma}_1^{t+\Delta} \text{ if } m_1^{t+\Delta} > 0
\]

\[
(1 + \rho\Delta)(1 + \mu\Delta)\left(\hat{\gamma}_2^t + \hat{\theta}_t\right) = \lambda \Delta \hat{\gamma}_1^{t+\Delta} + (1 - \lambda \Delta) \frac{\partial \hat{V}(x_t^{1+\Delta}, 2)}{\partial x_t^{1+\Delta}} \bigg|_{x_t^{1+\Delta}=0}
\]

if $x_t^{1+\Delta} = 0$
I.1 Boundary conditions

The boundary conditions are obtained when the unproductive agent has no money. From the budget constraint (under the conjecture that the agent spends all the transfer),

\[
\tau_t = \Delta c_t^1 \\
m_t \mu \Delta \bar{q}_t = \Delta c_t^1 \\
\frac{\mu}{2} q_t = c_t^1 \\
\frac{\mu}{2} \bar{q}_t = \frac{q_t}{\gamma_t^2}
\]

so that

\[
\gamma_t^2 = \frac{2}{\mu}
\]

Now we use the Euler equation for a productive agent when the unproductive agent has all the money,

\[
(1 + \rho \Delta)(1 + \mu \Delta) \gamma_t^1 = \lambda \Delta \gamma_{t+\Delta}^2 + (1 - \lambda \Delta) \gamma_{t+\Delta}^1
\]

notice that conditional on no state reversal the relative money holdings do not change so that \(\gamma_t^1 = \gamma_{t+\Delta}^1\). Some simple algebra provides that

\[
\frac{\gamma_{t+\Delta}^1}{\gamma_{t+\Delta}^2} = \frac{\lambda}{\rho + \lambda + \mu + \rho \mu \Delta}
\]

J The return on money

We define the stochastic interest rate (or net return on money) for a small time interval \(\Delta\) as

\[
\bar{r}(x) \Delta = \mathbb{E} \left[ \frac{\bar{q}_{t+\Delta}}{\bar{q}_t} - 1 \middle| x_t = x \right] = \mathbb{E} \left[ \frac{q_{t+\Delta}}{q_t} - \Delta \mu_t - 1 \middle| x_t = x \right]
\]

where \(\bar{q}_{t+\Delta} = \bar{q}(x_{t+\Delta})\) and \(\bar{q}_t = q(x_t)/m_t\). Without loss of generality consider the case where at time \(t\) agent 1 is productive with money holdings given by \(x_t^1\). Then,

\[
\bar{r}(x) \Delta = (1 - \lambda \Delta) \frac{\gamma(x_{t+\Delta}^1, 1)}{\gamma(x_t^1, 1)} + \lambda \Delta \frac{\gamma(x_{t+\Delta}^2, 1)}{\gamma(x_t^1, 1)} - 1 - \Delta \mu_t,
\]

where we used equation (5). We use that \(x_{t+\Delta}^i = x_t^i + \dot{x}^i(x_t^i, i) \Delta\) to do a Taylor expansion of first order of \(\gamma(x_{t+\Delta}^i, 1)\) to obtain

\[
\bar{r}(x) \Delta = \frac{\gamma_t(x_t^1, 1) \dot{x}^1(x_t^1, 1) \Delta}{\gamma(x_t^1, 1)} + \lambda \Delta \left( \frac{\gamma(x_t^2, 1) + \gamma_x(x_t^2, 1) \dot{x}^1(x_t^1, 2) \Delta}{\gamma(x_t^1, 1)} - \frac{\gamma(x_t^1, 1) + \gamma_x(x_t^1, 1) \dot{x}^1(x_t^1, 1) \Delta}{\gamma(x_t^1, 1)} \right) - \Delta \mu_t
\]
taking the limit as $\Delta$ approaches 0,

$$r(x) = \frac{\gamma_x(x^1, 1)\dot{x}^1(x^1, 1)}{\gamma(x^1, 1)} + \lambda \left( \frac{\gamma(x^2, 1)}{\gamma(x^1, 1)} - 1 \right) - \mu_t$$

We use equation (6), equation (7) and equation (9) to get

$$r(x) = \rho + \lambda \left( \frac{\gamma(x^2, 1)}{\gamma(x^1, 1)} - \frac{\gamma(1-x^2)}{\gamma(1-x^1)} \right)$$

or $r(x) = \rho + \lambda \left( \frac{\gamma(x^1, 1)}{\gamma(1-x^1)} - \frac{\gamma(1-x^2)}{\gamma(1-x^1)} \right)$ which immediately yields the expression in the text.

**K Derivation of the invariant wealth distribution**

The CDF for the money holdings, $F(x^i, s, t)$, with density $f(x^i, s, t)$ in states $s = 1, 2$ follows

$$F(x^i, 1, t + dt) = (1 - \lambda dt)F(x^i - \dot{x}^i(x^i, 1) dt, 1, t) + \lambda dt F(x^i - \dot{x}^i(x^i, 2) dt, 2, t)$$

$$F(x^i, 2, t + dt) = (1 - \lambda dt)F(x^i - \dot{x}^i(x^i, 2) dt, 2, t) + \lambda dt F(x^i - \dot{x}^i(x^i, 1) dt, 1, t)$$

Expanding $F(x^i, s, t)$ around $x^i$ gives (we only report the one for $s = 2$)

$$F(x^i, 2, t + dt) = (1 - \lambda dt) \left[ F(x^i, 2, t) - f(x^i, 2, t) \dot{x}^i(x^i, 2) dt \right] + \lambda dt \left[ F(x^i, 1, t) - f(x^i, 1, t) \dot{x}^i(x^i, 1) dt \right]$$

Subtracting $F(x^i, 2, t)$ from both sides and dividing by $dt$ and taking the limit for $dt \downarrow 0$

$$\lim_{dt \downarrow 0} \frac{F(x^i, 2, t + dt) - F(x^i, 2, t)}{dt} = \frac{\partial F(x^i, 2, t)}{\partial t} = -f(x^i, 2, t) \dot{x}^i(x^i, 2) - \lambda \left( F(x^i, 2, t) - F(x^i, 1, t) \right)$$

Using this equation together with the corresponding one for state $s = 1$ and imposing invariance give

$$0 = f(x^i, 2) \dot{x}^i(x^i, 2) + f(x^i, 1) \dot{x}^i(x^i, 1) \quad (22)$$

Taking the derivative w.r.t. $x$ delivers the Kolmogorov forward equation

$$\frac{\partial \partial F(x^i, 2, t)}{\partial x \partial t} = \frac{\partial f(x^i, 2, t)}{\partial t} = \frac{\partial \left[ -f(x^i, 2, t) \dot{x}^i(x^i, 2) - \lambda \left( F(x^i, 2, t) - F(x^i, 1, t) \right) \right]}{\partial x}$$

which, equated to zero (imposing invariance) gives the equation in the main text.

Using the expression in equation (9), together with $\dot{x}^1(x^1, 2) + \dot{x}^1(1 - x^1, 1) = 0$ to replace $\dot{x}^1(x^1, 1)$ and $\dot{x}^1(x^1, 2)$ into the KFE for the density function gives

$$\frac{f_x(x^1, 2)}{f(x^1, 2)} = \frac{(\lambda - \mu[x^1] - \frac{\partial \mu[x^1]}{\partial x^1}(x^1 - \frac{1}{2})\gamma(x^1, 2) + \frac{\gamma(x^1, 2)}{f(x^1, 2)} \lambda \gamma(x^1, 2)}{1 + \mu[x^1] (x^1 - \frac{1}{2}) \gamma(x^1, 2)}$$

Using equation (22) to replace the ratio $f(x^1, 2)/f(x^1, 1)$ in the above expression gives the equation for $\Omega(x^1)$ in the main text.
The value function $V\left(\frac{1}{2}; \mu\right)$ diverges when $\mu < 0$ or $\mu \to \infty$

We first state the claim and then we proceed to the proof.

**Claim 2** Under systematic extreme monetary expansions ($\mu \uparrow \infty$) or monetary contractions ($\mu < 0$), there is a break up in trade resulting in autarkic allocations. Then, $V\left(\frac{1}{2}; \mu\right)$ diverges to $-\infty$ either when $\mu \uparrow \infty$ or $\mu < 0$.

That $V(1/2; \mu)$ diverges to $-\infty$ when $\mu < 0$ is a direct implication of Proposition 1. As $x = 1/2$ the Lemma states that there is no monetary equilibrium and therefore there is no trade. As any given agent spends half of her life being unproductive, the agent spends half of her life with zero consumption (because of no trade). It follows immediately that $V(1/2; \mu)$ diverges to $-\infty$ when $\mu < 0$.

We next prove that $V(1/2; \mu)$ diverges to $-\infty$ when $\mu \uparrow \infty$. This proof has two pieces. First that when $\mu \uparrow \infty$ and there is no state reversal, $x = 0$. Second, that the consumption of an agent with no wealth approaches zero as $\mu \uparrow \infty$ (i.e. the real value of the monetary transfer becomes zero).

To prove the first part consider the discrete time version of equation (9), for a small period $\Delta$. Let $x_i^1$ be the (fraction of) money holdings at the beginning of period $t$, before the lump sum transfer $\mu \Delta/2$ is received and before any purchase is made. Without loss of generality, assume that agent of type 1 is unproductive with current money holdings $x_i^1$ so that,

$$x_{i+\Delta}^1(1 + \Delta \mu) = x_i^1 + \frac{1}{2}\mu \Delta - \frac{\Delta}{\gamma(x_t, 2)}$$

or,

$$x_{i+\Delta}^1 = \frac{x_i^1}{1 + \Delta \mu} + \frac{1}{2}\mu \Delta - \frac{\Delta}{\gamma(x_t, 2)(1 + \Delta \mu)}$$

Using that $\gamma(x, 2)$ is decreasing in $x$ (see Proposition 2), and that $\gamma(0, 2) = 2/\mu$ (see equation (12)) we can bound above $x_{i+\Delta}^1$,

$$x_{i+\Delta}^1 < \frac{x_i^1}{1 + \Delta \mu}$$

which immediately implies that

$$\lim_{\mu \uparrow \infty} x_{i+\Delta}^1 = 0$$

that is, an agent who remains unproductive begins the period with no money holdings as she spends the whole transfer she received from the government.

We next show that the consumption of an unproductive agent with no wealth approaches zero as $\mu \uparrow \infty$. Formally: $\lim_{\mu \uparrow \infty} c^1(0, 2) = 0$. To see this notice that From equation (14) we have that $c^1(0, 2) = \frac{\gamma(1, 1)}{\gamma(1, 2)} \frac{\lambda}{\rho + \lambda + \mu}$. Because $\gamma(x, 2)$ decreasing in $x$ (see Proposition 2),

$$c^1(0, 2) \leq \frac{\gamma(1, 1)}{\gamma(1, 2)} = \frac{\lambda}{\rho + \lambda + \mu}$$
where the last step uses equation (13). It is now immediate that \( \lim_{\mu \to \infty} c^1(0,2) = 0 \).

Notice that the above implies that \( \lim_{\mu \to \infty} V(x; \mu) = -\infty \) for any value of \( x \). This is easily seen from the discrete time version of equation (16),

\[
V(x; \mu) = \Delta \left( \ln c^1(x,2; \mu) - 1 - c^1(x,2; \mu) \right) + (1 - \rho \Delta)((1 - \lambda \Delta)V(\bar{x}; \mu) + \lambda \Delta V(1 - \bar{x}; \mu))
\]

where \( \Delta \) is the length of the time period and \( \bar{x} \) is the share of money of unproductive agents at the next date. We showed that as \( \mu \to \infty \) the unproductive agent will spend all her money, so that \( \lim_{\mu \to \infty} \bar{x} = 0 \). Then the agent consumption is expected to be zero if she remains unproductive. Thus the expected utility diverges.

\[ \text{M\ A policy function with discontinuity at } x = 0. \]

Recall that Proposition 1 states that a monetary equilibrium requires \( \mu[0] \geq 0 \); this precludes any systematic contractionary policy to be implemented. If the restriction, which comes from the tax solvency constraint, if what denies a policy similar to Friedman rule from being attainable, then a natural candidate for policy is one which allows for a discontinuity at \( x = 0 \). In this section we explore such a policy and we compare it with the best piecewise linear policy \( \hat{\mu}[x] \). In particular, we let \( \mu[0] = \omega_0 \) and \( \mu[x] = \omega \\forall \ x \in (0,1] \).

Because the discontinuity is at \( x = 0 \) the system of differential equations presented in equation (10) and (11), and the boundary conditions introduced in equation (12) and (13) still hold. This is a direct implication of Proposition 2: because an unproductive agent with zero wealth spends all of the government transfer, \( x \) remains at zero as long as there is no shock to the productive state. Therefore, the discontinuity of the policy at \( x = 0 \) does not introduce a "jump" in \( x \) which the system of differential equations should account for.\(^{18}\)

We compute the best discontinuous policy at \( x = 0 \) that maximizes ex-ante expected welfare as defined in equation (16), by searching on a bi-dimensional grid over \( \omega_0 \) and \( \omega \), where the set of parameters is constrained in order to satisfy the bounds on \( \mu[x] \) described in equation (8). We let \( \mu_d[x] \) denote the policy that maximizes welfare. The best policy prescribes expanding the money supply at zero at a rate above 7% (\( \omega_0 = 0.072 \)) and contracting the money supply otherwise at a rate higher than Friedman’s rule (\( \omega_1 = -0.058 \)). As with the piecewise policy, the contraction above the rate of discount is a result of the forward looking nature of agents’ decisions.

We now turn to compare the best discontinuous policy \( \mu_d[x] \) with the best piecewise linear policy \( \hat{\mu}[x] \). The piecewise linear policy provides higher welfare that the discontinuous policy as it can be inferred by \( \alpha \), the compensating variation relative to the complete markets allocation (see the first row of Table 3): the compensating variation is 13.9% under \( \hat{\mu}[x] \) while it is 15.5% under \( \mu_d[x] \). Next we aim to understand why the piecewise linear policy is preferable.

In Figure 6 we plot the consumption and expected return functions under the two different policies, \( \hat{\mu}[x] \) and \( \mu_d[x] \). It can be seen that, away from \( x \) small, the piecewise linear policy

---

\(^{18}\)Had the discontinuity being introduced at \( x > 0 \) the dimension of the problem would increase considerably: if \( \mu[x] \) is discontinuous at some value larger than zero, when \( x \) hits this value the state “jumps” which requires recasting the model to account for this new feature of the state variable.
Table 3: Summary statistics: piecewise linear vs discontinuous policy

<table>
<thead>
<tr>
<th>Best piecewise linear $\hat{\mu}[x]$</th>
<th>Best discontinuous $\mu_d[x]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\mu_0, \mu_1, \tilde{\mu}, \tilde{x}) = (0.0545, -0.083, -0.03, 0.028)$</td>
<td>$(\omega_0, \omega) = (0.072, -0.058)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Welfare cost$^a$ $(\alpha)$</th>
<th>13.9%</th>
<th>15.5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E} \left[ \ln (c^1(x, 2)) \right]$</td>
<td>-0.6</td>
<td>-0.7</td>
</tr>
<tr>
<td>(0.01)</td>
<td>(0.01)</td>
<td></td>
</tr>
<tr>
<td>std. dev. $\ln (c^1(x, 2))$</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>(0.005)</td>
<td>(0.003)</td>
<td></td>
</tr>
</tbody>
</table>

| $\mathbb{E}[x]$ | 0.38 | 0.36 |
| (0.006) | (0.002) |
| fraction of time $x < 0.02$ | 0.36 | 0.35 |
| (0.012) | (0.01) |
| Years to (first) hit $x = 0.02$ | 21.5 | 19.2 |
| (0.22) | (0.2) |
| fraction of time $x < 0.0001$ | 0.3 | 0.34 |
| (0.012) | (0.01) |
| years to (first) hit $x = 0.0001$ | 27.3 | 19.3 |
| (0.27) | (0.2) |

| $\mathbb{E}[r(x)]$ | 0.1 | 0.085 |
| (0.002) | (0.002) |
| $\mathbb{E}[\mu]$ | -0.02 | -0.036 |
| (0.001) | (0.000) |

$^a$ standard error in parenthesis. $^a$ Percentage loss (in consumption equivalent units) relative to a complete markets equilibrium. Computation does not require simulation. Parameters: $\lambda = 0.1$, $\rho = 0.05$. The Monte Carlo simulation uses a discretized time grid with a period length of 12 hours.

provides point wise higher consumption levels for the unproductive agent which translates into, point wise, higher expected returns. However, this does not imply that $\hat{\mu}[x]$ is a better policy: this analysis does not account for the endogenous transitions through the different values of $x$ which shape up the expected consumption profiles of the different agents in the economy. To this end, we produce simple unconditional statistics that will aid in the comparison.

As in the previous section we simulate the economy under $\hat{\mu}[x]$ and $\mu_d[x]$ and compute relevant unconditional statistics which we include in Table 3. Under the piecewise linear
policy, an unproductive agent’s instantaneous utility derived from consumption is on average higher but also more volatile than under the discontinuous policy. Notice that the way agents spend their money holdings is very different across models. It takes agents the same time to become poor under both policies (around 20 years to the first hit at \( x = 0.018 \)) and spend a similar fraction of time in poverty but, once there, agents run out of money much faster under the discontinuous policy; under \( \mu_d[x] \) it takes them only one month to deplete their money holdings from 0.018 to 0.0001, while 6 years under \( \hat{\mu}[x] \). Similarly, they spend much more time below 0.0001 under \( \mu_d[x] \) than under \( \hat{\mu}[x] \). This difference is what makes \( \hat{\mu}[x] \) better than \( \mu_d[x] \); because an unproductive agent’s consumption is very low for \( x \) close to zero (see Figure 6), a lower speed of money depletion provides a higher consumption stream.

Finally, the expected rate of return on money is lower and the average monetary contractions are higher under the discontinuous policy \( \mu_d[x] \). The lower return on money shows that \( \mu_d[x] \) provides lower production incentives than the piecewise linear policy \( \hat{\mu}[x] \) even though the average rate of monetary contraction under the discontinuous policy is higher and closer to Friedman rule. This exhibits an important property of our model: because markets are incomplete welfare is not maximized under the Friedman rule and therefore the optimal rate of monetary contractions does not need to be equal to the rate of discount \( \rho \).

### N A deterministic model with one period cycles

We want to evaluate a simple model where agents productive status switches deterministically every period. We will use this simple model to show that (i) Friedman’s rule can be supported in equilibrium at it attains the best symmetric allocation (as in Remark 2), and (ii) equilibria with rates of monetary expansion above Friedman’s rule can also be constructed.
Without loss of generality we assume that agent type 1 is productive in even periods and unproductive in odd periods. Agent type 1 faces the following problem,

\[
\max_{\{c^1_{2t}, l^1_{2t}, m^1_{2t+1}, m^1_{2t+2}\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^{2t} \left[ \ln c^1_{2t} - l^1_{2t} + \tilde{\gamma}^1_{2t} \left( m^1_{2t} + \frac{l^1_{2t} + \tau^2_{2t} - c^1_{2t}}{\tilde{q}_{2t}} - m^1_{2t+1} \right) + \tilde{\theta}^1_{2t} \ m^1_{2t+1} \right] + \sum_{t=0}^{\infty} \beta^{2t+1} \left[ \ln c^1_{2t+1} + \tilde{\gamma}^2_{2t+1} \left( m^1_{2t+1} + \frac{\tau^2_{2t+1} - c^1_{2t+1}}{\tilde{q}_{2t+1}} - m^1_{2t+2} \right) + \tilde{\theta}^2_{2t+1} \ m^2_{2t+2} \right]
\]

where the price of money $\tilde{q}_t$ and the lagrange multipliers $\tilde{\gamma}^s_t$ (budget constraint) and $\tilde{\theta}^s_t$ (borrowing constraint) (with $s \in \{1, 2\}$), are measured in “nominal” terms (units of consumption per 1 dollar). Note that $\tilde{\gamma}^1_t$ and $\tilde{\gamma}^2_t$ are the co-state variables when the agent is productive and unproductive, respectively.

The first order conditions give

\[
\begin{align*}
(c^1_{2t}) & : \frac{1}{c^1_{2t}} = \frac{\tilde{\gamma}^1_{2t}}{\tilde{q}_{2t}} \\
(l^1_{2t}) & : 1 = \frac{\tilde{\gamma}^1_{2t}}{\tilde{q}_{2t}} \\
(c^1_{2t+1}) & : \frac{1}{c^1_{2t+1}} = \frac{\tilde{\gamma}^2_{2t+1}}{\tilde{q}_{2t+1}} \\
(m^1_{2t+1}) & : \tilde{\gamma}^1_{2t} = \beta \tilde{\gamma}^2_{2t+1} + \tilde{\theta}^1_{2t} \\
(m^1_{2t+2}) & : \tilde{\gamma}^2_{2t+1} = \beta \tilde{\gamma}^1_{2t+2} + \tilde{\theta}^2_{2t+1}
\end{align*}
\]

where the multipliers $\tilde{\theta}$ are positive if the borrowing constraint binds.

We construct equilibria where nominal variables are homogeneous in the aggregate level of money: $\tilde{q}_t = q_t / m_t$, $\tilde{\gamma}^s_t = \gamma^s_t / m_t$, and $\tilde{\theta}^s_t = \theta^s_t / m_t$. The first order conditions reduce to

\[
\begin{align*}
(c^1_{2t}) & : \frac{1}{c^1_{2t}} = \frac{\gamma^1_{2t}}{q_{2t}} \\
(l^1_{2t}) & : 1 = \frac{\gamma^1_{2t}}{q_{2t}} \\
(c^1_{2t+1}) & : \frac{1}{c^1_{2t+1}} = \frac{\gamma^2_{2t+1}}{q_{2t+1}} \\
(m^1_{2t+1}) & : \gamma^1_{2t} = \beta \gamma^2_{2t+1} + \mu \theta^1_{2t} \\
(m^1_{2t+2}) & : \gamma^2_{2t+1} = \beta \gamma^1_{2t+2} + \mu \theta^2_{2t+1}
\end{align*}
\]

where we also used that monetary policy controls money growth rate $\mu$, $m_{t+1} = \mu m_t$.

It is immediate that $c^1_{2t} = 1$ (follows from combining equation (23) and equation (24)). Combining equation (24), equation (26) and equation (27) gives

\[
\gamma^1_{2t} = \frac{\beta}{\mu} \gamma^2_{2t+1} + \theta^1_{2t}, \quad \gamma^2_{2t+1} = \frac{\beta}{\mu} \gamma^1_{2t+2} + \theta^2_{2t+1}
\]
which gives

\[ \gamma_{2t}^1 = \left( \frac{\beta}{\mu} \right)^2 \gamma_{2t+2}^1 + \frac{\beta^2}{\mu} \theta_{2t+1}^2 + \theta_{2t}^1 \]

N.1 Friedman rule

Conjecture and verify that the following is an equilibrium:

\[ \mu = \beta, \theta_{1t}^2 = 0, c_{2t}^1 = c_{2t+1}^2 = \bar{c} = 1, \gamma_{2t}^1 = \gamma_{2t+1}^2 = q_{2t} = \bar{q}, \ \forall \ t \]

which is easy to verify that satisfies the first order conditions, i.e. equation (23) to equation (27) and sustain the complete markets allocation.

To pin down the price level \( \bar{q} \) (and therefore the value of the multipliers \( \gamma \)) we impose that the agent’s total nominal assets (inclusive of transfers) equals the total nominal consumption expenditures over the one period cycle. Without loss of generality we consider the first period and we assume that the unproductive agent starts will all the money,

\[ \frac{\bar{c}}{\bar{q}_0} = \beta m_0 \]

where \( \beta m_0 \) are the after-transfer money holdings of the unproductive agent. This equation can be solved to obtain,

\[ \bar{q} = \bar{\gamma} = \frac{1}{\beta} \]

N.2 Equilibria when \( \mu > \beta \)

Conjecture and verify that the following is an equilibrium:

\[ \theta_{2t}^1 = 0, c_{2t}^1 = 1, \gamma_{2t}^1 = \bar{\gamma}^1, \gamma_{2t+1}^2 = \frac{\mu}{\beta} \bar{\gamma}^1, \ \forall \ t = 0, 1, 2, 3, \ldots \]

which, using \( \gamma_{2t+1}^2 = \frac{\beta}{\mu} \gamma_{2t+2}^1 + \theta_{2t+1}^2 \), implies

\[ \theta_{2t+1}^2 = \left( \frac{\mu}{\beta} - \frac{\beta}{\mu} \right) \bar{\gamma}^1 > 0, \ \forall \ t = 0, 1, 2, 3, \ldots \]

In this case the borrowing constraint of the unproductive agent binds, i.e. \( \theta_{2t+1}^2 > 0 \). As before, it is readily verified that the proposed solution satisfies the first order conditions, i.e. equation (23) to equation (27), even though in this case the consumption allocations are not those obtained under complete markets. In particular, consumption when unproductive is

\[ c_{2t}^1 = \frac{\gamma_{2t+1}^1}{\gamma_{2t}^1} = \beta < 1, \ \forall \ t = 0, 1, 2, 3, \ldots \]

As before, to pin down the price level \( \bar{q} \) (and therefore the value of the multipliers \( \bar{\gamma} \)) we impose that the agent’s total nominal assets (inclusive of transfers) equals the total nominal
consumption expenditures over the one period cycle. Without loss of generality we consider
the first period and we assume that the unproductive agent starts will all the money,
\[
\frac{\bar{c}}{\bar{q}_0} = \mu m_0
\]
where \(\mu m_0\) are the after-transfer money holdings of the unproductive agent. This equation
can be solved to obtain,
\[
\bar{q} = \bar{\gamma} = \frac{1}{\mu}
\]

O Uncorrelated shocks (Bewley economy)

In this section we study a stationary economy where the productivity shocks are uncorrelated
across agents. Assume a unit mass of agents, indexed by \(i\), over the \([0,1]\) interval. As before,
the productivity state of each agent, \(s_i\) follows a Markov process, where \(\lambda\) denotes the rate
at which the state switches.

Let \(m_i^t\) be the money holdings of agent \(i\) at time \(t\), so that the total money supply is
\(m_t = \int_0^1 m_i^t\ di\). Let \(\tau_t\) denote the per capita transfer from the government. The government
budget constraint is
\[
\tau_t = \bar{q} t \int_0^1 \dot{m}_i^t\ di = \mu m_t \bar{q} t = \mu q_t
\]
where the last equality uses the homogeneity of \(\bar{q}\) with respect to \(m\). In what follows we let
\(\mu \geq 0\). The same argument developed in Proposition 1 shows that no monetary equilibrium
exists for \(\mu < 0\).

Obviously \(\int_0^1 x_i^t\ di = 1\) where \(x_i^t = m_i^t/m_t\). Notice that in this model with a continuum
of agents we have that \(x_i \in [0, +\infty)\), where \(x_1 = 1\) denotes the situation in which a single
agent money holding equal the economy’s average money, and \(x \uparrow \infty\) denotes the situation
in which one agent holds all of the money. Simple algebra gives that \(\dot{x}(x^i, s) = \dot{m}_i = m_i/m_t - x^i\mu\),
where we omit the time index for notation simplicity.

The agent’s first order conditions of this model are unchanged compared to the previous
model (equation (5)). The unproductive agent budget constraint and Euler equation (i.e.
when \(s = 2\)) give
\[
\dot{x}^1(x, 2) = \mu \left(1 - x^1\right) - \frac{1}{\gamma(x^1, 2)}
\]
This equation shows that money growth has no effect on the money share in the case where
the agent’s money holding equal the average money per capita in the economy, i.e. the ratio
is \(x^1 = 1\).\(^{19}\)

We assume the economy has a centralized competitive market where one unit of money
buys \(\bar{q} = \frac{1}{m} q\) units of consumption. This implies that all productive agents are willing to
produce in exchange for money as long as \(\gamma(x^1, 1) > q\), and that there is a level of money
holdings \(\bar{x}\) where productive agents are satiated with money balances: \(\gamma(\bar{x}, 1) = q\).

\(^{19}\)Note that the same property holds in the model with two agents, in which the total mass is 2, the index
\(x \in [0,1]\) and an equal distribution of money holdings implies that the ratio of the agent money to the
average (per capita) money holdings is 1/2, so that \(\mu\) does not affect \(\dot{x}^i\) when \(x^i = 1/2\).
Every agent works to save money balances $\bar{x}$ as soon as she gets productive. So the wealth share jumps from $x^1 < \bar{x}$ to $\bar{x}$ as soon as the agent become productive (this implies that $\dot{x}^1$ is infinite at the time of a jump).\textsuperscript{20} A productive agent aims to maintain the wealth share constant at $\bar{x}$ so that $\dot{x}^1(\bar{x}, 1) = 0$, which implies that for a productive agent $x^1 = \bar{x}$ then $\dot{m}_t/m_t = \mu \bar{x}$.

Before moving on with solving the model we define an equilibrium for this economy.

**Definition 2** A monetary equilibrium is a price function $\tilde{q}(m) = \frac{1}{m} q$, with $q \in \mathbb{R}^+$, and a stochastic process $x(t, \omega)$ with values in $[0, 1]$, such that a consumer $i$ maximizes expected discounted utility (1) subject to the budget constraints (2) and (3) with $q(t, \omega) = q(s)$, non-negativity (4), the government budget constraint constraint (28), and market clearing constraint (33).

In an internal solution the lagrange multipliers $\gamma(x^1, 1), \gamma(x^1, 2)$ solve the system of differential equations that we determined before (equation (6) and (7)), which under the assumptions of this section (using equation (29)) gives

$$
\gamma_x(x^1, 1) \dot{x}^1(x^1, 1) = (\rho + \mu) \gamma(x^1, 1) - \lambda (\gamma(\bar{x}, 2) - \gamma(\bar{x}, 1))
$$

$$
\gamma_x(x^1, 2) \left[ \mu (1 - x^1) - \frac{1}{\gamma(x^1, 2)} \right] = (\rho + \mu) \gamma(x^1, 2) - \lambda (\gamma(x^1, 1) - \gamma(x^1, 2))
$$

The system decouples in two ODEs which we discuss next.

Internal solution always applies for $x^1 \in (0, 1)$ for the unproductive agent. For the productive agent the solution is internal only at $x^1 = \bar{x}$, otherwise the state $x^1$ records a jump from $x^1$ to $\bar{x}$. At the replenishment level $\bar{x}$ the equation for the productive agent gives

$$
q = \gamma(\bar{x}, 1) = \frac{\lambda}{\lambda + \mu + \rho} \gamma(\bar{x}, 2)
$$

(30)

Since the disutility of labor is linear, the marginal utility of money for a productive agent is constant at the level $\gamma(\bar{x}, 1) = \gamma(x^1, 1)$ for $x^1 \in (0, \bar{x})$.\textsuperscript{21} The ODE for $\gamma(x^1, 2)$ can be rewritten as

$$
\gamma_x(x^1, 2) = \frac{(\rho + \lambda + \mu) \gamma(x^1, 2)^2 - \lambda q \gamma(x^1, 2)}{\gamma(x, 2) \mu (1 - x^1) - 1}
$$

(31)

with the following boundary conditions at $x^1 = 0$:

$$
\gamma(0, 2) = \frac{1}{\mu}
$$

(32)

where the boundary stems from the unproductive agent’s Euler equation and budget constraint at $x^1 = 0$.

Next we impose that at every $t$ demand equals supply in the asset market where money is exchanged for the consumption good. The asset demand originates from the productive

\textsuperscript{20}The assumption of linear disutility of labor is important for this result, as it implies that the productive agent immediately refills his money balances up to $\bar{x}$.

\textsuperscript{21}Hence the value (i.e. the utility level) for an productive agent with $x^1 \in (0, \bar{x})$ is linear in $x^1$. 


agents who aim to hold an amount of money $\bar{x}$. The supply comes from unproductive agents who exchange money for consumption. Let $f(x^1, 2)$ be the density function for the mass of unproductive agents with $\int_0^\bar{x} f(x^1, 2) dx^1 = 1$, so that $f(x^1, 2)$ is the measure of unproductive agent, who account for 1/2 of the total population under the invariant density. The market clearing equation is (see below)

$$\text{(33)} \quad (\mu + 2\lambda)(\bar{x} - 1) = \int_0^{\bar{x}} \frac{f(x^1, 2)}{\gamma(x^1, 2)} dx^1$$

Finally note that the density function for unproductive agents is obtained from the usual Kolmogorov forward equation. Using $f(x^1, 1) = 0$ and equation (29) to replace $\dot{x}^1(x^1, 2)$ into the KFE derived in Section 3.3

$$f_x(x^1, 2) = \frac{(\lambda - \mu)\gamma(x^1, 2) - \frac{(\rho + \lambda + \mu)\gamma(x^1, 2) - \lambda q}{1 + \mu(x^1 - 1)\gamma(x^1, 2)}}{1 + \mu (x^1 - 1)\gamma(x^1, 2)} \quad \text{(34)}$$

Notice that this problem has four unknowns: $q, \bar{x}$, and two constant of integration, one for the solution of equation (31) and one for the density function in equation (34). The four equations to solve for the four unknowns are: the boundary conditions in equation (30) and equation (32), the market clearing condition in equation (33), and the density function integrating to a unit mass.

**Derivation of market clearing equation**

We derive equation (33) as the limit of a discrete time model. Recall that each productive agent is aiming to keep his money balances at $m^i/m = \bar{x}$. Consider a time interval of length $\Delta$. In each period there is a fraction $\Delta \lambda$ of the mass of unproductive agents who become productive. The asset demand for one of these agents with wealth $x^1$ is given by a change in the stock, $\bar{x} - x^1$, and by a flow component that offsets the effect of money growth given by $\mu \Delta (\bar{x} - 1)$. The complementary fraction of productive agents, $1 - \lambda \Delta$, is formed by agents who were already productive in the previous period and held assets $\bar{x}$. The asset demand for these agents is the flow component that offsets the effect of money growth, given by $\mu \Delta (\bar{x} - 1)$. This gives

$$(1 - \Delta \lambda)\mu \Delta (\bar{x} - 1) + \lambda \Delta \int_0^{\bar{x}} \left[ \bar{x} - x^1 + \mu \Delta (\bar{x} - 1) \right] f(x^1, 2) \, dx^1 =$$

$$\left(1 - \Delta \lambda\right) \int_0^{\bar{x}} \frac{\epsilon (x^1, 2)\Delta}{q} f(x^1, 2) \, dx^1 + \lambda \Delta \frac{\epsilon(\bar{x}, 2)\Delta}{q}$$

Dividing by $\Delta$ and taking the limit as $\Delta \downarrow 0$ gives

$$\mu (\bar{x} - 1) + \lambda \int_0^{\bar{x}} (\bar{x} - x^1) \, f(x^1, 2) \, dx^1 = \int_0^{\bar{x}} \frac{\epsilon(x^1, 2)}{q} f(x^1, 2) \, dx^1$$
Using $\int_0^x f(x^1, 2) dx^1 = 1$ and $c^1(x^1, 2) = q/\gamma(x^1, 2)$ gives

$$
\mu(x-1) + \lambda x - \lambda \int_0^x x^1 f(x^1, 2) \, dx^1 = \int_0^x \frac{1}{\gamma(x^1, 2)} f(x^1, 2) \, dx^1
$$

Using that $\int x^i \, di = 1$ or, equivalently, $\frac{1}{2} \int_0^x f(x^1, 2) \, dx^1 + \frac{\bar{x}}{2} = 1$ gives

$$
\mu(x-1) + \lambda x - \lambda (2 - x) = \int_0^x \frac{1}{\gamma(x^1, 2)} f(x^1, 2) \, dx^1
$$

or

$$
\mu(x-1) + 2\lambda (x-1) = \int_0^x \frac{1}{\gamma(x^1, 2)} f(x^1, 2) \, dx^1
$$

which gives equation (33).

### Derivation of invariant wealth distribution

For $x^1 \in [0, \bar{x})$ we have $f(x^1, 1) = 0$. Using equation (29) to replace $\dot{x}^1(x^1, 2)$ in the KFE gives

$$
f_x(x^1, 2) = \frac{(\lambda - \mu)\gamma(x^1, 2) + \frac{\gamma_x(x^1, 2)}{\gamma(x^1, 2)}}{1 + \mu (x^1 - 1) \gamma(x^1, 2)}
$$

Using equation (31) to replace for $\frac{\gamma_x(x^1, 2)}{\gamma(x^1, 2)}$ in the above expression gives equation (34).

#### O.1 The case with constant money ($\mu = 0$)

It is interesting to analyze the case with $\mu = 0$ for the model with uncorrelated shocks because of its simplicity. We provide a characterization of the equilibrium and we prove existence and uniqueness.

After setting $\mu = 0$ equation (31) reduces to the following ODE: $\gamma_x(x^1, 2) = \lambda q\gamma(x^1, 2) - (\rho + \lambda)\gamma(x^1, 2)^2$, with boundary conditions that reduce to $lim_{x^1 \downarrow 0} \gamma(x^1, 2) = \infty$ and $\gamma(\bar{x}, 2) = \frac{q^{\rho+\lambda}}{\rho}$. Using the ODE and the first boundary provides an expression for the marginal value of money for an unproductive agent as a function of the price level $q$,

$$
\gamma(x^1, 2) = \frac{q\lambda}{\rho + \lambda} e^{q\lambda x^1} - 1
$$

which is strictly positive for every $q > 0$. This implies that the expected utility for an unproductive agent with wealth $x^1$, denoted by $v(x^1, 2)$, is given by the integral of $\gamma(x^1, 2)$, i.e.

$$
v(x^1, 2) = \frac{1}{\rho + \lambda} \log \left( e^{q\lambda x^1} - 1 \right) + \bar{v}
$$

where $\bar{v}$ is a finite constant.
Using the expression for \( \gamma(x^1, 2) \) and equation (30) gives an equation that relates \( \bar{x} \) and \( q \),

\[
\bar{x} = \frac{1}{\lambda q} \log \left( \frac{(\lambda + \rho)^2}{(\lambda + \rho)^2 - \lambda^2} \right)
\]  

(35)

which implies a unit elasticity of \( \bar{x} \) with respect to the price level \( q \).

Now we turn to evaluate the density of money holdings \( f(x^1, 2) \). Setting \( \mu = 0 \) in equation (34) reduces to

\[
\frac{f_x(x^1, 2)}{f(x^1, 2)} = \lambda q \left( 1 - \frac{\rho}{\lambda + \rho} \frac{e^{q\lambda x^1}}{(e^{q\lambda x^1} - 1)^{\frac{1}{\rho + \lambda}}} \right)
\]

where we used the expression we found for \( \gamma(x^1, 2) \). This expression is an ODE with solution

\[
f(x^1, 2) = Q_7 e^{q\lambda x^1} \left( e^{q\lambda x^1} - 1 \right)^{\frac{1}{\rho + \lambda}}, \quad \text{where } Q_7 \text{ is a constant to be determined next.}
\]

We have that

\[
\int_0^\bar{x} f(x^1, 2) dx^1 = 1
\]

We can use this equation to obtain \( Q_7 \). Therefore, the density \( f(x^1, 2) \) is

\[
f(x^1, 2) = \frac{q\lambda^2}{(\rho + \lambda)(e^{q\lambda x^1} - 1)^{\frac{1}{\rho + \lambda}}} \frac{e^{q\lambda x^1}}{(e^{q\lambda x^1} - 1)^{\frac{1}{\rho + \lambda}}}
\]

which is readily evaluated using equation (35).

An equilibrium exists if there exists a finite price level \( q \) such that the market clearing condition (see equation (33)) is satisfied. Using that \( \mu = 0 \) and by substituting the expressions we found for the marginal value of money \( \gamma(x^1, 2) \), density function \( f(x^1, 2) \), and that \( \bar{x} = \bar{x}(q) \), the market clearing condition reduces to

\[
\Upsilon_1(q) = \Upsilon_2(q)
\]

where \( \Upsilon_1(q) \equiv (\bar{x}(q) - 1)^2 \left( \frac{\lambda^2}{(\rho + \lambda)(x^1 - \lambda^2)} \right)^{\frac{1}{\rho + \lambda}} \) and \( \Upsilon_2(q) \equiv \int_{\bar{x}(q)}^{\bar{x}(q)} \left( e^{q\lambda x^1} - 1 \right)^{\frac{1}{\rho + \lambda}} dx^1 \), where both \( \Upsilon_1(q) \) and \( \Upsilon_2(q) \) are continuous and differentiable functions on \( q \). In order to check for existence and uniqueness of solution we do some analysis on these functions. Properties of \( \Upsilon_1(q) \): (i) \( \lim_{q \downarrow 0} \Upsilon_1(q) = +\infty \), (ii) \( \lim_{q \uparrow \infty} \Upsilon_1(q) < 0 \) and finite, (iii) strictly decreasing, (iv) strictly convex. With respect to \( \Upsilon_2(q) \) note that it is a function involving the hypergeometric function,

\[
\Upsilon_2(q) = \frac{\lambda + \rho}{\lambda q} \left( \frac{e^{q\lambda x^1(q)} - 1}{1 - e^{-q\lambda x^1(q)}} \right)^{\frac{1}{\rho + \lambda}} 2F1 \left( -\frac{\lambda}{\rho + \lambda}, -\frac{\lambda}{\rho + \lambda}; \rho + \lambda; e^{-q\lambda x^1(q)} \right)
\]

\[
- \lim_{q \downarrow 0} \frac{\lambda + \rho}{\lambda q} \left( \frac{e^{q\lambda x^1(q)} - 1}{1 - e^{-q\lambda x^1(q)}} \right)^{\frac{1}{\rho + \lambda}} 2F1 \left( -\frac{\lambda}{\rho + \lambda}, -\frac{\lambda}{\rho + \lambda}; \rho + \lambda; e^{-q\lambda x^1(q)} \right)
\]
which, using equation (35) and that \( \lim_{y \downarrow 0} \left( \frac{e^\lambda y - 1}{1 - e^{-\lambda y}} \right) = 1 \), reduces to

\[
\Upsilon_2(q) = \frac{\rho + \lambda}{\lambda q} \left( \frac{(\rho + \lambda)^2}{(\rho + \lambda)^2 - \lambda^2} \right) \quad 2 F_1 \left( -\frac{\lambda}{\rho + \lambda}, -\frac{\lambda}{\rho + \lambda}; \frac{\rho + \lambda}{\rho + \lambda}; 1 \right)
\]

with the following properties: (i) \( \Upsilon_2(q) < 0 \), (ii) \( \lim_{q \downarrow 0} \Upsilon_2(q) = -\infty \), (iii) \( \lim_{q \uparrow \infty} \Upsilon_2(q) = 0 \), (iv) \( \Upsilon_2(q) \) strictly increasing, and (v) \( \Upsilon_2(q) \) strictly concave.

Note now that given the listed properties of \( \Upsilon_1(q) \) and \( \Upsilon_2(q) \), there exists a unique value \( q \) such that \( \Upsilon_1(q) = \Upsilon_2(q) \), so that the market clearing condition is satisfied. This implies that when there is constant money in the economy, i.e. \( \mu = 0 \), there exists a unique monetary equilibrium.

O.2 Optimality of expansionary policy

In this section we discuss the optimality of a stationary expansionary policy, for an economy that has reached the invariant distribution of wealth. This exercise exploits the fact that we know the invariant distribution of \( x : f(x) \). The government wishes to maximize ex-ante expected welfare under the stationary density,

\[
W(\mu) = \mathbb{E}_x \left\{ u \left( c^1(x, 2; \mu) \right) + u \left( c^2(1 - x, 2; \mu) \right) - I^2(1 - x, 2; \mu) \right\} \\
= \int_0^1 f(x, 2; \mu) \left[ \ln \left( c^1(x, 2; \mu) \right) + \ln(1) - \left( 1 + c^1(x, 2; \mu) \right) \right] \, dx
\]

where the notation emphasizes that the consumption paths and the probability density of money holdings depend on the money growth rate \( \mu \), \( c^2(1 - x, 2; \mu) = 1 \) because of linear utility of labor, and \( I^2(1 - x, 2; \mu) = 1 + c^1(x, 2; \mu) \). The expression for \( W(\mu) \) measures the stationary ex-ante (expected) utility, i.e. the welfare of any given agent before her initial state is realized. Types are given equal weights because the symmetry of the Markov process for the shocks implies that agents are productive 1/2 of the time. It is assumed that initial money holdings \( x \) are drawn from the invariant distribution \( f(x; 2) \).

It is straightforward that monetary contractions (i.e. \( \mu < 0 \)) and extreme expansions (i.e. \( \mu \uparrow \infty \)) have ex-ante expected utility that diverges.\(^{22}\) That is, \( W(\mu) \downarrow -\infty \) in both cases.

\(^{22}\)The argument is analogous to the one developed in Appendix L. In both cases there is a brake in trade.
Figure 7: Derivative and level of ex-ante expected welfare $W(\mu)$ at $\mu = 0$

Parameters: $\rho = 0.05$. 