Large Contests

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Abstract

We consider contests with many, possibly heterogeneous, prizes and players that include many existing models as special cases. We show that the outcome of such contests is approximated by an appropriately defined set of incentive-compatible individually-rational single-agent mechanisms.

1 Introduction

There are many real-world competitions in which participants invest resources before they know whether they win or lose. Relevant settings include promotions within organizations (or professional awards), research and development races, political lobbying, sports, and college admissions.

Such competitions are often modelled in the economics literature as contest models, which generalize the all-pay auction with complete information to allow for multiple prizes and asymmetric players. The equilibria of such contests are not easy to derive, however, and typically have a complicated structure (for example, see Clark and Riis (1998) or Siegel (2010, 2011)). They necessary involve mixed strategies, because of the combination of complete information and the all-pay feature: if an agent is certain about the other

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agents’ bids, then he will either outbid them slightly or bid 0; in either case, the other players are not best responding to his strategy. In addition, players’ mixed strategies often include atoms at some bids, continuous distributions on some intervals of bids, and gaps on other intervals. Consequently, equilibrium characterizations of finite contests typically take the form of an algorithm (see, for example, Bulow and Levin (2006) or Siegel (2010, 2011)). This makes further analysis, such as comparative statics, difficult or impossible.

In many of these settings, there are many active participants. And even when the number of active participants is not large, the pool of potential competitors may be quite substantial. Therefore, a possible remedy for the difficulty in solving contest models with a finite number of competitors would be to study contests with continuum of prizes and agents. While the continuum approach is useful in other settings, it encounters substantial difficulties in contests. If agents play mixed strategies, i.e., each of a continuum of agents randomizes over a continuum of bids, then it is difficult to compute the resulting distribution over prizes, even under the assumption of joint measurability of strategies. But this is not the only difficulty. For example, in the case of half as many prizes as agents, it is easy to compute the distribution over prizes when a half of the agents bid 0 and the other half bid 1, but it is unclear what the distribution would be when a single agent deviates to bidding 1/2.\(^1\)

We take a different approach, and show that mechanism-design methods can be used to study large contests. We do this by proving that the equilibria of contests with a large number of agents and prizes converge to the outcomes of a single-agent incentive compatible (IC), individually rational (IR) mechanism with a continuum of types. Under a single-crossing condition on the agents’ payoff functions, we show that this mechanism implements a unique allocation, so one can apply standard techniques from the mechanism-design literature to approximate the equilibria of discrete large contests.

The intuition for the convergence result is the following. Every equilibrium along the sequence is IR, so the limit must also be IR. As the number of agents grows large, the

\(^1\)One may also wish to consider only deviations by sets of agents of positive measure. Then, however, the equilibria of finite contests would not converge to equilibria in the case of continuum (see the first example of Section 3).
competition they face becomes similar. That is, the mappings between bids and distributions of prizes that the agents face (given the other agents’ equilibrium strategies) become similar for all agents, and coincide in the limit. This means that, in the limit, each agent can mimic any other agent in terms of what they obtain. This yields a mapping $T$ from bids to (distributions of) prizes, such that agents choose their bids $t$ and corresponding prizes $T(t)$, as in a single-agent mechanism-design setting. In particular, the mechanism defined by this mapping is IC.

The notion of convergence plays an important role in our results. Our most general, but weakest, result shows the convergence of distributions over agents, prizes, and bids in the weak*-topology. This type of convergence enables us to approximate the average strategy (or the distribution over (bid,prize) pairs) of players whose types are close to a given type, but this may not be a good approximation of the equilibrium strategy of any single agent.

Under single crossing, however, we establish a much stronger form of convergence, which delivers a simultaneous approximation of all agents’ equilibrium strategies. In particular, we are able to say (approximately, but with an arbitrary degree of precision) how each agent will bid, and what prize she will obtain by making any given bid. This is because both the mapping $T$ from bids to prizes and the equilibrium strategies become deterministic when the number of agents grows large.

Our results are useful in the analysis of large contests in three, somewhat related, ways. First, they offer a simple approximate solution of models whose exact solution is complicated. Second, we approximate the solution of models for which there was no previously existing characterization of equilibria. Finally, and perhaps most importantly, our approach seems useful for further analysis of equilibria, such as welfare analysis and comparative statics. For example, our convergence results imply immediately that large contests are efficient under single crossing, a result which was derived by Bulow and Levin (2006) in a non-trivial manner, and only for a specific form of the agents’ payoff functions.

The rest of the paper is organized as follows. Section 2 introduces some basic terminology and notation. In Section 3, we present some examples. One purpose of having these examples is to illustrate our results and the arguments that support them in con-
crete scenarios. More importantly, the examples describe important applications, which were studied in the previously existing literature. Finally, in the case of these more specific models, we are able to show some additional properties of large contests, which do not always hold in the general case. Section 4 contains our main results, their proofs, and some discussion of other applications. In Section 5, we generalize our main results to some cases in which the proofs are slightly more involved.

2 Terminology and notation

2.1 Continuum of agents and prizes

There is a mass 1 of agents and a mass 1 of prizes. Each agent is characterized by a single parameter $x \in X = [0, 1]$ with a CDF $F$, where $F(x)$ is the mass of agents $z$ such that $z \leq x$. Each prize is characterized by a single parameter in $[0, 1]$, and we sometimes refer to prize 0 as “no prize.” The CDF $G$ describes the set of prizes, where $G(y)$ for $y \in Y = [0, 1]$ is the mass of prizes $z$ such that $z \leq y$.

Each agent has a utility over prizes and bids. The overall utility of an agent of type $x$ from bidding $t$ and obtaining a prize of type $y$ is

$$v_x(y) - c_x(t),$$

where $v_x(y) \in [0, 1]$ is the utility from obtaining prize $y$, and $c_x(t) \geq 0$ is the cost of bidding $t$. The agent’s utility from obtaining no prize is 0, and the cost of bidding 0 is 0.

We assume that $v_x(y)$ is continuous in $x$ and $y$ and strictly increasing in $y$, and $c_x(t)$ is continuous in $x$ and $t$ and strictly decreasing in $t$.

A consistent allocation is a probability distribution $H$ on $X \times Y$, whose marginal on $X$ coincides with distribution $F$, and whose marginal on $Y$ coincides with distribution $G$. The conditional distribution $H_x$, $x \in X$, will be interpreted as the lottery over prizes faced by an agent of type $x$.

An allocation $H$ is efficient if it maximizes the aggregate utility of all agents, i.e., it

$^2$All probability measures are assumed to be defined on the $\sigma$-algebra of Borel sets.
maximizes
\[ \int_{x \in X} \int_{y \in Y} v_x(y) dH(x, y) \]
across all allocations.

A (direct) mechanism \( M \) prescribes for each announced type \( x \in X \) a joint probability distribution \( Q_x(y, t) \) over prizes \( y \in Y \) and bids \( t \in R \). We refer to the cost of a prescribed bid as transfer. A mechanism is incentive compatible (IC) if the expected utility of each type of agent \( x \) is maximized by reporting \( x \) truthfully, i.e.,
\[ \int_{y \in Y} \int_{t \in R} [v_x(y) - c_x(t)] dQ_x(y, t) \]
is maximized at \( z = x \).

An IC mechanism is individually rational (IR) if this highest expected utility is at least 0, i.e.,
\[ \int_{y \in Y} \int_{t \in R} [v_x(y) - c_x(t)] dQ_x(y, t) \geq 0. \]
We restrict attention to mechanisms such that this inequality is an equality for at least one type \( x \).

We say that an IC, IR mechanism implements an allocation \( H \) if the marginal of \( Q_x \) on \( Y \) coincides with \( H_x \) for almost every \( x \). This does not imply that \( H \) and \( \{Q_x : x \in X\} \) determine a probability distribution \( D \) on \( X \times Y \times [0, \infty) \).

3 To see why, take some non-measurable function \( f : X \to [0, \infty) \), have \( H \) distributed uniformly on, and assign probability 1 to, the diagonal \( \{(x, x) : x \in X\} \), and have \( Q_x \) assign probability 1 to the pair \((f(x), x)\). That is, type \( x \) bids \( x \) and obtains prize \( f(x) \). Note that if the costs of bidding and utilities of all type are 0 then this mechanism is IC, IR.

4 In general, some mechanisms may not determine a probability distribution on \( X \times Y \times [0, \infty) \). Suppose for example that \( H \) is the uniformly distributed on the diagonal of \( X \times Y \), and take a nonmeasurable function \( f : X \to [0, \infty) \). Then, take the mechanism that prescribes to agent \( x \) prize \( x \) and transfer \( f(x) \).

However, in applications (or under some mild conditions on the primitives of the model) mechanisms will be regular. We disregard the discussion under what conditions they are regular, as this issue is orthogonal to the objectives of our paper. Instead, we assume directly that a mechanism determines a probability distribution on \( X \times Y \times [0, \infty) \).
2.2 Finite number of agents and prizes

We approximate the setting with continuum of agents and prizes by using complete-information contests with \( n \) (ordered) agents and \( n \) (ordered) prizes, some of which may be worth 0 (these correspond to “no prize”). In such a contest, all agents make their bids simultaneously and pay the associated costs. The agent who makes the highest bid obtains the highest prize, the agent with the second-highest bid obtains the second-highest prize and so on, until all prizes are exhausted. In case of a tie, the highest tied agent obtains the highest (relevant) prize, the second-highest tied agent obtains the second-highest (relevant) prize and so on, until all the relevant prizes are exhausted.\(^5\)

The agents and prizes correspond to the \( n \)-quantiles of the distributions of agents and prizes in the continuum setting. That is, we set the utility of agent \( i \) from obtaining prize \( j \) to \( v_x(y) \), where \( F(x) = i/n \) and \( G(y) = j/n \), and we set agent \( i \)’s cost of bidding \( t \) to \( c_x(t) \), where \( F(x) = i/n \). The primitives of the game are commonly known.\(^6\)

To formalize our notion of approximation, we transform the equilibrium outcomes in the discrete case to probability distributions on \( X \times Y \times [0, \infty) \). We then relate these distributions to probability distributions \( D \) that describes the outcomes of regular mechanisms. To begin, note that an equilibrium of an all-pay auction determines for every agent a joint distribution over her bids and the prizes she obtains. Denote by \( Q^n_{i}(j, t) \) agent \( i \)’s equilibrium probability of obtaining prize \( j \) if she bids \( t \). We “smooth out” the mass \( 1/n \) associated with each prize by defining distribution \( \overline{Q}^{n}_{i} \) on \( Y \times [0, \infty) \) so that with probability \( Q^{n}_{i}(j, t) \) when agent \( i \) bids \( t \), she obtains a prize \( y \) such that \((j-1)/n < G(y) \leq j/n \). More precisely, the measure that \( \overline{Q}^{n}_{i} \) assigns to any Borel subset of the set of \( y \)’s such that

\(^5\)We choose this tie-breaking rule for its expositional convenience. Any other tie-breaking rule (deterministic or random) would work equally well, as long as it is specified in advance.

\(^6\)The special case in which \( c_x(t) = t \) corresponds to the all-pay auction with complete information and (possibly) heterogeneous prizes.
Finally, we “smooth out” the mass $1/n$ associated with each player by setting the distribution of bids and prizes for every type $x$ with $(i-1)/n < x \leq i/n$ to coincide with $Q^n_i$. More precisely, we define a distribution $D^n$ on $X \times Y \times [0, \infty)$ by letting its marginal with respect to $x$ coincide with $Q^n_i$ where $(i-1)/n < x \leq i/n$.9

**Definition 1** The (specific selection of) equilibria of discrete contests approximate the outcome of a regular mechanism $M$ if $D^n \rightarrow D$ in the weak*–topology.

Recall that the weak*–topology in the set of all probability measures on a 3-dimensional cube that consists of all unions of finite intersections of sets of the form\[ \{Q : |E^P f - E^Q f| < \varepsilon\}, \]
where $E$ stands for the expected-value operator $P$ and $Q$ are probability measures, $\varepsilon > 0$, and $f$ is a real-valued and continuous function on the cube. We refer the reader to Rudin (1973) for additional details on the weak*-topology.

Our results say that discrete contests approximate mechanisms with continuum of agents and prizes in the sense of Definition 1. It is therefore useful to interpret this definition properly. Suppose we are interested in the strategy (or the distribution over (bid,prize) pairs) of an agent of some (rational) type $x$ in the $n$-th contest (where $n$ is large). We cannot simply take the conditional of $D$ on $\{x\} \times Y \times [0, \infty)$ as an approximation, because the convergence in weak*–topology is only up to sets of measure zero. More generally, we cannot pin down the strategy of a single type (even in approximation) when we know only the limit distribution.

We can, however, approximate the distribution over (bid,prize) pairs of types close to $x$. Indeed, take a closed interval $I$ that contains $x$, and a slightly larger open interval

Expressed as a CDF,\[ \overline{Q}^n_i (y, t) = \sum_{k=0}^{j-1} Q^n_i(k, t) + Q^n_i(j, t) n \left( G(y) - \left( \frac{(j-1)}{n} \right) \right) \]
and $\overline{Q}^n_i (t, 0) = Q^n_i (t, 0)$.\[ \text{That } \overline{Q}^n_i \text{ is well defined as a probability distribution on } Y \times [0, \infty) \text{ follows from standard arguments.} \]

That $D^n$ is well defined as a probability distribution on $X \times Y \times [0, \infty)$ follows from standard arguments.

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9 That $D^n$ is well defined as a probability distribution on $X \times Y \times [0, \infty)$ follows from standard arguments.
J, and “average out” the distribution D across these two intervals. More precisely, we should average out the conditional distributions on I × Y × [0, ∞) and J × Y × [0, ∞).

These two averages can be interpreted as some bounds on the distribution of types close to x in large contests. The validity of this approach is guaranteed by the convergence in weak*-topology, if we take as f a function with values in the interval [0, 1], which takes value 1 on I × Y × [0, ∞) and takes value 0 on the complement of J × Y × [0, ∞).

Finally, the convergence in the sense of Definition 1 may not guarantee that the outcomes of mechanisms with continuum agents and prizes approximate jointly the equilibrium strategies of all agents in large discrete contests. The reason is that in order to obtain a good approximation of the strategies of types in a neighborhood of some x, we need to take a sufficiently large n. Then, the neighborhood may well many other types of the form i/n, and form the limit distribution we only learn about the average strategy across all these types. Therefore, we will seek results which establish convergence in a stronger sense than that in the sense of Definition 1.

3 Examples

Example 1 (based on Clark and Riis (1998), see also Siegel (2010)) Suppose that each agent values all prizes equally, but different agents may value the prizes differently. More specifically, let x denote the utility of an agent of type x from obtaining a positive prize, i.e., v_x(y) = x for all y > 0. The cost of bidding is the same across all agents; more specifically, let c_x(t) = t for all x and t. For the sake of this example, we assume a uniform type distribution, that is, F(x) = x for all x ∈ X. The prize distribution is G(y) = 1/2 for all y ∈ [0, 1), and G(1) = 1. That is, there is a mass 1/2 of non-zero prizes. We make these simplifying assumptions for ease of exposition.

In this case, all efficient allocations assign prizes to agents who value them most; for example, this can be accomplished by the probability measure H which assigns probability 1 to, and is distributed uniformly on, the set

\{(x, 0) : x ≤ 1/2\} ∪ \{(x, 1) : x > 1/2\}.

10 We provide an example that illustrates this point in Section 3.
Any IC, IR mechanism that implements this allocation prescribes transfer 0 for all agents of type $x < 1/2$, and prescribes transfer $1/2$ for all agents of type $x \geq 1/2$. In this, and the three other examples discussed later, it is easy to see that the IC, IR mechanisms that implements efficient allocations are regular.

We now show that this outcome is approximated by the equilibria of contests with a large number of agents and prizes. This is useful because the equilibria of contests necessary involve mixed strategies and are therefore not simple to derive. Mixed strategies arise because of the combination of complete information and the all-pay feature: if an agent is certain about the other agents’ bids, then he will either outbid them slightly or bid 0; in either case, the other players are not best responding to his strategy. In contrast, it is straightforward to find the efficient allocation and the IC, IR mechanism that implements this allocation in the case of continuum.

Our approach is to partially characterize the equilibria for large $n$, using equilibrium properties similar to those derived by Clark and Riis and Siegel. This direct approach from first principles does not rely on a complete characterization of equilibrium, and is therefore in the spirit of our general approach, although it uses arguments specific to this example. The partial characterization shows that for every $\varepsilon > 0$, if $n$ is sufficiently large, then:

(a) the fraction of agents $i$ such that $i/n < 1/2$ who obtain no prize with probability higher than $1 - \varepsilon$ is higher than $1 - \varepsilon$; (b) the fraction of agents $i$ such that $i/n > 1/2$ who obtain a prize with probability higher than $1 - \varepsilon$ is higher than $1 - \varepsilon$;

(c) the fraction of agents $i$ such that $i/n < 1/2$ who bid $t \in [0, \varepsilon]$ with probability higher than $1 - \varepsilon$ is higher than $1 - \varepsilon$;

(d) the fraction of agents $i$ such that $i/n > 1/2$ who bid $t \in [1/2 - \varepsilon, 1/2]$ with probability higher than $1 - \varepsilon$ is higher than $1 - \varepsilon$.

\[11\] An alternative approach is to use the complete equilibrium characterization of Clark and Riis and Siegel, and take the limit as the number of players tends to infinity. This is what we do in later examples. This alternative approach is by no means simpler, and the more basic approach we take in this example better illustrates how increasing the number of players affects the relevant aspects equilibrium characteristics.
We will say that an agent bids close to $t$ with positive probability when for every $\delta > 0$, the probability of bidding $s \in [t - \delta, t + \delta]$ is positive. We denote by $m$ the number of non-zero prizes ($m$ differs from $n/2$ by at most 1). The following properties, which we will use to prove (a)-(d), hold in any equilibrium of the contest.

**Property 1.** At least $n - m + 1$ agents bid close to 0 with positive probability.

Let $b_i = \min \{b > 0 : \text{agent } i \text{ bids close to } b \text{ with positive probability}\}$. Suppose that there are only $k \leq n - m$ agents who bid close to zero with positive probability. Then $\{b_i : b_i > 0\} \neq \emptyset$; let $b^* = \min \{b_i : b_i > 0\}$. No player bids in $(0, b^*)$, because the probability of winning by bidding any $t \in (0, b^*)$ is 0.

Let $l$ be the number of agents whose strategies have an atom at $b^*$. If $k + l > n - m$, then any of these $l$ agents could profitably deviate by bidding slightly more than $b^*$, thereby increasing the probability of obtaining a prize discretely. If $k + l \leq n - m$, then the payoff of any agent $i$ such that $b^* = b_i$ would be negative, because the probability of winning by bidding close to $b^*$ would be close to 0.

**Property 2.** Exactly $n - m$ agents’ strategies have atoms at 0. These are agents 1, 2, ..., $n - m$.

Suppose first that the strategies of more than $n - m$ agents had atoms at 0. Thus, with positive probability all these players bid 0 simultaneously. Therefore, any of these agents would profitably deviate by bidding slightly more than 0, thereby increasing the probability of obtaining a prize discretely.

Suppose now that the strategies of fewer than $n - m$ agents had atoms at 0. This implies that the equilibrium payoff of any agent who bids close to 0 with positive probability is 0. By Property 1, there are at least $n - m + 1$ agents who make such bids. However, the equilibrium payoff of the $m$ agents $n - m + 1, n - m + 2, ..., n$ must be positive. Indeed, none of the agents $1, 2, ..., n - m$ bids more than $1/2$ in equilibrium, and therefore any agent $i > n - m$ can ensure a positive payoff by bidding $t \in (1/2, 1/2 + 1/n)$.

Since the payoff of every agent whose strategy has an atom at 0 is equal to 0, these are agents 1, 2, ..., $n - m$. 

Property 3. Exactly $m+1$ agents bid close to $1/2$. These are agents $n-m, n-m+1, \ldots, n$.

Take the highest bid $b$ such that some agent bids close to it with positive probability. Bids close to $b$ must be made with positive probability by at least $m+1$ agents; otherwise, agents who bid close to $b$ with positive probability could profitably deviate by refraining from bidding sufficiently close to $b$.

If $b$ were higher than $1/2$, then only the $m$ agents $n-m+1, n-m+2, \ldots, n$ could bid close to $b$ with positive probability, because agents $1, 2, \ldots, n-m$ never bid more than $1/2$. Thus, $b$ is at most $1/2$.

If $b$ were strictly lower than $1/2$, agent $n-m$ could obtain a positive payoff by making any bid between $t \in (b, 1/2)$. However, by Property 2 the equilibrium payoff of this agent is $0$. Thus, $b = 1/2$.

Since agents $1, 2, \ldots, n-m-1$ never bid more than $1/2 - 1/n$, the agents who bid close to $1/2$ are agents $n-m, n-m+1, \ldots, n$.

In order to establish (a)-(d), we first show that for every $\varepsilon > 0$, if $n$ is sufficiently large, then the atoms at 0 of the strategies of agents $i = 1, \ldots, n-m$ are larger than $1-\varepsilon$ for all but a fraction lower than $\varepsilon$ of those agents. This, and the fact that only $n-m$ agents’ strategies have atoms at 0, imply conditions (a) and (c). Denote these atoms by $p_1, \ldots, p_{n-m}$.

Suppose to the contrary that for arbitrarily large values of $n$, a fraction $\varepsilon$ of those agents have atoms lower than $1-\varepsilon$ at 0. By Property 1, there is an agent $i \geq n-m+1$ who bids close to 0 with positive probability. The payoff of that agent is

$$p_1 \cdot \ldots \cdot p_{n-m} \cdot (1/2 + (i-n+m)/n) \leq (1-\varepsilon)^{n-m} \cdot 1^{(1-\varepsilon)(n-m)} \cdot (1/2 + (i-n+m)/n).$$

This payoff must be at least $(i-n+m)/n$, since agent $i$ can obtain a payoff arbitrarily close to $(i-n+m)/n$ by bidding slightly more than $1/2$. Thus,

$$\frac{(i-n+m)/n}{1/2 + (i-n+m)/n} \geq (1-\varepsilon)^{n-m} \cdot 1^{(1-\varepsilon)(n-m)} 
\geq (1-\varepsilon)^{n-m} \cdot 1^{(1-\varepsilon)(n-m)} \cdot (1/2 + (i-n+m)/n).$$

This, however, contradicts the fact that the right-hand side of this inequality tends to 1 as $n$ tends to $\infty$. To see the convergence, notice that the right-hand side takes the highest
value at \( i = n \) and the lowest value at \( i = n - m + 1 \). Together with the assumption that \( n = 2m \), this yields

\[
\sqrt[n]{\frac{1/2m}{1/2 + 1/2m}} \leq \sqrt[n]{\frac{(i - n + m)/n}{1/2 + (i - n + m)/n}} \leq \sqrt[n]{\frac{1/2}{1/2 + +1/2}}.
\]

Now, the convergence follows from the fact that \( \sqrt{m} \to_{m \to \infty} 1 \), and \( \sqrt{c} \to_{m \to \infty} 1 \) for any constant \( c > 0 \).

To see (d), suppose to the contrary that for arbitrarily large values of \( n \), a fraction \( \varepsilon \) of agents \( i \) such that \( i/n > 1/2 \) bid less than \( 1/2 - \varepsilon \) with probability \( \varepsilon \). There are \( n\varepsilon/2 \) of these agents. Call a success a single agent’s bid of \( t < 1/2 - \varepsilon \). Therefore the probability of success is \( \varepsilon \). The agents bid independently. So by the law of large numbers (LLN) the fraction of these agents who succeed, i.e., bid less than \( 1/2 - \varepsilon \), is at least \( 3\varepsilon/4 \) with probability that approaches 1 as \( n \to \infty \).

On the other hand, (c) with \( \varepsilon^2/4 \) in place of \( \varepsilon \) implies that the fraction of agents \( i \) such that \( i/n < 1/2 \) who bid no more than \( \varepsilon^2/4 \) with probability higher than \( 1 - \varepsilon^2/4 \) is higher than \( 1 - \varepsilon^2/4 \). There are \( n(1 - \varepsilon^2/4)/2 \) of these agents. By the LLN the fraction of these agents who bid no more than \( \varepsilon^2/4 \) is at least \( 1 - \varepsilon^2/2 \) with probability that approaches 1 as \( n \to \infty \).

The two applications of the LLN imply that the fraction of all agents \( i \) who bid no more than \( \max\{1/2 - \varepsilon, \varepsilon^2/4\} \) is higher than

\[
\frac{1}{2} \left(1 - \frac{\varepsilon^2}{4}\right) \left(1 - \frac{\varepsilon^2}{2}\right) + \frac{1}{2} \frac{3\varepsilon}{4} > \frac{1}{2}
\]

with probability that approaches 1 as \( n \to \infty \).

Therefore, for sufficiently large values of \( n \), agent \( i = n - m \) can obtain a payoff bounded away from 0 (by a bound independent of \( n \)) by bidding \( t \in (\max\{1/2 - \varepsilon, \varepsilon^2/4\}, 1/2) \). This contradicts the fact that the payoff of this agent is 0.

By (d), a fraction arbitrarily close to 1 of agents \( i \) such that \( i/n > 1/2 \) bid slightly less than \( 1/2 \) with probability arbitrarily close to 1. Since each of these agents can secure a prize by bidding slightly above \( 1/2 \), each of them must obtain a prize with probability arbitrarily close to 1 when bidding slightly less than \( 1/2 \). This yields (b).

Conditions (a)-(d), together with the definition of weak-*-convergence, guarantee the convergence from Definition 1.
We end the discussion of Clark and Riis’ contest with a remark on Definition 1. Suppose that \( v_x(y) = 1 \) instead of \( v_x(y) = x \), and let \( n = 2k + 1 \). Then the \( n \)-th contest has an equilibrium in which the \( k \) even agents \( 2, 4, \ldots, 2k \) bid 0 and obtain no prize, and the \( k + 1 \) odd agents \( 1, 3, \ldots, 2k + 1 \) use the same mixed strategy on \([0, 1]\). In this equilibrium, as Example 1 shows, as \( n \) increases, the mixing agents bid close to 1 and obtain prize 1 with probability close to 1. Therefore, the distributions \( D^n \) converge in weak*-topology to distribution \( D \) in which every type \( x \) bids 0 and obtains 0 with probability 1/2 and pays 1 and obtains 1 with probability 1/2. Consequently, the strategies of all agents in the discrete contests qualitatively differ from the conditionals of distribution \( D \).

**Example 2** (based on Bulow and Levin (2006)) Suppose that distributions \( F \) and \( G \) are uniform.\(^{12}\) All agents have identical linear costs of bidding, \( c_x(t) = t \) for all \( x \) and \( t \), and the utility of agent of type \( x \) from prize \( y \) is \( v_x(y) = xy \).

The efficient allocation is assortative, that is, agent \( x \) obtains prize \( x \). Because this allocation is non-decreasing, it can be implemented by an IC mechanism. That is, there exist transfers such that an agent of type \( x \) will report her type truthfully and obtain prize \( x \). These transfers are pinned down up to the transfer of the lowest type, which in an IR mechanism has to be 0 (because type 0 obtains 0). From the envelope theorem, we have that the derivative of the utility of an agent of type \( x \) is \([x^2 - t(x)]' = x \). Therefore, \( t(x) = x^2 - \int_0^x zdz = x^2/2 \). We will show that this outcome is approximated by equilibria of contests with a large number of agents and prizes. As in the previous examples, the simplicity of the allocation and the IC, IR mechanisms that achieves it contrasts with the relative complexity of players’ equilibrium mixed strategies. Moreover, while the IC, IR mechanism are given explicitly, players’ mixed strategies are derived by an algorithm and are not described in closed form.

To provide some intuition for the approximation, we provide a heuristic argument that portrays approximately the equilibrium strategies when the number of agents and prizes is large. This argument makes use of some equilibrium properties demonstrated by Bulow and Levin, and does not require their full algorithm for constructing the equilibrium. A

\(^{12}\)Bulow and Levin make this assumption in Section 7, in which they study the all-pay auction for \( n \rightarrow \infty \).
complete, but somewhat less illuminating, proof of the approximation can be obtained by adapting the proof of Theorem 1 below.

The outline of the argument is as follows. Each agent chooses a bid from an interval, and the intervals are staggered so that the intervals of higher agents have higher lower and upper bounds. The number of agents that have a given bid in their interval is known. This tells us we can divide each bidding interval to a known number of subintervals of equal length, such that the density on each subinterval is known and constant. This gives us the length of the interval and also how this length changes across players. This also shows that the intervals shrink to 0. Taken together, these observations imply efficiency and pin down the bid of each type.

We now describe the argument in greater detail. Bulow and Levin show that agent $i$’s strategy is continuously distributed on an interval $[b^n_i, d^n_i]$. The intervals have the property that $b^n_i < b^n_j$ and $d^n_i \leq d^n_j$ for any $i < j$ (except $i = 1$ and $j = 2$, in which case we have that $b^n_i = b^n_j = 0$).

In particular, if a bid $t$ is contained in some agent’s bidding interval, then it is contained in the bidding intervals of agents $l(m), l(m) + 1, \ldots, m$, where $l(m)$ is the lowest agent whose interval contains $t$, and $m$ is the highest agent whose interval contains it. Bulow and Levin show that

$$l(m) = \arg\min_l \left\{ \frac{1}{m - l} \sum_{k=l}^{m} \frac{n^2}{k} - \frac{n^2}{l} > 0 \right\},$$

and that the density of the strategy of agent $l$, $l(m) \leq l \leq m$, at bid $t$ belonging to her bidding interval is

$$\frac{1}{m - l(m)} \sum_{k=l(m)}^{m} \frac{n^2}{k} - \frac{n^2}{l}.$$  

(see their Lemma 2 and the paragraph following the lemma).

Choose a rational $x \in (0, 1)$, and take a sequence of $n, m \to \infty$ such that $x = m/n$. Partition the bidding interval of agent $m$ into subintervals such that agent $m$ is the lowest bidder on the rightmost subinterval, the second lowest bidder on the second rightmost subinterval, and so on, until the leftmost subinterval, on which she is the highest bidder. It follows from (1) that $m - l(m)$ differs from $\sqrt{2l(m)}$ by at most 1 (see their Lemma 3). In particular, $l(m)$ is of order $m$. Therefore, the number of subintervals in the partition is
approximately $\sqrt{2m}$.

From (2), the density of agent $m$’s strategy on these subintervals is (approximately)

$$d, d + \frac{n^2}{l(m)[l(m) + 1]}, d + \frac{2n^2}{l(m)[l(m) + 2]}, \ldots, d + \frac{\sqrt{2mn^2}}{l(m)[l(m) + \sqrt{2m}]}$$

respectively, where $d$ is the density on the rightmost subinterval. Moreover, (1) implies that $d$ cannot exceed $\frac{n^2}{l(m)[l(m) - 1]}$.

The lengths of these subintervals are (approximately) equal. Denote this common length by $\Delta$. Since

$$d\Delta + \left[d + \frac{n^2}{l(m)[l(m) + 1]}\right]\Delta + \left[d + \frac{2n^2}{l(m)[l(m) + 2]}\right]\Delta + \ldots + \left[d + \frac{\sqrt{2mn^2}}{l(m)[l(m) + \sqrt{2m}]}\right]\Delta \approx 1$$

and $l(m)$ is of order $m$, we have that $\Delta$ is of order $x^2/m$.

This implies that the length of each bidding interval tends to 0 as $n, m \to \infty$. Passing to a subsequence if necessary, these shrinking intervals converge to a number $t(x)$. Since $t(x)$ and $t(x - 1/n)$ differ (approximately) by the length of one interval, $\Delta$, the derivative of $t(x)$ at $x$ is

$$t'(x) = \frac{\Delta}{1/n} = x. \quad (3)$$

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In particular, in the limit agents of higher types obtain higher prizes. This implies that agent \( x \) obtains prize \( x \). Since type 0 obtains 0, (3) implies that agent \( x \) bids \( t(x) = x^2/2 \).

**Example 3** (based on Barut and Kovenock (1998)) Suppose all agents have identical linear costs of bidding, \( c_x(t) = t \) for all \( x \) and \( t \). They also value prizes in the same way, but the value of lower-ranked prizes is lower, \( v_x(y) = y \) for all \( x \) and \( y \). We assume uniform distribution of agents and prizes: \( F(x) = x \) for all \( x \in X \), and \( G(y) = y \) for all \( y \in [0,1] \). We make this simplifying assumptions for ease of exposition.

In this case, all allocations are efficient. In particular, so is the uniform allocation whose density is \( h(x,y) = 1 \) for all values of \( x \) and \( y \). The unique IC, IR mechanism that implements this allocation assigns an agent prize \( y \) if the agent bids \( y \). In response, agents randomize uniformly across all bids \( t \in [0,1] \). That is, \( Q_x(t,y) \) is distributed uniformly on the diagonal \( y = t \), for every type \( x \).

When there are \( n \) agents and prizes, we assume that each agent’s value of prize \( j = 1, \ldots, n \) is \( (j-1)/(n-1) \).\(^{16}\)

The contest has a unique equilibrium. In equilibrium, all agents randomize uniformly across all bids \( t \in [0,1] \). To see that these strategies constitute an equilibrium, note that the payoff to bidding \( t \), given that other agents randomize uniformly, is

\[
\left( \begin{array}{c} n-1 \\ n-1 \end{array} \right) t^{n-1} \left( \begin{array}{c} n-1 \\ n-2 \end{array} \right) t^{n-2} (1-t) \left( \begin{array}{c} n-2 \\ n-1 \end{array} \right) + \cdots + \left( \begin{array}{c} n-1 \\ 0 \end{array} \right) (1-t)^{n-1} \frac{0}{n-t} = \left( \begin{array}{c} n-2 \\ n-2 \end{array} \right) t^{n-1} + \cdots + \left( \begin{array}{c} n-2 \\ 0 \end{array} \right) (1-t)^{n-2} - t = t(1 + 1) - t = 0.
\]

In particular, the agents are indifferent across all bids, given that other agents randomize uniformly across all bids \( t \in [0,1] \). For the proof of uniqueness of this equilibrium, see Barut and Kovenock (1998).

Thus, the agents behave in the contest exactly as they do in the case of continuum. Consequently, each agent obtains each prize with the same probability, exactly as in the case of continuum. Moreover, by the LLN, for every \( \varepsilon > 0 \), if \( n \) is sufficiently large, then

\(^{16}\)This is a departure from the convention adopted throughout the paper according to which the value of prize \( j \) should be \( j/n \). The departure is inessential but enables us to substantially simplify calculations.
an agent who bids $y$ obtains a prize in $[y - \varepsilon, y + \varepsilon]$ at least with probability $1 - \varepsilon$. This implies that Definition 1 is satisfied.

4 Main Result??

Throughout this section, we will assume that distributions $G$ and $F$ are strictly increasing, that $G(1) = F(1) = 1$, and that $F$ is continuous with $F(0) = 0$. In particular, $G$ may have atoms, so some prizes (and the 0 prize) may have positive measure.\(^{17}\) We will also assume that $v_x(y)$ is continuous in $x$ and $y$, and strictly increasing in $y$, and that $c_x(t)$ is continuous in $x$ and $t$, and strictly increasing in $t$. We will also restrict the range of bids $t$ that can be made by the players to $B = [0, b_{\text{max}}]$, where $b_{\text{max}}$ is some fixed rational higher than $\max \{c_x^{-1}(v_x(1))\}$. This last assumption is without loss of generality, since the assumption that $v_x(y)$ is increasing in $y$ implies that for all types bids higher than $\max \{c_x^{-1}(v_x(1))\}$ are strictly dominated by 0. This also implies that in any finite approximating contest a player who bids $b_{\text{max}}$ in equilibrium wins the highest prize with probability 1 (otherwise he would be tying for the highest prize, so a slightly higher bid would be better, a contradiction).

We will say that weak single-crossing is satisfied if for any $x_1 < x_2$, $t_1 < t_2$, and $y_1 < y_2$ we have that $v_{x_1}(y_2) - c_{x_1}(t_2) \geq v_{x_1}(y_1) - c_{x_1}(t_1)$ implies $v_{x_2}(y_2) - c_{x_2}(t_2) \geq v_{x_2}(y_1) - c_{x_2}(t_1)$. That is, if a lower $x_1$ type prefers to obtain a higher prize $y_2$ at a higher bid $t_2$ to obtaining a lower prize $y_1$ at a lower bid $t_1$, then so does any higher type $x_2$. If the higher type strictly prefers to obtain the higher prize at the higher bid, i.e., the second inequality is strict, then we will say that strict single-crossing is satisfied.

**Theorem 1** Suppose that strict single-crossing is satisfied. Then, there exist increasing and continuous functions $T : B \to Y$ and $br : X \to B$ such that for every $\varepsilon > 0$, there is an $N$ such that for every $n \geq N$ in any equilibrium of the $n$-th contest,

1. with probability 1 the bid of agent $i = 1, \ldots, n$ differs from $br(i/n)$ by at most $\varepsilon$;
2. if agent $i = 1, \ldots, n$ bids $t$, then with probability at least $1 - \varepsilon$ she obtains a prize that differs from $T(t)$ by at most $\varepsilon$.

\(^{17}\)Note that $G$ being strictly increasing guarantees that $a > 0$, otherwise $G(0) = 1$ and $G$ would be constant on $[0, 1]$. 
Moreover, the mechanism that prescribes for type $x$ bid $br(x)$ and prize $T(br(x))$ is a regular IC-IR mechanism that implements the unique efficient consistent allocation.

By “the unique efficient consistent allocation” we mean the one that assortatively matches types and prizes, so that type $x$ obtains prize $y = G^{-1}(F(x))$, where $G^{-1}(z) = \inf \{y : G(y) \geq z\}$ for $z \in [0, 1]$. Strict single crossing implies that when higher types have weakly higher costs, this allocation is efficient in the standard sense, i.e., it maximizes the aggregate value from the prizes. This is because in this case $v_{x_1}(y_2) > v_{x_2}(y_2)$ for any $x_1 < x_2$ and $y_2 > 0$.\(^{18}\) If players’ cost are linear, i.e., $c_x(t) = t$, then because $G^{-1}(F(x))$ is non-decreasing, it can be implemented by a mechanism that satisfies IC. That is, there exist transfers such that an agent will choose to report her type truthfully and will obtain prize $G^{-1}(F(x))$. In fact, IC tells us that the transfers are pinned down up to the payment of the lowest type, which in an IR mechanism has to be 0 (because the lowest type obtains the lowest prize). More specifically, the envelope theorem shows that the transfer required to obtain prize $G^{-1}(F(x))$ is

$$t(x) = v_x(G^{-1}(F(x))) - \int_0^x v'_x(G^{-1}(F(z)))dz,$$

(4)

where $v'_x(y)$ denotes the partial derivative with respect to the agent’s type $x$. This show the uniqueness of the IC, IR, mechanism that implements the unique efficient consistent allocation. This mechanism is obviously regular.

The burden of the proof of Theorem 1 is in showing the existence of functions $T$ and $br$ with properties (1) and (2), such that $br(x)$ coincides with $t(x)$ given by (4) and $T(br(x)) = G^{-1}(F(x))$.

To formulate the next result, we need to redefine IC and IR. A mechanism is almost surely IC if the agent maximizes her ex-ante expected utility by reporting her type truthfully, i.e.,

$$\int_{x \in X} \left[ \int_{y \in Y} \int_{t \in R} [v_x(y) - c_x(t)]dQ_z(y,t) \right]dF(x)$$

is maximized by taking $z = x$ for all $x \in X$. In other words, only a measure 0 of types $x$ may strictly prefer reporting some other type $z$ to reporting her type truthfully.

\(^{18}\)To see this, apply strict single-crossing with $y_1 = t_1 = 0$ and $t_2 = c_{x_1}^{-1}(v_{x_1}(y_2))$.
An IC mechanism is almost surely IR if this highest ex-ante expected utility is at least 0, i.e.,
\[ \int_{x \in X} \left[ \int_{y \in Y} \int_{t \in T} \left[ v_x(y) - c_x(t) \right] dQ_x(y, t) \right] dF(x) \geq 0, \]
or equivalently, only a measure 0 of types \( x \) may strictly prefer obtaining the payoff 0 to participating in the mechanism.

**Theorem 2** Even if strict single-crossing is not satisfied, any sequence of equilibria of discrete contests contains a subsequence that approximates the outcome of a regular almost surely IC-IR mechanism that implements a consistent allocation.

If weak single-crossing is satisfied, then this consistent allocation is also efficient.

It is important to note that the proof of the result applies not only to sequences of equilibria \((E_n)_{n=1}^{\infty}\), where \( n \) is the number of agents in a contest, but also to all subsequences of such sequences. That is, any subsequence of \((E_n)_{n=1}^{\infty}\) has an approximating subsequence.

We conjecture that this is the strongest general convergence result one can expect. The reason is that some contests have multiple equilibria, and \((E_n)_{n=1}^{\infty}\) may be chosen in an “irregular” manner so that two different subsequences approximate two different outcomes.

Note, however, that this result already implies uniform convergence (across all equilibria) of finite-contests to the set of outcomes of regular IC-IR mechanisms that implement consistent allocations. More precisely, for any \( \varepsilon > 0 \) there exists an \( N \) such that for every \( n \geq N \), equilibrium \( E_n \) is \( \varepsilon \)-close\(^{19}\) to the outcome of a regular IC-IR mechanism that implements some consistent allocation. Indeed, if for some \( \varepsilon > 0 \) and arbitrarily large \( n \) we could find an equilibrium \( E_n \) that was not \( \varepsilon \)-close to the outcome of any such mechanism, then we would have a subsequence that contains no converging subsequence.

As we pointed out in Section 2, convergence in weak*-topology has only limited applicability; it enables us to approximate the strategies (or the distribution over (bid,prize) pairs) of types close to a given type in large discrete contests, but we may not be able to approximate the equilibrium strategy of any agent.

\(^{19}\)To measure the distance use any metrization of weak*-topology.
It is therefore important to establish (if possible) convergence in a stronger sense. We can show that under the assumptions of Theorem 2, there exists (for a subsequence) an increasing and continuous functions \( T : B \to Y \) such that the IC-IR mechanism prescribes to every type \( x \) with probability 1 (prize,bid) pairs \((T(t),t)\) such that bid \( t \) maximizes \( v_x(T(t)) - c_x(t) \).

In addition, if it happens that for every type \( x \) there is a unique optimal bid \( t \), then we the convergence in weak*-topology implies convergence in a sense similar to that from Theorem 1. Namely, for all except an \( \varepsilon \)-fraction of agents \( i = 1, \ldots, n \), with probability \( 1 - \varepsilon \) the bid of agent \( i = 1, \ldots, n \) differs from the optimal \( t \) by at most \( \varepsilon \), and also with probability \( 1 - \varepsilon \) she obtains a prize that differs from \( T(t) \) by at most \( \varepsilon \).

### 4.1 Discussion

Theorem 1 applies to Example 2. In this example, the uniform distribution of types and prizes implies that the assortative allocation assigns prize \( x \) to every type \( x \). By the envelope theorem, the transfer assigned to every type \( x \) is \( t(x) = x^2 - \int_0^x (y) dy = x^2/2 \). Therefore, we have that \( T(br(x)) = x \) and \( br(x) = x^2/2 \). Figure 1 illustrates the functions \( T \) and \( br \). Figure 2 illustrates the function \( T \circ br \).

![Figure 1: The inverse tariff \( T \) and the best-response function \( br \) in Example 2](image-url)
In addition to describing the outcome of large contests that have been solved in the literature, Theorem 1 applies to many contests for which there is no existing equilibrium characterization. To demonstrate this, consider the following example.

Suppose that types and prizes are distributed uniformly, so $F(x) = x$, $G(y) = y$, and let $v_x(y) = xh(y)$ and $c_x(t) = t$ for some strictly increasing and continuous function $h$ with $h(0) = 0$. As we saw in Example 2, the case $h(y) = y$ corresponds to the setting of Bulow and Levin (2006). The case $h(y) = y^2$ corresponds to Xiao’s (2012) quadratic prize sequence. The case $h(y) = e^y$ corresponds to Xiao’s geometric prize sequence. For other, non-trivial functions $h$ (including $h(y) = y^m$ for $m > 2$), however, there is no current result that characterizes equilibrium behavior in finite contests.

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20 This is equivalent to setting $v_x(y) = h(y)$ and $c_x(t) = t/x$. Indeed, for any finite contest, dividing every player $i$’s Bernoulli utility by $i/n$ does not change the set of equilibria. For the continuum case, dividing the Bernoulli utility of an agent of type $x$ by $x$ does not change his preferences over (prize,bid) pairs, and therefore does not change the set of IC, IR, consistent mechanisms.

21 This is because $v_x((j + 1)/n) - v_x(j/n) = x(2j + 1)/n^2$, so

$$v_x\left(\frac{j+1}{n}\right) - v_x\left(\frac{j}{n}\right) - \left(v_x\left(\frac{j+1}{n}\right) - v_x\left(\frac{j}{n}\right)\right) = x\frac{2j}{n^2} - x\frac{2j - 1}{n^2} = x\frac{2}{n^2}.$$

22 This is because $v_x((j + 1)/n)/v_x(j/n) = xe^{1/n}$.
Strict single crossing holds because for any $x' > x$, $y' > y$, and $t' > t$ we have that

$$xh(y') - t' \geq xh(y) - t \Rightarrow x(h(y') - h(y)) \geq t' - t$$

$$\Rightarrow x(h(y') - h(y)) > t' - t \Rightarrow x'h(y') - t' > x'h(y) - t.$$ 

The assortative allocation assigns prize $x$ to every type $x$. By the envelope theorem and the fact that type 0 gets prize 0, the bidding function $t(x)$ that implements the assortative allocation satisfies for every type $x$

$$xh(x) - t(x) = \int_{0}^{x} h(y) \, dy \Rightarrow t(x) = xh(x) - \int_{0}^{x} h(y) \, dy. \quad (5)$$

Thus, even though no equilibrium characterization currently exists for most functions $h$, Theorem 1 shows that for large $n$ a player with type $x$ bids something close to $xh(x) - \int_{0}^{x} h(y) \, dy$ and with high probability obtains a prize close to $x$.

Theorem 1 also applies to contests that combine identical and differing prizes, which are not accommodated by the existing literature. Such contests correspond to distributions $G$ that have atoms. For example, let $G(y) = \begin{cases} \frac{y}{2} & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$, so that half the prizes have value 1. Setting $h(y) = y$, we obtain a setting similar to that of Example 2, which differs from that example in that there are many identical best prizes. In this case, the assortative allocation assigns prize $2x$ to every type $x < 1/2$, and prize 1 for every type $x \geq 1/2$. Similarly to (5), for $x < 1/2$ we obtain

$$2x^2 - t(x) = \int_{0}^{x} 2y \, dy \Rightarrow t(x) = x^2,$$

and for $x \geq 1/2$ we obtain

$$x - t(x) = \int_{0}^{1/2} 2y \, dy + \int_{1/2}^{x} 1 \, dy \Rightarrow t(x) = x - \frac{1}{4} - \left(x - \frac{1}{2}\right) = \frac{1}{4}.$$ 

Therefore, Theorem 1 shows that for large $n$ a player with type $x$ bids something close to $\min\{x^2, 1/4\}$ and with high probability obtains a prize close to $\min\{2x, 1\}$.

Theorem 1 does not apply to Example 3 because strict single crossing fails. Theorem 2 applies, however, and because weak single crossing holds, the allocation is efficient.

22
Moreover, as the proof of Theorem 2 shows, the limiting mechanism is generated by a deterministic inverse tariff $T$. That is, there is a continuous, non-decreasing function $T : B \to Y$ and correspondence $br : X \to B$ such that (i) $br(x)$ is the set of optimal bids for type $x$ given $T$, (ii) in the limiting mechanism each type $x$ is assigned bids from $br(x)$ with probability 1, and (iii) in the limiting mechanism by choosing $t$ an agent obtains prize $T(t)$ with probability 1. This implies that $T(t) = t$, because agents must be indifferent between all prizes (this follows from the fact that they have the same utility function).

Even though Theorem 2 specifies $T$ and $br$, it cannot pin down the limiting mechanism, since there is a continuum of efficient allocations and transfers that are consistent.

Figure 3 illustrates the function $T$ and correspondence $br$. Figure 4 illustrates the limiting mechanism, which we derived in Example 3.

![Figure 3: The inverse tariff $T$ and the best-response function $br$ in Example 3](image)

Figure 3: The inverse tariff $T$ and the best-response function $br$ in Example 3
4.2 Proof of Theorem 1

The proof will show that any subsequence of the initially given sequence of equilibria contains a subsequence for which there exist functions $T$ and $br$ with the required properties. Since $br(x)$ and $T(br(x))$ coincide with the unique regular IC-IR mechanism that implements the unique efficient consistent allocation, $T$ and $br$ will be the same for all these subsequences. Therefore, $T$ and $br$ have the required properties for the initially given sequence of equilibria (otherwise there would be a subsequence with no subsequence for which $T$ and $br$ have the required properties).

Consider the sequence of approximating contests, indexed by $n$, and a corresponding sequence of equilibria $\sigma^n$, where $\sigma^n = (\sigma^n_1, \ldots, \sigma^n_n)$ and $\sigma^n_i$ is player $i$’s equilibrium strategy (i.e., a distribution over bids) in the $n$-th contest. Denote by $R^n_i(t)$ the random variable that is the percentile location of player $i$ in the ordinal ranking of the players in the $n$-th contest if she bids slightly above $t$ and the other players employ their equilibrium
strategies. That is, 
\[ R^n_i (t) = \frac{1}{n} \left( 1 + \sum_{k \neq i} 1_t (\sigma^n_k) \right), \]
where \(1_t (\sigma)\) is 1 if \(\sigma \leq t\) and 0 otherwise. Let 
\[ A^n_i (t) = \frac{1}{n} \left( 1 + \sum_{k \neq i} \Pr (\sigma^n_k \leq t) \right) \]
be the expected percentile location of player \(i\). Then, by Hoeffding’s inequality, for all \(t\) in \(B\) we have
\[ \Pr (|R^n_i (t) - A^n_i (t)| > \delta) < 2 \exp \{ -2\delta^2 (n - 1) \}. \] (6)
Finally, let 
\[ A^n (t) = \frac{1}{n} \sum_{i=1}^{n} A^n_i (t) \]
be the average of the expected percentiles locations of the players in the \(n\)-th contest if they bid \(t\) and the other players employ their equilibrium strategies.

Now, note that \(A^n : B \rightarrow [0, 1]\), and let \(T^n\) be the mapping from bids to prizes induced by \(A^n\). That is, \(T^n (t) = G^{-1} (A^n (t))\); recall that \(G^{-1} (z) = \inf \{ y : G (y) \geq z \}\). Note that \(G^{-1}\) is continuous, because \(G\) is strictly increasing and right-continuous. Also, \(G^{-1}\) coincides with the inverse of \(G\) wherever \(G\) is continuous.

Because \(A^n\) is (weakly) increasing, so is \(T^n\). Take an ordering of all rationals in \(B\), denoted by \(q_1, q_2, \ldots\). Take a converging subsequence of the sequence \(T^n (q_1)\), denote it by \(T^{n_1} (q_1)\), and denote its limit by \(T (q_1)\). Take a converging subsequence of the sequence \(T^{n_2} (q_2)\), denote it by \(T^{n_2} (q_2)\), and denote its limit by \(T (q_2)\). Continue in this fashion to obtain a function \(T : \{q_1, q_2, \ldots\} \rightarrow [0, 1]\). In addition, define a subsequence of \(T^n\) such that its \(k\)-th element is the \(k\)-th element in the sequence \(T^{n_k}\). For the rest of the proof, denote this new sequence by \(T^n\).

We now describe some properties of \(T\):

1. \(T\) is (weakly) increasing, because every \(T^n\) is (weakly) increasing.
2. \(T (0) = 0\).

---

23 This is the infimum of her ranking if she bids above \(t\), which is equivalent to bidding \(t\) and winning any ties there. If players’ strategies are continuous, then this is equivalent to bidding \(t\).
Indeed, suppose to the contrary that $T(0) > 0$. This implies that for some $\delta > 0$ and large enough $n$, we would have that $A^n(0) > 1 - a + \delta$. This means, in turn, that the strategies of a fraction of at least $1 - a + \delta$ agents in the $n$-th contest have atoms at $0$. Therefore, there is a positive probability that these players tie for a fraction $\delta$ of prizes of positive value, so any one of them would be better off bidding slightly above $0$ instead of bidding $0$.

(3) $T(b_{\text{max}}) = 1$, because $A^n(b_{\text{max}}) = 1$, and therefore $T^n(b_{\text{max}}) = 1$.

We will now show that $T$ can be extended uniquely to a continuous function on the entire interval $B$.

**Lemma 1** For any $t \in B$ (not necessarily rational) and any two sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in $B$, we have $\lim T(q^m) = \lim T(r^m)$.

**Proof.** Suppose the contrary for some $t \in (0, b_{\text{max}})$, $q^m \uparrow t$, and $r^m \downarrow t$. Let $y' = \lim T(q^m)$ and $y'' = \lim T(r^m)$ (the limits exist by the monotonicity of $T$), and let $\gamma = (y'' - y')/4$. In what follows, the indexes $N_M$, $N$, and $n$ are assumed large enough so that $t + 1/N_M \leq b_{\text{max}}$ and $t - 1/N_M \geq 0$, and similarly for $N$ and $n$.

We will first show that for every $M$ there exists $N_M$ such that for any $n \geq N_M$ and any type $x$,

$$\frac{v_x(y'' - \gamma) - v_x(y' + \gamma)}{c_x(t + 1/n) - c_x(t - 1/n)} > M.$$  

(7)

Observe first that the function $D(x) = v_x(y'' - \gamma) - v_x(y' + \gamma)$ of variable $x$ is continuous, and so it attains a minimum at a certain type $x'$. Because $v_x$ strictly increases in $y$, we have that $D(x') > 0$.

Now, let $C_n = \max_x \{ c_x(t + 1/n) - c_x(t - 1/n) \}$ (assume that $n$ is large enough so that $t - 1/n > 0$). The maximum exists because $c_x(t)$ is a continuous function of $x$. Note that $C_n$ decreases in $n$. Suppose that $\lim_{n \to \infty} C_n = \Delta > 0$, and denote by $x_1, x_2, \ldots$ a sequence of types such that $x_n$ attains the maximum in the definition of $C_n$. Take a converging subsequence of types, and denote its limit by $x''$. Because $c_{x''}$ is continuous at $t$, there exists $N$ such that

$$c_{x''}(t + 1/N) - c_{x''}(t - 1/N) < \Delta/3.$$
Moreover, because the cost is a continuous function of \( x \) there is a large enough \( n \) such that

\[
|c_{x''} (t + 1/N) - c_{x''} (t - 1/N) - (c_{x_n} (t + 1/N) - c_{x_n} (t - 1/N))| < \Delta/3.
\]

The last two inequalities yield

\[
c_{x_n} (t + 1/N) - c_{x_n} (t - 1/N) < 2\Delta/3.
\]

Since

\[
C_n = c_{x_n} (t + 1/n) - c_{x_n} (t - 1/n) < c_{x_n} (t + 1/N) - c_{x_n} (t - 1/N) < 2\Delta/3,
\]

for every \( n > N \) we obtain a contradiction to the assumption that \( \lim_{n \to \infty} C_n = \Delta \).

Thus, \( \lim_{n \to \infty} C_n = 0 \). This, together with that fact that \( D (x') > 0 \), yields (7).

Let \( M = 2b_{\text{max}} / (b_{\text{max}} - D (x')) \), choose an element \( t' \) in the sequence \( q^n \) such that \( t' \in (t - 1/2N_M, t) \), and choose an element \( t'' \) in the sequence \( r^n \) such that \( t'' \in (t, t + 1/2N_M) \).

Choose \( n > N_M \) large enough so that:

1. \( |T^n (t'') - T (t'')| < \gamma/3 \) and \( |T^n (t') - T (t')| < \gamma/3 \).

2. For all \( i \), \( \Pr ((A^n (t'') - R^n_i (t'')) > \alpha) < D (x') / 2v_1 (1) \) and \( \Pr ((R^n_i (t') - A^n (t')) > \alpha) < D (x') / 2v_1 (1) \), (see (6)) where \( \alpha \) is small enough so that \( T^n (t'') - G^{-1} (A^n (t'') - \alpha) < \gamma/3 \) and \( G^{-1} (A^n (t') + \alpha) - T^n (t') < \gamma/3 \) (recall that \( G^{-1} \) is continuous).

Then, in the equilibrium of the \( n \)-th contest that corresponds to \( T^n \), no agent bids in \([t - 1/N_M, t]\) with positive probability, because such bids give lower payoff than bidding slightly above \( t'' \). To see why, note that the payoff of any agent \( i \) from obtaining a prize for bids in \([t - 1/N_M, t]\) is at most the payoff from obtaining a prize when bidding \( t' \), which is at most

\[
\left( 1 - \frac{D (x')}{2v_1 (1)} \right) \left( v_{i/n} \left( T (t') + \frac{2\gamma}{3} \right) \right) + \frac{D (x')}{2v_1 (1)} v_1 (1)
\]

\[
< \left( 1 - \frac{D (x')}{2v_1 (1)} \right) \left( v_{i/n} \left( y' + \frac{2\gamma}{3} \right) \right) + \frac{D (x')}{2},
\]

and the payoff from obtaining a prize when bidding slightly above \( t'' \) is at least

\[
\left( 1 - \frac{D (x')}{2v_1 (1)} \right) \left( v_{i/n} \left( T (t'') - \frac{2\gamma}{3} \right) \right) > \left( 1 - \frac{D (x')}{2v_1 (1)} \right) \left( v_{i/n} \left( y'' - \frac{2\gamma}{3} \right) \right).
\]
The difference in the payoffs is therefore at least
\[
\left(1 - \frac{D(x')}{2v_1(1)}\right) \left(v_{i/n} \left(y'' - \frac{2\gamma}{3}\right)\right) - \left(1 - \frac{D(x')}{2v_1(1)}\right) \left(v_{i/n} \left(y' + \frac{2\gamma}{3}\right)\right) - \frac{D(x')}{2}
\]

\[
= \left(1 - \frac{D(x')}{2v_1(1)}\right) \left(v_{i/n} \left(y'' - \frac{2\gamma}{3}\right) - \left(v_{i/n} \left(y' + \frac{2\gamma}{3}\right)\right)\right) - \frac{D(x')}{2}
\]

\[
\geq \left(1 - \frac{D(x')}{2v_1(1)}\right) \left(v_{i/n} (y'' - \gamma) - \left(v_{i/n} (y' + \gamma)\right)\right) - \frac{D(x')}{2}
\]
and because \(D(x') \leq \left(v_{i/n} (y'' - \gamma) - \left(v_{i/n} (y' + \gamma)\right)\right)\), we have that this last expression is at least
\[
\left(\frac{1}{2} - \frac{D(x')}{2v_1(1)}\right) \left(v_{i/n} (y'' - \gamma) - \left(v_{i/n} (y' + \gamma)\right)\right)
\]

\[
> \left(\frac{v_1(1) - D(x')}{2v_1(1)}\right) \left(c_{i/n} \left(t + 1/N_M\right) - c_{i/n} \left(t - 1/N_M\right)\right)
\]

\[
= c_{i/n} \left(t + 1/N_M\right) - c_{i/n} \left(t - 1/N_M\right) \geq c_{i/n} \left(t''\right) - c_{i/n} \left(t - 1/N_M\right),
\]

by the definition of \(M\).

This shows that no agent bids in \([t - 1/N_M, t']\) with positive probability. Consider the largest interval of bids that contains \([t - 1/N_M, t']\) in which no agent bids with positive probability. Then, the only way for any agent to bid slightly above the top of this interval is that some other agent has an atom exactly at the top of this interval. But the agent with the atom would be better off lowering his bid (by bidding the atom he cannot be tying with other agents, otherwise he would increase his bid). Therefore, no agent bids more than \(t - 1/N_M\), which implies that \(T^n(t') = 1\). But \(T^n(t') \rightarrow T(t) \leq y' < y'' \leq 1\), a contradiction.

For the case \(t = b_{\text{max}}\), set \(t'' = b_{\text{max}}\) and repeat the argument above.\(^{24}\)

Suppose now that \(t = 0\). Then the above proof, with \(t' = t\) instead of \(t' = t - 1/2N_M\), shows that for large \(n\) no agent bids \(t' = 0\) with positive probability. This means, in turn, that sufficiently small bids give lower payoff than the bid \(t''\). Thus, no agent bids makes such small bids with positive probability, and a contradiction is obtained by an argument analogous to that for \(t > 0\). ■

\(^{24}\)The only difference is that bidding “slightly above \(b_{\text{max}}\)” is impossible. But by bidding \(b_{\text{max}}\) a player wins with probability 1, because \(b_{\text{max}}\) is strictly dominated by 0 for all players.
Now extend $T$ to the entire interval $B$ by setting $T(t) = \lim T(q^m)$ for some sequence $q^m \to t$ of rationals in $B$. Lemma 1 shows that $T(t)$ is the same regardless of the chosen sequence $q^m$ (if two different sequences approach $x$ from the same direction, consider a third sequence the approaches from the other direction and apply Lemma 1). In addition, Lemma 1 shows that this is indeed an extension, by choosing $q^m = t$ for any rational $t$. Finally, the extended $T$ is continuous. Otherwise, there would be some $t, \varepsilon > 0$, and a sequence $t^m \to t$ such that $|T(t^m) - T(t)| > \varepsilon$ for every $m$; construct a sequence of rationals $q^m$ such that $|T(q^m) - T(t^m)| < \varepsilon/2$ and $|q^m - t^m| < 1/m$, which would imply that $q^m \to t$ and $|T(q^m) - T(t)| > \varepsilon/2$, a contradiction.

**Lemma 2** $T^n$ converges to $T$ uniformly on $B$.

**Proof.** Suppose the contrary. Then there is some $\delta > 0$ and a sequence of integers $n_1, n_2, \ldots, n_k, \ldots$ such that for every $n_k$ there is some bid $t_k$ with $|T^{n_k}(t_k) - T(t_k)| > \delta$. Passing to a subsequence if necessary we can assume that the sequence $(t_k)_{k=1}^\infty$ is convergent; denote its limit by $t$.

Consider rationals $q'$ and $q''$ such that $q' < t < q''$ and $T(q'') - T(q') < \delta/3$; such numbers exist because $T$ is continuous. For large enough values of $k$, we have that $|T^{n_k}(q') - T(q')| < \delta/3$ and $|T^{n_k}(q'') - T(q'')| < \delta/3$.

For any $t' \in [q', q'']$, either (1) $T^{n_k}(t') \geq T(t')$, or (2) $T^{n_k}(t') \leq T(t')$.

By the monotonicity of $T$ and $T^{n_k}$, we have

$$T^{n_k}(t') - T(t') \leq T^{n_k}(q'') - T(q') \leq |T^{n_k}(q'') - T(q'')| + |T(q'') - T(q')| < 2\delta/3$$

in case (1), and

$$T(t') - T^{n_k}(t') \leq T(q'') - T^{n_k}(q') \leq |T(q'') - T(q')| + |T(q') - T^{n_k}(q')| < 2\delta/3$$

in case (2).

Since $t_k \in [q', q'']$ for large enough values of $k$, we obtain a contradiction to the assumption that $|T^{n_k}(t_k) - T(t_k)| > \delta$ for all such $k$. $\blacksquare$

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25 If $t = 0$ set $q' = 0$ and if $t = b_{\text{max}}$ set $q'' = b_{\text{max}}$. 

29
For every $x$, denote by $BR_x$ type $x$’s set of optimal bids given $T$, i.e., the bids $t$ that maximize $v_x(T(t)) - c_x(t)$. Denote by $BR(\varepsilon)$ the $\varepsilon$-neighborhood of the graph of the correspondence that assigns to every $x \in [0, 1]$ the set $BR_x$, i.e., $BR(\varepsilon)$ is the union over all types $x$ and bids $t \in BR_x$ of the open ball of radius $\varepsilon$ centered at $(x, t)$. For every type $x$ denote by $BR_x(\varepsilon)$ the set of bids $t$ such that $(x, t) \in BR(\varepsilon)
$. Note that $BR(\varepsilon)$ is a 2-dimensional open set, while each $BR_x(\varepsilon)$ is a 1-dimensional “slice.” Note also that $BR(\varepsilon)$ is in general larger than the union, across $x$, of the set of bids whose distance from some bid in $BR_x$ is less than $\varepsilon$. In particular, $BR_x(\varepsilon)$ may contain bids whose distance from every bid in $BR_x$ is more than $\varepsilon$.

**Lemma 3** For every $\varepsilon > 0$, there is an $N$ such that for every $n \geq N$, the equilibrium bid of each agent $i = 1, ..., n$ in the $n$-th contest belongs to $BR_{i/n}(\varepsilon)$ with probability 1, i.e.,

$$
\sigma^n_i(BR_{i/n}(\varepsilon)) = 1.
$$

**Proof.** Suppose to the contrary that for arbitrarily large $n$, the strategy of some agent $i$ in the $n$-th contest assigns a positive probability to the complement of $BR_{i/n}(\varepsilon)$. Let $x_n = i/n$. Passing to a convergent subsequence if necessary, we assume that $x_n$ converges to some $x^*$. (Note that $x^*$ may not be a rational number.)

Now, for every $x$ there is a $\delta_x > 0$ such that any bid from the complement of $BR_x(\varepsilon)$ gives type $x$ a payoff lower than any element of $BR_x$ by at least $\delta_x$. Otherwise, by taking a suitable subsequence, we would show that there exists an element of $BR_x$ that belongs to the complement of $BR_x(\varepsilon)$. Let $\delta = \delta_x^*$.

Observe that:

1. The maximal payoff of type $x$, attained at any bid from $BR_x$, is continuous in $x$.

   Indeed, by continuity of the payoff function, any bid $t \in BR_x$ yields to any type close enough to $x$, a payoff close to that obtained by type $x$ by bidding $t$. Thus, the maximal payoff function is lower semi-continuous.\textsuperscript{26}

\textsuperscript{26}Lower semi-continuity says that the value of the limit is no lower than the limit of values, while upper semi-continuity says that the value of the limit is no higher than the limit of values. Of course, a function is continuous if it is upper and lower semi-continuous.
Suppose now that the function is not upper semi-continuous. That is, there is a \( \rho > 0 \) and a sequence of types \( x_k \to x \) such that by bidding a \( t_k \) type \( x_k \) obtains a payoff at least \( \rho \) higher than the maximal payoff of type \( x \). Passing to a convergent subsequence if necessary, we can assume that \( t_k \to t \) for some bid \( t \). By continuity of the payoff functions, by bidding \( t \) type \( x \) obtains a payoff higher by at least \( \rho > 0 \) than her maximal payoff, a contradiction.

2. For every \( \rho > 0 \), there exists an \( N \) such that for \( n \geq N \) the highest payoff that type \( x \) can obtain by bidding in the complement of \( BR_{x_n}(\varepsilon) \) is higher by at most \( \rho \) than the highest payoff that type \( x^* \) can obtain by bidding in the complement of \( BR_{x^*}(\varepsilon) \).

Indeed, suppose that for \( n_k > k \) type \( x_{n_k} \) obtains by bidding some \( t_k \) in the complement of \( BR_{x_{n_k}}(\varepsilon) \) a payoff at least \( \rho \) higher than the highest payoff that type \( x^* \) can obtain by bidding in the complement of \( BR_{x^*}(\varepsilon) \). Passing to a convergent subsequence if necessary, we can assume that \( t_k \to t \) for some bid \( t \). Since every \( (x_{n_k}, t_k) \) belongs to the complement of \( BR(\varepsilon) \), so does \( (x^*, t) \); thus, \( (x^*, t) \) belongs to the complement of \( BR_{x^*}(\varepsilon) \). However, by continuity of the payoff functions, bidding \( t \) gives type \( x^* \) a payoff at least \( \rho \) higher than the highest payoff that type \( x^* \) can obtain by bidding in the complement of \( BR_{x^*}(\varepsilon) \), a contradiction.

By 1 and 2, for sufficiently large \( n \) any bid in the complement of \( BR_{x_n}(\varepsilon) \) gives type \( x_n \) a payoff lower by at least \( \delta/2 \) than any bid in \( BR_{x_n} \). By uniform convergence of \( T^n \) to \( T \), the analogous statement (with \( \delta/2 \) replaced with some smaller positive number) is true for \( T \) replaced with \( T^n \). This means, however, that for sufficiently large \( n \), type \( x_n \) would be strictly better off bidding slightly above any bid in \( BR_{x_n} \) than bidding in the complement of \( BR_{x_n}(\varepsilon) \). This is because, similarly to the proof of Lemma 1, Hoeffding’s inequality (see (6)) implies that for sufficiently large \( n \), by bidding slightly above \( t \) the agent obtains a prize arbitrarily close to \( T^n(t) \) with probability arbitrarily close to 1.

**Lemma 4** If strict single-crossing is satisfied, then for all \( x \), the set \( BR_x \) is a singleton. In addition, the function that assigns to \( x \) the single element of \( BR_x \) is continuous and weakly increasing.

**Proof.** Observe that for any \( x' < x'' \), strict single-crossing implies that if \( t' \in BR_{x'} \) and
$t'' \in BR_{x''}$, then $t' \leq t''$. Suppose that $BR_{x'}$ contained two bids, $t_1 < t_2$, for some type $x'$. Then for any $0 < \varepsilon < (t_2 - t_1)/3$, for sufficiently large $n$, the first observation and Lemma 3 imply that only agents with types in $I = [\max \{x' - \varepsilon, 0\}, \min \{x' + \varepsilon, 1\}]$ may bid in the interval $[t_1 + (t_2 - t_1)/3, t_2 - (t_2 - t_1)/3]$. Let

$$\delta = \min_{x \in I} \{c_x(t_2 - (t_2 - t_1)/3) - c_x(t_1 + (t_2 - t_1)/3)\} > 0$$

be the minimal cost difference across these types between bidding at the top and the bottom of the smaller interval $[t_1 + (t_2 - t_1)/3, t_2 - (t_2 - t_1)/3]$. For any $\Delta > 0$, let

$$\gamma(\Delta) = \max_{x \in I} \max_{z \in [0,1]} \{v_x(G^{-1}(\max\{z + \Delta, 1\})) - v_x(G^{-1}(z))\}$$

be the maximal benefit across these types from winning against an additional fraction $\Delta$ of players. Then $\lim_{\Delta \to 0} \gamma(\Delta) = 0$.\footnote{Indeed, notice that $\gamma \geq 0$ is monotonic in $\Delta$, and suppose that $\lim_{\Delta \to 0} \gamma(\Delta) = \beta > 0$. Consider a sequence $(x_n, z_n)$ such that $v_{x_n}(G^{-1}(\max\{z_n + 1/n, 1\})) - v_{x_n}(G^{-1}(z_n)) > \beta/2$. Passing to a subsequence if necessary, consider the limit $(x^*, z^*)$ of $(x_n, z_n)$. By continuity of $v$ and $G^{-1}$, we would then have that $v_{x^*}(G^{-1}(z^*)) - v_{x^*}(G^{-1}(z^*)) \geq \beta/2$, a contradiction.}

Taking $\Delta = 2\varepsilon$, for any agent with type in $I$ the gain of bidding in $[t_1 + (t_2 - t_1)/3, t_2 - (t_2 - t_1)/3]$ instead of bidding $t_1 + (t_2 - t_1)/3$ is at most $\gamma(2\varepsilon)$, whereas the cost is at least $\delta$. Therefore, for small enough $\varepsilon$ no agent bids in the interval $[t_1 + (t_2 - t_1)/3, t_2 - (t_2 - t_1)/3]$. But then, by the same argument as in the second to last paragraph of the proof of Lemma 1, no agent bids more than $t_2 - (t_2 - t_1)/3$. This implies that $T^n(t_2 - (t_2 - t_1)/3) = T^n(t_2) = 1$, so $T(t_2 - (t_2 - t_1)/3) = T(t_2) = 1$. Therefore, $t_2$ is not in $BR_{x'}$, because bidding slightly above $t_2 - (t_2 - t_1)/3$ gives type $x'$ a higher payoff.

It follows that $BR_x$ is a singleton for any $x$. Thus, the function that assigns to $x$ the single element of $BR_x$ is weakly increasing. We will denote this function by $br(x)$. Given the singleton property, a simpler version of the argument that showed the singleton property shows that $br(x)$ is continuous in $x$. \Halmos

The mechanism that prescribes for agent $x$ prize $T(br(x))$ and transfer $c_x(br(x))$ is IC and IR, because $br(x)$ is type $x$’s optimal bid given $T$; in particular, each type is at least as well off bidding her optimal bid as bidding 0 and obtaining $T(0) = 0$.\footnote{Indeed, notice that $\gamma \geq 0$ is monotonic in $\Delta$, and suppose that $\lim_{\Delta \to 0} \gamma(\Delta) = \beta > 0$. Consider a sequence $(x_n, z_n)$ such that $v_{x_n}(G^{-1}(\max\{z_n + 1/n, 1\})) - v_{x_n}(G^{-1}(z_n)) > \beta/2$. Passing to a subsequence if necessary, consider the limit $(x^*, z^*)$ of $(x_n, z_n)$. By continuity of $v$ and $G^{-1}$, we would then have that $v_{x^*}(G^{-1}(z^*)) - v_{x^*}(G^{-1}(z^*)) \geq \beta/2$, a contradiction.}
This allocation is consistent with respect to types, by construction. Consistency with respect to prizes means that for any prize \( y \) the mass of prizes no higher than \( y \) that are allocated is \( G(y) \). In the mechanism, this allocated mass is the mass of types \( x \) for whom \( T(br(x)) \leq y \). Thus, consistency means that for every \( y \) we have

\[
F(\max \{ x : T(br(x)) \leq y \}) = G(y).
\]  

(8)

It suffices to show that for every type \( x \) we have \( G^{-1}(F(x)) = T(br(x)) \). To see this, choose any \( y \) and let \( x' = \max \{ x : G^{-1}(F(x)) = y \} \), which is well defined because \( G^{-1} \) and \( F \) are continuous. We then have

\[
F(\max \{ x : T(br(x)) \leq y \}) = F(\max \{ x : G^{-1}(F(x)) \leq G^{-1}(F(x')) \}) = F(x')
\]

and \( G(y) = G(G^{-1}(F(x'))) = F(x') \), so (8) holds. Therefore, the following lemma proves consistency.

**Lemma 5** \( G^{-1}(F(x)) = T(br(x)) \) for any type \( x \).

**Proof.** Consider first any type \( x \) such that \( x_{min} = \min \{ z : br(z) = br(x) \} > 0 \) and \( x_{max} = \max \{ z : br(z) = br(x) \} < 1 \) (\( x_{min} \) and \( x_{max} \) are well defined because \( br \) is continuous). Take any \( \delta \in (0, \min \{ x_{min}, 1 - x_{max} \}) \). By Lemma 3 applied to \( \varepsilon = |br(x_{min}) - br(x_{min} - \delta)|/2 \), if \( n \) is sufficiently large, then the equilibrium bids of each agent \( i = 1, ..., n \) of with \( i/n \) lower than \( x_{min} - \delta \) are lower than \( br(x_{min} - \delta) + \varepsilon \). Therefore, an agent who bids \( br(x_{min}) \) outbids all agents of types lower than \( x_{min} - \delta \) and obtains a prize \( y \geq G^{-1}(F(x_{min} - \delta)) \). Consequently, \( T(br(x_{min})) \geq G^{-1}(F(x_{min} - \delta)) \), and because \( F \) and \( G^{-1} \) are continuous, by taking \( \delta \) to 0 we obtain \( T(br(x_{min})) \geq G^{-1}(F(x_{min})) \).

Similar arguments show that \( T(br(x_{min} - \delta)) \leq G^{-1}(F(x_{min})) \), and because \( T \) and \( br \) are continuous, by taking \( \delta \) to 0 we obtain \( T(br(x_{min})) \leq G^{-1}(F(x_{min})) \). Similarly, for sufficiently large \( n \), the bids of all agents of types higher than \( x_{max} + \delta \) are higher and bounded away from \( br(x_{max}) \), so \( T(br(x_{max})) \leq G^{-1}(F(x_{max} + \delta)) \), and by taking \( \delta \) to 0 we obtain \( T(br(x_{max})) \leq G^{-1}(F(x_{max})) \); and similar arguments show that \( T(br(x_{max})) \geq G^{-1}(F(x_{max})) \). Since \( br(x) = br(x_{min}) = br(x_{max}) \), this yields that \( T(br(x)) = G^{-1}(F(x_{min})) = G^{-1}(F(x_{max})) \). Therefore, \( T(br(x)) = G^{-1}(F(x)) \) (because \( G^{-1} \circ F \) is monotonic).

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Suppose that \( x_{\min} > 0 \) and \( x_{\max} = 1 \). Then, \( T(br(x_{\min})) = G^{-1}(F(x_{\min})) \) by an analogous argument to the previous case. And because \( T(br(x_{\min})) = T(br(1)) \) and \( G^{-1} \circ F \) is weakly increasing, we have \( T(br(z)) \leq G^{-1}(F(z)) \) for any \( z \in [x_{\min}, 1] \). To show the reverse inequality, it suffices to show that \( T(br(1)) = 1 \), because \( G^{-1}(F(z)) \leq 1 \) for any \( z \). Suppose to the contrary that \( T(br(1)) = y < 1 \). Because no bidder bids above \( b_{\max} \), for any \( n \) we have \( A^n(b_{\max}) = 1 \), so \( T^n(b_{\max}) = 1 \) and \( T(b_{\max}) = 1 \). By monotonicity of \( T \), \( br(1) < b_{\max} \); let \( \varepsilon = 1 - y > 0 \). Because \( T \) is continuous, there is a bid \( t \in (br(1), b_{\max}) \) such that \( T(t) = y + \varepsilon/2 < 1 \). By the monotonicity of \( br \) and Lemma 3 with \( \varepsilon = t - br(1) \), for large enough \( n \) all bidders bid less than \( t \), so \( T^n(t) = 1 \). But then \( T(t) = 1 \), a contradiction.

Now suppose that \( x_{\min} = 0 \) and \( x_{\max} < 1 \). Then, the arguments above show that \( T(br(x_{\max})) = G^{-1}(F(x_{\max})) \). And because \( T(br(0)) = T(br(x_{\max})) \) and \( G^{-1} \circ F \) is weakly increasing, we have \( T(br(z)) \geq G^{-1}(F(z)) \) for any \( z \in [0, x_{\max}] \). To show the reverse inequality, it suffices to show that \( T(br(0)) = 0 \), because \( G^{-1}(F(z)) \geq 0 \) for any \( z \). Because \( T(0) = 0 \), it suffices to show that \( br(0) = 0 \). Suppose to the contrary that \( br(0) = t > 0 \).

Notice first that the proof of Lemma 3 actually gives a stronger result that not only equilibrium bids but also all best responses of agent \( i/n \) (to equilibrium strategies of other agents) belong to \( BR_{i/n}(\varepsilon) \). Thus, by the monotonicity of \( br \) and and this stronger result with \( \varepsilon = t/2 \), for large enough \( n \) no bidder has best responses lower than \( t/2 \). For such an \( n \), denote by \( t_{\inf} \geq t/2 > 0 \) the infimum of the union of players’ best response sets. Suppose that some agent \( i \) has an atom at \( t_{\inf} \). Then he obtains the lowest prize by bidding \( t_{\inf} \). Indeed, the bids of all other agents are weakly higher than \( t_{\inf} \) with probability 1, so agent \( i \) could obtain a prize higher then the lowest one as a result of favorable tie-breaking at \( t_{\inf} \). But then agent \( i \) would be better of by bidding slightly above \( t_{\inf} \) instead of bidding \( t_{\inf} \). However, if any positive bid that gives the lowest prize cannot be any agent’s best response. Therefore, no agent has an atom at \( t_{\inf} \). By the definition of \( t_{\inf} \), there is an agent who makes bids arbitrarily close to \( t_{\inf} \) with positive probability. Because no agent has an atom at \( t_{\inf} \), the probability that the agent obtains prize 0, by making bids converging to \( t_{\inf} \) approaches 1. Therefore, such bids cannot be the agent’s best response either.
Finally, it cannot be that $x^{\text{min}} = 0$ and $x^{\text{max}} = 1$. This is because by the previous parts of the lemma, $x^{\text{max}} = 1$ implies $T(br(1)) = 1$ and $x^{\text{max}} = 0$ implies $T(br(0)) = 0$, which contradicts the definitions of $x^{\text{min}}$ and $x^{\text{max}}$. 

Thus, this mechanism coincides with the unique IC-IR mechanism that implements the efficient consistent allocation.

To complete the proof, it remains to show (2) in the statement of the theorem. It follows from Lemma 3 that for sufficiently large $n$ a bidder obtains a prize no higher than $T(t) + \varepsilon$ with arbitrarily high probability by bidding slightly above $t$. On the other hand, the bidder obtains a prize no lower than $T(t') - \varepsilon$ with arbitrarily high probability by bidding slightly above any $t' < t$ (for $t = 0$ we already know that $T(0) = 0$). So, (2) in the statement of the theorem follows from continuity of $T$.

**Remark 1** It follows from continuity of functions $br$ and $T$, but we omit the proof, that the convergence established in Theorem 1 implies the convergence according to Definition 1. That is, under the assumptions of Theorem 1, any sequence of equilibria of discrete contests approximates the outcome of the unique IC-IR regular mechanism that implements the unique efficient consistent allocation in the sense of Definition 1.

### 4.3 Proof of Theorem 2

From the subsequence of contests that correspond to the subsequence $T^n$ that converges uniformly to $T$, choose a subsequence such that $D^n$ converges to some probability distribution $D$ in weak*-topology. We will show that $D$ assigns probability 1 to the set $A = \{(x, y, t) : t \in BR_x \text{ and } y = T(t)\} \subset X \times Y \times B$.

By standard arguments the correspondence that assigns $BR_x$ to type $x$ is upper semi-continuous. Therefore, the set $\{(x, t) : t \in BR_x\} \subset X \times B$ is closed, and by continuity of $T$, $A$ is also closed. Suppose to the contrary that $D$ assigns a positive measure to the complement of $A$. Then, for some $\varepsilon > 0$, $D$ assigns a positive measure to the complement of a $2\varepsilon$-neighborhood $U$ of $A$, that is, to the set $X \times Y \times B - U$. Consider the $\varepsilon$-neighborhood $V$ of $A$ and its closure $\bar{V}$ (which is contained in $U$), and take a continuous function $f :
\[ X \times Y \times B \to [0, 1] \] such that \( f(\bar{V}) = 1 \) and \( f(X \times Y \times B - U) = 0 \). Then,

\[ \int f dD < 1. \]

However, by Lemma 3, uniform convergence of \( T^n \) to \( T \), and Hoeffding’s inequality (see (6)), for sufficiently large \( n \) every player \( i \) in the \( n \)-th contest bids \( t \) and with probability higher than \( 1 - 1/n \) obtains \( y \) such that \((i/n, y, t) \in V\). Thus,

\[ \int f dD^n \to 1, \]

a contradiction.

Thus, each of a measure 1 of types \( x \) is assigned by the “limit” mechanism bids \( t \in BR_x \) (with the corresponding prizes \( T(t) \)) with probability 1. This implies that \( D \) determines a regular almost surely IC-IR mechanism. It remains to show that this mechanism implements a consistent allocation, but this follows from the fact that \( D^n \) implement consistent allocations. We show this for the marginal with respect to \( x \); the proof for the marginal with respect to \( y \) is analogous. Consider a continuous function \( f_k : X \times Y \times B \to [0, 1] \) whose value is 1 on the set of all \((x, y, t)\) such that \( x \leq x^* \) and 0 on the set of all \((x, y, t)\) such that \( x \geq x^* + 1/k \). Then, by the definition of \( k \)-convergence,

\[ \int f_k dD^n \to \int f_k dD. \]

For large enough \( n \) the integrals on the left-hand side belong to interval \([F(x^*), F(x^* + 1/k)]\). Therefore, integral on the right-hand side also belongs to interval \([F(x^*), F(x^* + 1/k)]\). Taking the limit for \( k \to \infty \), and applying right-continuity of cumulative distribution functions, we obtain that \( F \) is the marginal of \( D \) with respect to \( x \).

This completes the proof of the first part of the theorem. To show the second part, recall that the “limit” mechanism prescribes to each type \( x \) only bids \( t \in BR_x \) (together with prizes \( T(t) \)), i.e., such bids are prescribed with probability 1.

If for any \( x' < x'' \) type \( x' \) is prescribed \( t' \) and type \( x'' \) is prescribed \( t'' \) such that \( t' > t'' \), then \( x' \) must weakly prefer \( t' \) to \( t'' \) and \( x'' \) must weakly prefer \( t'' \) to \( t' \). By weak single-crossing, \( x'' \) must weakly prefer \( t' \) to \( t'' \), and \( x' \) must weakly prefer \( t'' \) to \( t' \). That is, each of the two types is indifferent between the two bids. Thus, if a lower type is prescribed a higher prize, this is without loss of efficiency.
5 Extensions

In this section, we generalize our results to the case of any CDF $G$. This more general result is important to capture applications such as the contests studied by Clark and Riis (see Example 1). The additional feature of the general distribution $G$, compared to the distributions studied in the previous section is that some open intervals of types may have measure zero. Let $I_1 = (y_{l1}^1, y_{u1}^1)$ be the largest (in terms of length) such interval;\footnote{In the case of multiplicity, choose an arbitrary largest interval.} let $I_2 = (y_{l2}^2, y_{u2}^2)$ be the largest such interval which is disjoint with $I_1$, and so on. Then, every open interval of types that has measure zero is contained in one of the intervals $I_1, I_2, \ldots$; moreover for any given $\delta > 0$, only a finite number of intervals $I_k$ have length higher than $\delta$.

We conjecture the following counterpart of Theorem 1:

**Conjecture 1** Suppose that strict single-crossing is satisfied. Then, there exist a (weakly) increasing right-continuous function $T : B \rightarrow Y$, which is continuous except a countable set of bids $B^*$, and a (weakly) increasing upper hemi-continuous correspondence $br : X \rightarrow B$, which takes single values from $B - B^*$ except a countable set of types $X^*$, such that for any $\varepsilon > 0$ and open set $U^* \supset X^*$ and a left-open set $U^s \supset B^s$, there is an $N$ such that for every $n \geq N$ in any equilibrium of the $n$-th contest,

1. with probability 1 the bid of agent $i = 1, \ldots, n$ such that $i/n \notin U^*$ differs from $br(i/n)$ by at most $\varepsilon$;

2. if agent $i = 1, \ldots, n$ bids $t \notin V^*$, then with probability at least $1 - \varepsilon$ she obtains a prize that differs from $T(t)$ by at most $\varepsilon$.

Moreover, any mechanism that prescribes for type $x$ bid $br(x)$ and prize $T(br(x))$ is a regular IC-IR mechanism that implements the unique efficient consistent allocation.

In addition, for any $t \in B$ (not necessarily rational) one of the following holds:

1. For any two sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in $B$, we have $\lim T(q^m) = \lim T(r^m)$.
2. There is some $k = 1, 2, \ldots$ such that for any two sequences $q^m \uparrow t$ and $r^m \downarrow t$ of rationals in $B$, we have $\lim T(q^m) = y^l_k$ and $\lim T(r^m) = y^u_k$.

Conjecture 1 applies to Example 1. Figures 5 illustrates the corresponding function $T$ and correspondence $br$. Figure 6 illustrates the correspondence $T \circ br$.

Figure 5: The inverse tariff $T$ and the best-response function $br$ in Example 1

Figure 6: The equilibrium bids and allocation in Example 1
We conjecture that the convergence conjectured in Conjecture 1 implies the convergence according to Definition 1. We also conjecture that Theorem 2 generalizes to an arbitrary CDF $G$.

Another possible extension of our results is to allow the cost function $c_x(t)$ to be only a weakly increasing function of an agent’s bid $t$. We find this kind of result rather difficult to obtain. Many of our arguments rely on the assumption that a higher bid is inferior if does not increase the chance of obtaining higher prizes, or it increases it only marginally (by an arbitrarily small probability for sufficiently large number of bidders). However, we are able to demonstrate the relation between large discrete contests and mechanism design for a continuum of agents and prizes for an application in which the cost of bidding remains constant over an interval of bids, namely, for head-start contests.

**Example 4** (based on Siegel (2011)) Suppose that all agents have the same utility function $v_x(y) = 1$ for all $y > 0$. The cost of bidding, however, varies across agents; more specifically, let $c_x(t) = \max\{0, t - x\}$ for all $x$ and $t \geq x$, and restrict the set of bids of agent of type $x$ to $[x, \infty)$. In Siegel’s terminology, an agent of type $x$ has a head start $x$. We assume a uniform type distribution, that is, $F(x) = x$ for all $x \in X$. The prize distribution is $G(y) = 1/2$ for all $y \in [0, 1)$, and $G(1) = 1$. That is, there is a mass $1/2$ of non-zero prizes. We make these simplifying assumptions for ease of exposition. To simplify a little further, in the discrete case we consider $n = 2k + 1$, so there are $k + 1$ identical non-zero prizes. We make these simplifying assumptions for ease of exposition.

Obviously, all allocations are efficient. Not all of them, however, can be implemented by an IC, IR mechanism; one example is the allocation that assigns prizes to the agents with the lowest head starts. The assortative allocation can be implemented and assigns prizes to the agents with the highest head starts. This allocation can be described as the probability measure $H$ that assigns probability 1 to, and is distributed uniformly on, the set

$$\{(x,0) : x \leq 1/2\} \cup \{(x,1) : x > 1/2\}.$$
Any IC, IR mechanism that achieves this allocation requires any agent of type $x \leq 1/2$ to bid $x$, and any agent of type $x > 1/2$ to bid $t = 3/2$. We will now show that this outcome is approximated by equilibria of contests with a large number of agents and prizes. As in the previous example, the simplicity of the allocation and the IC, IR mechanisms that achieve it contrasts with the relative complexity of players’ equilibrium mixed strategies. Moreover, while the IC, IR mechanism are given explicitly, players’ mixed strategies are derived by an algorithm and are not described in closed form.

To show the approximation we make use of some equilibrium properties demonstrated by Siegel, and do not require a complete equilibrium characterization. We first observe that in equilibrium

(a) agents $i = 1, 2, \ldots, k-1$ (i.e., the agents with the lowest head starts) bid no higher than their head starts and lose the auction (see Corollary 1 in Siegel (2011)).

Now consider the other agents. We will show that

(b) for any $\varepsilon > 0$, if $k$ is sufficiently large, then the fraction of agents $k, k+1, \ldots, 2k+1$ who bid $t \in [(3k+1)/(2k+1) - \varepsilon, (3k+1)/(2k+1)]$ with probability higher than $1 - \varepsilon$ is higher than $1 - \varepsilon$.

Indeed, note first that because agents $1, 2, \ldots, k-1$ do not bid more than $(k-1)/(2k+1)$ and there are $k+1$ prizes, an agent $i = k, k+1, \ldots, 2k+1$ wins a prize if he bids more than $(k-1)/(2k+1)$ and in addition outbids at least one of the other agents $k, k+1, \ldots, 2k+1$. This implies that no agent $i = k+1, \ldots, 2k+1$ bids more than $(3k+1)/(2k+1)$, because such bids are strictly dominated by 0 for agent $k$. Another observation is that the equilibrium payoff of agent $2k+1$ is $(k+1)/(2k+1)$, which is the difference between his head start and that of player $k$ (see the second paragraph on page 15 of Siegel (2011)).

Now suppose that for some $\varepsilon > 0$, for any $k > 0$ a fraction $\varepsilon$ of the $k+1$ agents $k, k+1, \ldots, 2k+1$ bid $t \in [(3k+1)/(2k+1) - \varepsilon, (3k+1)/(2k+1)]$ with probability at most $1 - \varepsilon$. These agents therefore bid less than $(3k+1)/(2k+1) - \varepsilon$ with probability at least $\varepsilon$ (because agents $k, k+1, \ldots, 2k+1$ do not bid more than $(3k+1)/(2k+1)$). But then by bidding $(3k+1)/(2k+1) - \varepsilon$ player $2k+1$ wins with a probability no lower than $1 - (1 - \varepsilon)^{(k+1)}$ (because he wins whenever not all these agents bid more than $(3k+1)/(2k+1) - \varepsilon$), so he
can obtain a payoff no lower than

\[ 1 - (1 - \varepsilon)^{(k+1)} - \left[ \frac{(3k + 1)}{(2k + 1)} - \varepsilon - \frac{(2k + 1)}{(2k + 1)} \right] = \frac{(k + 1)}{(2k + 1)} - (1 - \varepsilon)^{(k+1)} + \varepsilon, \]

which exceeds \((k + 1) / (2k + 1)\) for sufficiently large \(k\).

Conditions (a) and (b), together with the fact that all but one of the agents \(k, k + 1, \ldots, 2k + 1\) obtain a prize, guarantee the convergence from Definition 1.
References


