Dominance solvable contests

Alberto Vesperoni*

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Abstract

Consider a two players all pay contest with success functions of the ratio type and with countable and arbitrarily fine effort space. This work shows that when the costs or valuations are symmetric the contest is always iteratively dominance solvable. If the contest is asymmetric this is not necessarily the case, since a simple condition on the best reply functions must be fulfilled. As an example we show that the Tullock contest fulfills such condition, and hence it is always dominance solvable. Since a dominated strategy is never played with positive probability in a mixed strategy equilibrium this work rules out the existence of mixed strategy equilibria in a certain class of contests.

Contests have often been studied for their analytic tractability compared to auctions: they would usually present a pure strategy equilibrium instead of mixed strategy one (see Konrad (2009) for a review of the literature). Although the existence of such pure strategy equilibrium has been shown before (see Perez-Castrillo and Verdier (1992), Okuguchi and Szidarovszky (1997) and Nti (1999)), previous studies did not rule out the existence of multiple equilibria in the mixed strategy extension of the game. This work deals with this question, showing that if the effort space is countable and sufficiently fine under certain conditions the pure strategy equilibrium is the unique one in the mixed strategy extension of the game.

More precisely we will show that if the costs or valuations are symmetric then the contest is always iteratively dominance solvable. If the contest is asymmetric then this is not necessarily the case. In this work we identify a simple condition on the best reply functions such that if it is fulfilled the contest is iteratively dominance solvable. Moreover we show that the classical Tullock contest fulfills such condition and hence is always iteratively dominance solvable.

Since dominated strategies are never played with positive probability in a mixed strategy equilibrium, we can conclude that when the contest is it-

*Department of Economics, Stockholm School of Economics. Alberto.Vesperoni@hhs.se
eratively dominance solvable there are no Nash equilibria in mixed strategies.

The paper develops as follows. The first section develops the model and the main results in two subsections about symmetric and asymmetric contests. The second section concludes.

1 The model

Consider a two players \( i \in \{A, B\} \) all pay contest with success functions of the ratio type (see Skaperdas (1996)). The utility of player \( i \) is

\[
  u_i(x_i, x_{-i}) = \frac{f(x_i)}{f(x_i) + f(x_{-i})} - c_i x_i
\]

where \( c_i \in \mathbb{R}^+ \) and \( c_A \leq c_B \), and the efforts are \( \{x_A, x_B\} \in \mathbb{R}^2_+ \). The function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is such that \( f(0) = 0 \), strictly increasing (for all \( x \): \( f'(x) > 0 \)), concave (for all \( x \): \( f''(x) \leq 0 \)) and twice differentiable. Notice that if \( f \) is concave then the utility of player \( i \) is necessarily concave, hence the first order condition of player \( i \)

\[
  \frac{f'(x_i) f(x_{-i})}{[f(x_i) + f(x_{-i})]^2} = c_i
\]

implicitly defines his best reply function \( BR_i : X_{-i} \rightarrow X_i \). It is easy to verify that the best replies cross in a unique point where the equilibrium efforts are such that

\[
  \frac{f(\hat{x}_A)}{f'(\hat{x}_A)} = \frac{c_B}{c_A} \frac{f(\hat{x}_B)}{f'(\hat{x}_B)}
\]

It follows that \( \hat{x}_A > \hat{x}_B \) and hence the player with the lowest cost exerts the highest equilibrium effort.

We have seen that the best replies cross in a unique point, hence there exists a unique Nash equilibrium in pure strategies. Are there Nash equilibria in mixed strategies? In order to answer this question we follow Baye et al. (1994) by making the effort space countable and arbitrarily fine. The effort space of each player is \( X_i = \{k + \delta t\}_{t \in \mathbb{Z}_+} \) where \( \{k, \delta\} \in \mathbb{R}^2_+ \) and \( k \) such that \( \hat{x}_i \in X_i \).

Let us develop some tools that will be useful in our proofs. Firstly we will identify sufficient conditions for a strategy \( x_i \) dominating or being dominated
by $x_i + \delta$. Secondly we will show that all strategies of player $i$ which are never best replies to any pure strategy are strictly dominated.

Consider the incentive of player $i$ to rise his effort of one step $\delta$, that is

$$\frac{f(x_i + \delta)}{f(x_i + \delta) + f(x_{-i})} - \frac{f(x_i)}{f(x_i) + f(x_{-i})} > \delta c_i \tag{2}$$

It is easy to verify that for every $x_{-i}$ the utility function of player $i$ is increasing for $x_i < BR_i(x_{-i})$ and decreasing otherwise. Since $f$ is twice differentiable it follows that there always exist a $\delta$ such that for all $\delta < \delta$ if $x_i < BR_i(x_{-i})$ then condition 2 holds and if $x_i > BR_i(x_{-i})$ it does not.

If equation 2 holds for the $x_{-i}$ which minimizes the LHS then $x_i$ is strictly dominated by $x_i + \delta$. On the other hand if equation 2 does not hold for the $x_{-i}$ which maximizes the LHS then $x_i + \delta$ is strictly dominated by $x_i$.

It is easy to verify that for $x_{-i} \in \mathbb{R}_+$ the first derivative of the LHS of equation 2 is zero at $x^*_{-i}$ such that $f(x^*_{-i}) = \sqrt{f(x_i + \delta)f(x_i)}$, and hence $x_i < x^*_{-i} < x_i + \delta$. We will consider $\delta$ to be small, hence for simplicity we will assume $x^*_{-i} = x_i$. Notice that the LHS is 0 when $x_{-i} = 0$ and $x_{-i} \to +\infty$. Moreover the LHS is strictly positive for all $x_{-i} \in \mathbb{R}_+$. It follows that the LHS is maximized at $x_{-i} = x_i$ and it is always increasing in $x_{-i}$ for $x_{-i} < x_i$ and decreasing otherwise.

Let us see which $x_{-i}$ maximizes the LHS when $x_{-i}$ belongs to a bounded interval $[\underline{x}_{-i}, \overline{x}_{-i}]$. If $x_i \in [\underline{x}_{-i}, \overline{x}_{-i}]$ then the LHS is maximized at $x_{-i} = x_i$.

If $x_i < \underline{x}_{-i}$ the LHS is maximized at $\underline{x}_{-i}$ and if $x_i > \overline{x}_{-i}$ the LHS is maximized at $\overline{x}_{-i}$. The following proposition summarizes the results.

**Proposition 1** Suppose $x_{-i} \in [\underline{x}_{-i}, \overline{x}_{-i}]$. There always exist a $\delta$ such that for all $\delta < \delta$ the strategy $x_i$ strictly dominates $x_i + \delta$ if one of the following three conditions holds.

1. $x_i < \underline{x}_{-i}$ and $x_i > BR_i(\underline{x}_{-i})$
2. $x_i \in [\underline{x}_{-i}, \overline{x}_{-i}]$ and $x_i > BR_i(x_i)$
3. $x_i > \overline{x}_{-i}$ and $x_i > BR_i(\overline{x}_{-i})$

Let us consider for which $x_{-i} \in [\underline{x}_{-i}, \overline{x}_{-i}]$ the LHS is minimized. It is easy to verify that for $x_i$ such that $f(x_i) < \sqrt{f(\underline{x}_{-i})f(\overline{x}_{-i})}$ the LHS is minimized at $\overline{x}_{-i}$, while for for $x_i$ such that $f(x_i) > \sqrt{f(\underline{x}_{-i})f(\overline{x}_{-i})}$ the LHS is minimized at $\underline{x}_{-i}$. The following proposition summarizes the results.

**Proposition 2** Suppose $x_{-i} \in [\underline{x}_{-i}, \overline{x}_{-i}]$. There always exist a $\tilde{\delta}$ such that for all $\delta < \tilde{\delta}$ the strategy $x_i + \delta$ strictly dominates $x_i$ if one of the following two conditions holds.

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1. \( f(x_i) < \sqrt{f(x_{-i})f(\bar{x}_{-i})} \) and \( x_i < BR_i(\bar{x}_{-i}) \)

2. \( f(x_i) > \sqrt{f(x_{-i})f(\bar{x}_{-i})} \) and \( x_i < BR_i(x_{-i}) \)

Let us see the last of our tools. It is easy to verify that the maximum value of the best reply function \( BR_i(x_{-i}) \) of player \( i \) is for \( x_{-i} \) such that

\[
\frac{f(x_i)}{f'(x_{-i})} = \frac{1}{4c_i}
\]

and takes value \( \bar{x}_{i1} = x_{-i} \). It follows that the best reply function is maximized at its intersection with the 45 degree line, it is always increasing for \( x_{-i} < \bar{x}_{i1} \) and decreasing otherwise. Moreover the lower is the cost \( c_i \) the higher is the maximum value \( \bar{x}_{i1} \) of the best reply function. The following proposition shows that all \( x_i > \bar{x}_{i1} \) are strictly dominated by \( \bar{x}_{i1} \).

**Proposition 3** For all \( \delta \in \mathbb{R}_{++} \) the strategy \( \bar{x}_{i1} \) strictly dominates all \( x_i > \bar{x}_{i1} \).

**Proof:** A necessary and sufficient condition for the claim in the proposition being true is that for all \( x_i > \bar{x}_{i1} \) and all \( x_{-i} \in X_{-i} \)

\[
\frac{f(x_i)}{f(x_{-i})} - \frac{f(\bar{x}_{i1})}{f(\bar{x}_{i1}) + f(x_{-i})} < x_i - \bar{x}_{i1}
\]

Holding \( x_i \) constant, call the function at the LHS of 4 \( g(x_{-i}) \). Notice that \( g(0) = 0 \) and \( \lim_{x_{-i} \to +\infty} g(x_{-i}) = 0 \). Since \( x_i > \bar{x}_{i1} \) then for all \( x_{-i} \in X_{-i} \) it holds \( g(x_{-i}) \geq 0 \). Moreover notice that the first derivative \( g'(x_{-i}) \) is equal to zero in a unique point \( x_{-i} = x^* \) which is such that \( f(x^*) = \sqrt{f(x_i)f(\bar{x}_{i1})} \). Since \( g(x_{-i}) \) is always positive, minimized at the boundaries of the considered domain and its first derivative is zero at a unique point \( x^* \) then \( g(x_{-i}) \) is maximized at such point.

Notice that if condition 4 holds at \( x_{-i} = x^* \) then it holds for all \( x_{-i} \in X_{-i} \), since \( x^* \) maximizes the LHS. Condition 4 evaluated at \( x_{-i} = x^* \) becomes

\[
\frac{f(x_i)^{1/2} - f(\bar{x}_{i1})^{1/2}}{f(x_i)^{1/2} + f(\bar{x}_{i1})^{1/2}} < x_i - \bar{x}_{i1}
\]

Notice that the LHS and the RHS are both equal to 0 if evaluated at \( x_i = \bar{x}_{i1} \). Moreover notice that the LHS is concave and its first derivative evaluated at \( x_i = \bar{x}_{i1} \) is equal to 1, as the one of the RHS. It follows from the concavity of the LHS that for all \( x_i > \bar{x}_{i1} \) the LHS is smaller than the RHS and hence condition 5 always holds.
The three propositions above give us useful tools to show that, under certain conditions, contests are iteratively dominance solvable. Let us first analyze the case with symmetric costs. The reason for which we analyze this case separately is that the proof is particularly simple and the result particularly robust.

1.1 The symmetric case

If \( c_A = c_B = c \) the equilibrium efforts are such that \( f(\hat{x}_i)/f'(\hat{x}_i) = 1/(4c) \). It follows by equation 3 that these efforts are equal to \( \bar{x}_{i1} \), which is the upper bound of the best reply function of player \( i \). Since the equilibrium strategies are symmetric and equal to the upper bound of the best replies let us call \( \hat{x}_i = \bar{x}_{i1} = \hat{x} \).

Consider the strategies \( x_i > \hat{x} \), by proposition 3 all such strategies are strictly dominated by \( \hat{x} \) in the first round of iterative elimination of strictly dominated strategies (IESDS).

Consider a strategy \( x_i < \hat{x} \). By condition 1 of proposition 2 for all strategies of the opponent \( x_i \leq x_{-i} \leq \hat{x} \) the strategy \( x_i \) is strictly dominated by \( x_i + \delta \). Consider the lowest strategy of each player, \( x_i = k \). Considering only the strategies of the opponent which have not been eliminated in the first round of IESDS, by condition 1 of proposition 2 \( x_i = k \) is strictly dominated by \( x_i = k(\delta) + \delta \) and hence is ruled out in the second round of IESDS. Applying recursively condition 1 of proposition 2 to all the remaining strategies from the lowest to the highest brings us to the result that \( \{\hat{x}, \hat{x}\} \) is the unique strategy profile which survives IESDS, and hence is the unique rationalizable one. The following proposition summarizes the results.

**Proposition 4** If \( c_A = c_B \) there always exist a \( \bar{\delta} \) such that for all \( \delta < \bar{\delta} \) the unique rationalizable strategy profile is the Nash equilibrium in pure strategies.

1.2 The asymmetric case

Suppose \( c_A < c_B \). By proposition 3 all \( x_B > \bar{x}_{B1} \) are strictly dominated, and hence eliminated at the first round of IESDS. Notice that \( \bar{x}_{B1} < \bar{x}_{A1} \). It follows that \( BR_A(\bar{x}_{B1}) > \bar{x}_{B1} \) since for \( x_i < \bar{x}_{i1} \) the best reply function is always such that \( BR_i(x_{-i}) > x_{-i} \) due to its concavity. By condition 3 of proposition 1 if \( x_B \leq \bar{x}_{B1} \) for all \( x_A > BR_A(\bar{x}_{B1}) \) the strategy \( x_A \) is strictly dominated by \( x_A - \delta \). It follows that all \( x_A > BR_A(\bar{x}_{B1}) \) are eliminated at the second round of IESDS. Since \( \bar{x}_{B1} < \bar{x}_{A1} \) then \( BR_A(\bar{x}_{B1}) < \bar{x}_{A1} \) due
to \( BR_A \) being increasing for all \( x_B < \bar{x}_{A1} \). Since \( BR_A(\bar{x}_{B1}) < \bar{x}_{A1} \) and all strategies above \( BR_A(\bar{x}_{B1}) \) are strictly dominated then \( BR_A(\bar{x}_{B1}) \) is the new upper bound for the rationalizable strategies of \( A \). Let us call the new upper bound \( \bar{x}_{A2} = BR_A(\bar{x}_{B1}) \).

We have seen that after two rounds of IESDS the rationalizable strategies are such that \( k \leq x_A \leq \bar{x}_{A2} \) and \( k \leq x_B \leq \bar{x}_{B1} \), hence we found upper bounds for the rationalizable strategies. For now the lower bound of the rationalizable strategies of player \( i \in \{A, B\} \) is \( x_{i1} = k \). Let us look for more restrictive lower bounds in a recursive fashion, where \( \underline{x}_{it} \) is the lower bound of player \( i \) after \( t \) rounds of IESDS.

Consider player \( B \) at round \( t \). The rationalizable strategies of \( A \) are \( \underline{x}_{At} \leq x_A \leq \bar{x}_{A2} \). It is easy to verify that by condition 1 of proposition 2 the new lower bound of the rationalizable strategies of \( B \) is

\[
\underline{x}_{Bt+1} = \min \left\{ BR_B(\bar{x}_{A2}), f^{-1} \left[ \sqrt{f(\underline{x}_{At})f(\bar{x}_{A2})} \right] \right\} \tag{6}
\]

For convenience let us call the second element of the set of possible lower bounds \( \underline{x}_{Bt+1} \).

Consider player \( A \) at round \( t \). The rationalizable strategies of \( B \) are \( \underline{x}_{Bt} \leq x_B \leq \bar{x}_{B1} \). It is easy to verify that by proposition 2 the new lower bound of the rationalizable strategies of \( A \) is

\[
\underline{x}_{At+1} = \max \left\{ BR_A(\underline{x}_{Bt}), f^{-1} \left[ \sqrt{f(\underline{x}_{Bt})f(\bar{x}_{B1})} \right] \right\} \tag{7}
\]

For convenience let us call the second element of the set of possible lower bounds \( \underline{x}_{At+1} \).

Let us develop our recursive argument for increasing the lower bounds of rationalizable strategies. Consider \( k \) arbitrarily small. Since \( k \) is small at round 3 of IESDS \( \underline{x}_{B4} = \bar{x}_{B4} \). It is easy to verify that by equation 7 since the lower bound of \( B \) increased then the lower bound of \( A \) will necessarily increase. As far as \( \underline{x}_{Bt+1} = \bar{x}_{Bt+1} \) an increase in the lower bound of \( A \) also induces an increase in the lower bound of \( B \), which then induces an increase in the one of \( A \) and so on. At some round \( \tau \) of IESDS it will be the case that \( \underline{x}_{A_{\tau-1}} \) is high enough to induce \( \underline{x}_{B\tau} = BR_B(\bar{x}_{A2}) \).

Let us now make our crucial assumption for the proof. Suppose

\[
BR_A(BR_B(\bar{x}_{A2})) > \bar{x}_{B1} \tag{8}
\]

which means that the best reply of \( A \) to the lower bound of \( B \) at \( \tau \) is higher than \( \bar{x}_{B1} \), which is the value of \( x_A \) above which the best reply of \( B \) becomes
decreasing. Since $\bar{x}_A$ is a transformation of the geometric mean of $x_B$ and $\bar{x}_B$ then it is never higher than $\bar{x}_B$. By equation 7 it follows that

$$x_{A\tau+1} = BR_A(BR_B(\bar{x}_{A2}))$$

Call the best reply of $B$ to the new lower bound of $A$ $\bar{x}_{B2} = BR_B(x_{A\tau+1})$, which by condition 1 of proposition 1 is the new upper bound for the rationalizable strategies of $B$.

Let us summarize the results. We started by identifying by proposition 3 an upper bound for the rationalizable strategies of $B$. By iteratively eliminating strategies of both players and assuming $BR_A(BR_B(\bar{x}_{A2})) > \bar{x}_B$ we found a new upper bound for the rationalizable strategies of $B$.

Let us now complete our proof. Notice that the best reply function of $B$ is decreasing and the one of $A$ is increasing in the domain at round $\tau + 1$ of rationalizable strategies. Under our assumption in equation 8 this guarantees the iterative procedure described before to be a contraction mapping. By iteratively applying such procedure the unique strategy profile which will not be eliminated is the Nash equilibrium $\{\hat{x}_A, \hat{x}_B\}$. The following proposition summarizes the results.

**Proposition 5** Consider $c_A < c_B$. If $BR_A(BR_B(BR_A(\bar{x}_B))) > \bar{x}_B$ there always exist a $\delta$ such that for all $\delta < \delta$ the unique rationalizable strategy profile is the Nash equilibrium in pure strategies.

As an example consider the Tullock success function, where $f(x_i) = x_i$. It is easy to verify that the condition

$$BR_A(BR_B(BR_A(\bar{x}_B))) > \bar{x}_B$$

is always fulfilled for the Tullock contest, and hence it is always iteratively dominance solvable.

2 Conclusion

In this work we have seen that if the strategy space is discrete and arbitrarily fine 2 players symmetric contests are always dominance solvable. This is not necessarily the case if costs or valuations are asymmetric. We have seen that asymmetric contests are dominance solvable if a simple condition on the best reply functions is fulfilled, and that the classical Tullock contest fulfills such condition.
For all the contests which are dominance solvable we can conclude that there are no Nash equilibria in mixed strategies, and hence that the Nash equilibrium in pure strategies is the unique one also in the mixed strategy extension of the game.

References


