The transitive core of preference relations

Hiroki Nishimura∗

April 2012

Preliminary version

1 Introduction

Much evidence suggests that a preference relation of a decision maker is often not transitive in reality. Armstrong [1, 2, 3] and Luce [11] argued that, in contrast to the standard utility theory, a preference relation of a decision maker is suffered from nontransitive indifferences due to the imperfect powers of discrimination. Kahneman and Tversky [8] presented the extensive experimental results that show that subjects violate the expected utility hypothesis in a consistent manner leading to cyclic preferences over prospects. To accommodate such choice anomalies, Loomes and Sugden [10] developed a nontransitive evaluation of prospects that accounts for the experience of regret. Roelofsma and Read [15], followed by Read [14] and Ok and Masatlioglu [13] studied intertemporal choice problems in which a decision maker exhibits cyclic choice patterns. In the context of the social choice theory, the classical paradox by Condorcet shows that a collective preference relation can be cyclic under the majority voting rule even when individual preferences behave nicely.

While nontransitive preference relations are recognized as a convincing descriptive phenomenon as such, many economists on the other hand appear not to view them as the normative concept. In addition, due to the issues of preference cycles, especially the inability of choices and the instability to the framing effects, the nontransitive preference relations are seldom, if not ever, employed in applied studies. Consequently, there seems to be a serious gap between the efforts made from the descriptive perspective and the applied works adopting the normative assumption within their foundation. Facing this situation, the following questions is

∗Department of Economics, New York University, 19th West 4th Street, New York, NY 10012. Email: hiroki@nyu.edu.
posed in the paper: how a decision maker should behave if his preference relation is in fact not transitive? If his preference relation is not conceived as rational, what can the economic theory, as the normative theory of decision making, teach him to do?\footnote{In the proposition of the prescriptive analysis, Bell et al.\cite{4} wrote “how can real people – as opposed to imaginary, idealized, super-rational people without psyches – make better choices in a way that does not do violence to their deep cognitive concerns?” In the spirit of seeking the economic guidance on a basis of the recognition that people in reality typically do not confirm the normative assumption, the present paper has much in common with their objective.}

This paper attempts to answer these questions by studying certain maps, called rationalization rules, that transform each possibly cyclic preference relation into a transitive order. Under this formulation, the previous question is rephrased to the search of a justifiable rationalization rule. Importantly, in order to understand a rationalization rule as a method of “rationalizing” cyclic preference relations, it would not be enough that the rule simply converts nontransitive relations into transitive relations. Rather, the rule must provide a consistent and meaningful guidance of the decision making that respects the decision maker’s preference relation.

The transitive closure, defined as the smallest binary relation that extends the original preference relation, is a possible candidate of a rationalization rule that we are after. However, it turns out that the transitive closure brings a result far from what we can accept as a normative method of rationalization in many contexts where nontransitive preference relations naturally arise. In fact, it is shown that, in certain interesting classes of nontransitive preference relations, all the “guidance” obtained by the transitive closure is to treat any pair of alternatives indifferently regardless of what the decision maker initially manifested. As a substitute, we may define the transitive interior as the dual of the transitive closure, namely, as the largest transitive binary relation that refines the original preference relation. However, the hope of adopting this notion as a rationalization rule is also crumbled by a simple but unfortunate fact: the transitive interior never exists for a cyclic preference relation.\footnote{This fact can be easily proved. Let $\succ$ be a cyclic binary relation on a set $X$ of alternatives, and take any $x, y$ with $x \succ y$. Then, $\{(x, y)\}$ is a binary relation that refines $\succ$, and therefore, as the largest transitive binary relation that refines $\succ$, the transitive interior must include $\{(x, y)\}$. As $x, y$ with $x \succ y$ is arbitrary in the above argument, the transitive interior must agree with $\succ$, contradicting with the definition that the transitive interior is transitive.}

These observations render that the best transitive approximations of the decision maker’s preference relation are not suitable for, at least in the classes of nontransitive preference relations of interest, a good method of rationalization. This is where this paper proposes an alternative approach: the transitive core. After all, the best rationalization rule does not have to be the best approximation of a preference relation. While it is important that a rationalization rule respects the decision maker’s preference, the rule should nevertheless treat this criterion in
accordance with the consistency of a preference relation. Put differently, a rationalization rule should be able to alter a preference relation to a good extent if it is necessary to break inconsistent cycles within it. The transitive core is designed to capture a consistent part of a preference relation. To be precise, an alternative \( x \) is evaluated at least as good as another alternative \( y \) by the transitive core of a preference relation \( \succeq \) if any third alternative \( z \) weakly better than \( x \), namely \( z \succeq x \), is weakly better than \( y \), and any \( z \) weakly worse than \( y \), namely \( y \succeq z \), is weakly worse than \( x \). It is shown that the transitive core can be characterized by the principles each of which has a normative justification. Furthermore, when applied to the classes of nontransitive preference relations of interest, the transitive core is shown to preserve (i) the unanimity order for the majority voting, (ii) the statewise dominance order for the regret theory, (iii) the constant-time preference relation for intertemporal decision problems, and (iv) recovers the true preference relation in the theory of imperfect discrimination.

The paper is structured as follows. In the next section, a few preliminary definitions are introduced. In Section 3, the transitive core is formally defined, and its axiomatic characterization is presented. In addition, the axiom system is studied from the normative perspective. Subsequently, Section 4 examines the behavior of the transitive core in various classes of nontransitive preference relations. In particular, it is observed that the transitive core provides a result much intuitive relative to the transitive closure.

2 Preliminaries

A binary relation \( \succeq \) on \( X \) is said to be reflexive if \( x \succeq x \) for all \( x \in X \), transitive if \( x \succeq y \) and \( y \succeq z \) imply \( x \succeq z \) for all \( x, y, z \in X \), complete if either \( x \succeq y \) or \( y \succeq x \) holds for all \( x, y \in X \), and antisymmetric if \( x \succeq y \) and \( y \succeq x \) imply \( x = y \) for any \( x, y \in X \). A reflexive and transitive binary relation on \( X \) is referred to as a preorder on \( X \). In
turn, a preorder on $X$ is called a weak order if it is complete, a partial order if it is antisymmetric, and a linear order if it is complete and antisymmetric. A cycle of a binary relation $\succeq$ is a finite sequence $(x_i)_{i=1}^k$ such that $x_1 \succeq x_2 \succeq \cdots \succeq x_k \succeq x_1$ with at least one relation holding strictly. We say that $\succeq$ is acyclic if there is no cycle of $\succeq$ and that it is cyclic otherwise. A binary relation is acyclic whenever it is transitive, but the converse is not true except when the relation is complete.

A cycle of a binary relation $\triangleright$ is a finite sequence $(x_i)_{i=1}^k$ such that $x_1 \triangleright x_2 \triangleright \cdots \triangleright x_k \triangleright x_1$ with at least one relation holding strictly. We say that $\triangleright$ is acyclic if there is no cycle of $\triangleright$ and that it is cyclic otherwise. A binary relation is acyclic whenever it is transitive, but the converse is not true except when the relation is complete. 

Let $X$ be the set of conceivably all the alternatives that a decision maker may consume. The decision maker exhibits a preference relation over alternatives, and this information is observed as a complete binary relation $\succeq$ on $X$. Importantly, we do not assume that $\succeq$ is transitive. Therefore, the preference relation $\succeq$ may have cycles within it in general. As a method of rationalizing a possibly cyclic preference relation $\succeq$, we define a binary relation on $X$ that is referred to as the transitive core of $\succeq$. The transitive core is designed to capture a consistent part of a preference relation. Put precisely, an alternative $x$ is evaluated at least as good as another alternative $y$ by the transitive core of $\succeq$ if any third alternative $z$ weakly better than $x$, namely $z \succeq x$, is weakly better than $y$, and any $z$ weakly worse than $y$, namely $y \succeq z$, is weakly worse than $x$.\footnote{While the motivation is different, the definition of the transitive core is similar to one studied by Galaabaatar and Karni [7] and Karni [9] who, from a strict preference relation $\succ$, defined a binary relation $\succ^*$ such that $x \succ y$ iff $z \succ x$ implies $z \succ y$ for all $z \in X$. Also, as we shall see in Section 4, Luce [11] defined an order that has a seemingly different definition but shares a close idea with the}
**Definition.** Let $\succeq$ be a complete binary relation on $X$. Then, the transitive core of $\succeq$ is a binary relation $\partial \succeq$ on $X$ such that, for all $x, y \in X$,

$$x \partial \succeq y \quad \text{if and only if} \quad \begin{cases} y \succeq z & \text{implies} \ x \succeq z \\ z \succeq x & \text{implies} \ z \succeq y \end{cases} \quad \text{for every} \ z \in S.$$ 

As a plain observation, the transitive core of $\succeq$ is transitive. Moreover, it refines the original preference relation $\succeq$, meaning that an alternative $x$ is evaluated weakly higher than another alternative $y$ by the transitive core of the preference relation $\succeq$ only if so is the case for $\succeq$. This point, which reflects a rather conservative feature of the transitive core, draws a sharp contrast to the transitive closure according to which any alternative $x$ is evaluated weakly higher to $y$ so far as $x \succeq y$. Finally, we can readily see that the transitive core of $\succeq$ is $\succeq$ itself if $\succeq$ is initially transitive. This means that the transitive core is not a notion that imposes a particular value to the decision maker beyond the rationality of a preference relation.

### 3.2 Axiomatic characterization

In this section, we shall view the transitive core as a map that assigns a preorder to a complete binary relation on various domains. In order to understand the transitive core as a new method of rationalizing cyclic preference relations, it would not be enough for the core to be a rule that simply converts nontransitive relations into transitive relations. Instead, it must provide a consistent and meaningful guidance of the decision making that respects the decision maker’s original preference relation. The objective of this section is to clarify in what sense the transitive core can be understood as a normative rule from the economist’s point of view and in what sense it is meaningful for the decision makers who potentially receive its guidance.

Let $X$ be the collection of all nonempty subsets of $X$, and $\mathcal{P}$ the set of all complete binary relations defined on a set in $X$. Each $\succeq \in \mathcal{P}$ is viewed as a preference relation of a decision maker in an environment where the feasibility of alternatives is possibly restricted. A *rationalization rule* is a map $\sigma$ such that (i) for every preference relation $\succeq \in \mathcal{P}$, $\sigma$ associates it with a preorder $\sigma(\succeq)$ on the same domain as $\succeq$ and (ii) $\succeq$ is transitive. Therefore, a rationalization rule converts every cyclic preference relation into a transitive preference relation while it respect a preference relation as it is if the relation is originally transitive. The definition of rationalization rules captures a necessary but otherwise inadequate requirement as the sensible method of rationalizing preference relations. At this level, the notion is general (or empty) enough to include, for instance, a rule that assigns the trivial weak order $X \times X$ to any cyclic preference relations. Note that the observations

---

transitive core.
from the last section permits the transitive core to be well defined as a rationalization rule.

**Definition.** The transitive core is a rationalization rule $\partial$ such that

$$x \partial \succeq y \text{ if and only if } \begin{cases} y \succ z \text{ implies } x \succ z \\ z \succ x \text{ implies } z \succ y \end{cases} \text{ for every } z \in S$$

for any $S \in \mathcal{X}$ and any complete binary relation $\succeq$ on $S$.

Besides the basic requirement as a rationalization rule, the transitive core satisfies additional desirable properties. Notable among them are the five axioms listed below. Where $\sigma$ is a rationalization rule, the following axioms apply for any $S, T \in \mathcal{X}$ with $T \subseteq S$ and any complete binary relations $\succeq, \succeq'$ on $S$.

**Axiom 1.** If there exists a bijection $\pi$ on $S$ such that $x \succeq y$ if and only if $\pi(x) \succeq' \pi(y)$ for all $x, y \in S$, then $x \sigma(\succeq) y$ if and only if $\pi(x) \sigma(\succeq') \pi(y)$ for all $x, y \in S$.

**Axiom 2.** For any $x, y \in S$, $x \sigma(\succeq) y$ if and only if $y \sigma(\preceq) x$.

**Axiom 3.** For any $x, y \in S$, $x \sigma(\succeq) y$ if and only if $x \sigma(\succeq \cup \{(y, x)\}) y$.

**Axiom 4.** For any $x, y \in T$, $x \sigma(\succeq) y$ implies $x \sigma(\succeq T) y$.

**Axiom 5.** For any $x, y \in S$, if $x \sigma(\succeq_{xyz}) y$ for all $z \in S$, then $x \sigma(\succeq) y$.

The first axiom is the *anonymity* principle that requires the rationalization rule to function based solely on the order relationship between alternatives. The existence of a bijection $\pi$ such that $x \succeq y$ if and only if $\pi(x) \succeq' \pi(y)$ means that $\succeq$ and $\succeq'$ are equivalent upon relabelling of the alternatives. If this is the case, the axiom claims that $\sigma(\succeq)$ and $\sigma(\succeq')$ also behave in the same way upon the same relabelling. The second is the *symmetry* axiom that assumes the interchangeability of order of taking the inverse and the rationalization rule. Suppose that, for alternatives $x$ and $y$ in $X$, a rationalization rule breaks a preference of $x$ to $y$ on a basis of a “bad” cycle $x \succ y \sim z \succ x$. Should the same rule suggest to break a preference of $y$ to $x$ if it were the case that $x \prec y \sim z \prec x$ instead? The axiom, which is the requirement of symmetry with respect to preference relations, says yes to this.

What comes next is the *neutrality* axiom between weak preferences and strict preferences. Consider a preference relation $\succeq$ such that $x \succ y$ for some alternatives $x$ and $y$. Then, $\succeq \cup \{(y, x)\}$ is another preference relation formed from $\succeq$ by replacing the strict preference between $x$ and $y$ with their indifference. The axiom is a requirement on the rationalization rule for such a pair of preference relations to break the strict preference $x \succ y$ from the former if and only if the weak preference
$x \succ y$ from the latter. This axiom, as discussed later, characterizes the qualitative aspect of rationalization rules.

The last two axioms define the reduction property and the extension property of a rationalization rule. Note that, whenever $T \subseteq S$, a preference relation $\succ$ on $S$ is “more cyclic” than its restriction on $T$ in a sense that $\succ$ bears all the cycles of $\succ_{T}$ and possibly more. Based on this fact, the reduction axiom requires that any preference that survives the rationalization rule on $\succ$ should survive its application on the restriction. On the other hand, the extension axiom guarantees a preference on the larger domain to survive the rule provided that it survives for all the restriction on the triangle domains.

While Axioms 1 through 5 are presented here as the properties admitted by the transitive core, they in fact exhaust all the properties it must satisfy. The main result of this section, Theorem 1, shows that the five axioms provide a precise characterization of the transitive core.

**Theorem 1.** Let $\sigma$ be a rationalization rule such that $\sigma(\succ)$ refines $\succ$ for all $\succ \in \mathcal{P}$. Then, $\sigma$ satisfies Axioms 1 to 5 if and only if it is the transitive core.

Consequently, Theorem 1 provides the theoretical foundation for the examination of the transitive core from normative viewpoints. The first two axioms appear to be fairly acceptable requirements on a rationalization rule. The reduction and extension axioms are not entirely innocuous but reasonable to a good extent. From my perspective, the neutrality requirement (Axiom 3) seems to be the most controversial yet connotative axiom. It guarantees the impartiality of a rationalization rule between strict preferences and weak preferences in that any reasoning that breaks a strict preference must break a weak preference, and vice versa. Therefore, it gives validity to the statement that a problematic preference is problematic because, regardless of whether it is strict or weak, it is causing inconsistency in the preference relation. This appears to be a sensible reasoning. On the other hand, the condition excludes viewpoints such that strict preferences are the stronger version of weak preferences or that the extent to which one alternative is preferred to another can be measured. Based on these ideas, strict preferences should be preserved over weak preferences. The transitive core, as an advocate of the neutrality axiom, adopt the former principle and reject the latter.

As a dual of the transitive closure, which is defined as the smallest transitive extension of a preference relation, the transitive core can be characterized as the largest transitive refinement in a certain class of rationalization rules. The next proposition shows that the extension axiom (Axiom 5) may be dropped from the characterization of the transitive core in place of the maximum requirement of a rationalization rule.
Proposition 2. Let $\sigma$ be a rationalization rule such that $\sigma(\succeq)$ refines $\succeq$ for all $\succeq \in \mathcal{P}$. If $\sigma$ satisfies Axioms 1 to 4, then $\sigma(\succeq) \subseteq \partial \succeq$ for all $\succeq \in \mathcal{P}$.

Loosely speaking, for any refining rationalization rule $\sigma$, the larger $\sigma(\succeq)$ is, the closer $\sigma(\succeq)$ is from the original preference relation $\succeq$. Therefore, Proposition 2 shows that the transitive core respects the decision maker’s preference relation very well. As a matter of fact, it gives the best approximation of the original preference relation compared to any rationalization rule that admits Axioms 1 to 4. This is another desirable property the transitive core offers.

4 Applications

4.1 Majority voting

Let $n$ be any natural number, and $X$ any nonempty set of alternatives. Consider an environment in which, facing a pair of options from $X$, $n$ many voters vote to decide an alternative they choose. Each voter $i$, where $i \in \{1, \ldots, n\}$, evaluates alternatives according to a linear order $\succeq_i$ on $X$ and has an equal share of votes. The majority voting rule in such an environment defines a collective preference relation $\succeq$ on $X$ such that, for all $x, y \in X$,

$$x \succeq y \text{ if and only if } \left| \{i : x \succeq_i y\} \right| \geq \left| \{i : y \succeq_i x\} \right|.$$  \hspace{1cm} (1)

It is well known that the collective preference relation $\succeq$ by the majority voting is, while complete, not necessarily transitive.

For any $x, y \in X$, we shall say that $x$ is unanimously preferred to $y$ if $x \succeq_i y$ for all voters $i$. If $x$ is unanimously preferred to $y$, then $x$ is chosen over $y$ by the majority voting and thus $x \succeq y$. In fact, as the term suggests, $x$ is chosen over $y$ not only by the majority of voters but by all the voters. In this sense, the unanimity preference is a part of the collective preference supported by the strong consistency on the individual preferences. Also, notice that the unanimity preference is transitive in a sense that if $x$ is unanimously preferred to $y$, and so is $y$ to $z$, then $x$ must be unanimously preferred to $z$. These observations suggest that asking a normative guidance of rationalization to preserve at least the unanimity part of the collective preference seems to be a modest request.

Example 1. The transitive closure taken on the collective preference $\succeq$ induced by the majority voting rule does not always preserve the unanimity preference. To see this, consider the following example where three voters ($n = 3$) vote for pairs of options from four alternatives ($X = \{x, x', y, z\}$). Voters preference relations over alternatives are defined as in Table 1. Note that an alternative $x$ is unanimously
preferred to the inferior alternative $x'$ so that $x > x'$. However, as $x' \succeq y$, $y \succeq z$, and $z \succeq x$, we have $x' \text{tr}(\succeq) x$ by the transitive closure. On the other hand, any alternative $\alpha \in X$ satisfies $\alpha \succeq x$ only if $\alpha \succeq x'$, and $x' \succeq \alpha$ only if $x \succeq \alpha$, while the same does not hold when we replace the role of $x$ and $x'$. Therefore, we have $x \partial \succeq x'$ and not $x' \partial \succeq x$, which shows that the transitive core preserves the unanimity preference.

In the example above, we observed that the transitive core works well in confirming the unanimity preference of the society as a part of the rationalized preference ordering. This observation holds in general: for any number of voters and for any composition of individual preferences, what is chosen unanimously over another is chosen by the transitive core of a collective preference relation induced by the majority voting.

**Proposition 3.** Let $n \in \mathbb{N}$, and Let $\succeq_i$ be any linear order on $X$ for each $i$. Also, let $\succeq$ be a complete binary relation defined by (1). Then, for any $x, y \in X$, if $x$ is unanimously preferred to $y$, then $x \partial \succeq y$ and not $y \partial \succeq x$.

As the last remark of this section, we note that both the transitive closure and the transitive core of a social preference induced by the majority voting behave nicely when the Condorcet winner happens to exist. Assume that $n$ is an odd number for the rest of the argument for simplicity. An alternative $x \in X$ is called a Condorcet winner if it is chosen over any other alternatives by the pairwise majority voting, that is, $x \succeq y$ for all $y$ in $X$, where $\succeq$ is a social preference defined by (1). A Condorcet winner may not exist, but it is unique if it does. Moreover, if the Condorcet winner exists, it is strictly preferred to any other alternatives by both the transitive closure $\text{tr}(\succeq)$ and the transitive core $\partial \succeq$ of the social preference. So, these two rationalization rules choose the Condorcet winner as the unique maximum from the set of all candidates.

### 4.2 Theory of imperfect discrimination

An attention to the observation that the indifference relation of a decision maker may not be transitive due to an inability of discrimination between alternatives has
been long paid in the literature. Armstrong [1, 2, 3] questioned the assumption of transitive indifferences and first introduced the utility model of imperfect discrimination. Luce [11] brought a notion of semiorders into the economic theory and provided an axiomatic foundation to it. Subsequently, a number of generalizations of semiorders, including the interval orders by Fishburn [5], are developed on the seek of further descriptive models of intransitive indifference.

In this section, consider the following simplified representation of semiorders studied by Luce [11]. Let $X$ be a connected metric space and $\succeq$ be a complete binary relation on $X$. A semiorder representation is a pair $(u, \epsilon)$ of a continuous utility function $u : X \to \mathbb{R}$ and a nonnegative number $\epsilon \geq 0$ that satisfies

$$x \succeq y \text{ if and only if } u(x) \geq u(y) - \epsilon$$

for all $x, y \in X$. We say that a complete binary relation $\succeq$ on $X$ admits a semiorder representation if such a pair $(u, \epsilon)$ exists. Note that the semiorder representation reduces to the standard utility representation when $\epsilon = 0$. On the other hand, in case $\epsilon$ has a strictly positive value, an alternative may be evaluated as indifferent to other alternatives even when their utility values are different. Figure 1 illustrates the regions of preferred, indifferent, and dispreferred options relative to an alternative $x \in X$.

The condition (2) can be equivalently restated as $x \sim y$ if and only if $|u(x) - u(y)| \leq \epsilon$, and $x > y$ if $u(x) > u(y) + \epsilon$. Therefore, the decision maker following a semiorder representation exhibits a strict preference only if one of the alternatives under consideration is sufficiently better than the other. The coefficient $\epsilon$ is, with this reason, called a measurement of the just noticeable difference.

Now, consider any semiorder representation with a strictly positive $\epsilon > 0$. We
can easily observe that, for any \( x, y \in X \), there is a finite sequence \((z_i)_{i=1}^{k}\) of alternatives such that every adjacent pair along this sequence is declared to be indifferent while \( z_1 = x \) and \( z_k = y \). This observation shows that any cyclic preference relation \( \succsim \) admitting a semiorder representation is extremely cyclic. Put differently, the transitive core taken for a preference relation induced by the semiorder representation is the trivial relation \( X \times X \) except when \( \epsilon = 0 \). The result seems to impair the role of the transitive closure as the normative guidance of rationalization, for the statement that the decision maker should be indifferent for a cup of coffee with any number of sugar cubes, unless he has a perfect sense of tastes to discern a just grain of difference, is hardly acceptable.

Suppose that we know that a decision maker follows a semiorder representation but cannot directly observe a function \( u \) that represents his utility evaluations. Facing the problem of recovering the decision maker’s “true” preference ordering represented by \( u \) from the information on an obscured preference relation \( \succsim \) in (2), Luce [11] invented a method of inferring an order that takes out the distortion due to the imperfect discrimination of alternatives (Definition 1). Where \( \succsim \) is a complete binary relation on \( X \), this order, which we shall refer to as the Luce’s order induced by \( \succsim \), is defined as a pair of binary relations \((\succsim_L, \sim_L)\) on \( X \) such that \( x \succsim_L y \) iff either

(i) \( x \succ y \),

(ii) \( x \sim y \) and there exists \( z \in X \) such that \( x \sim z \) and \( z \succ y \), or

(iii) \( x \sim y \) and there exists \( z \in X \) such that \( x \succ z \) and \( z \sim y \),

and \( x \sim_L y \) iff neither \( x \succsim_L y \) nor \( y \succsim_L x \) holds. Subsequently, he proved that the Luce’s order is a weak order on \( X \) provided that a preference relation \( \succsim \) admits a semiorder representation (Theorem 1), and moreover it fully recovers the true preference relation of the decision maker under certain conditions (Theorem 3), so that for any \( x, y \in X \),

\[
    x \succsim_L y \text{ if and only if } u(x) > u(y).
\]  

The condition that allows the derivation of the equivalence (3) in the current simplified setup can be stated as a property that a preference relation is globally non-trivial.

**Strong nontriviality.** For any \( x \in X \), there is a \( y \in X \) such that not \( x \sim y \).

The main finding of this section reveals how the transitive core behaves on preference relations under the semiorder representation. It shows that, in the present framework, the transitive core reduces to the Luce’s order and consequently that
the transitive core fully obtains the utility ordering represented by $u$ under the same condition that carries the representation (3).

**Proposition 4.** Let $\succ$ be a complete binary relation on $X$ that admits a semiorder representation $(u, \epsilon)$. Then, the transitive core of $\succ$ coincides with the Luce’s order induced from $\succ$, i.e. $x \partial \succ y$ if and only if $x \succ^L y$ or $x \sim^L y$ for all $x, y \in X$. Moreover, provided that $\succ$ is strongly nontrivial,

$$x \partial \succ y \quad \text{if and only if} \quad u(x) \geq u(y)$$

for any $x, y \in X$.

Note that, while the definition of the Luce’s order can be extended for preference relations on the general domain discussed in Section 3, it does not provide a well defined weak order in general in that both $x \succ^L y$ and $y \succ^L x$ may hold simultaneously. On the other hand, the transitive core always yields a well defined transitive binary relation, and it in particular reduces to the Luce’s order for the present model of the semiorder representation. In this sense, the transitive core may be viewed as a proper extension of the Luce’s order as a rationalization rule on the general domain.

### 4.3 Regret theory

Let $\{1, \ldots, n\}$ be a finite set of states of the world. Each state $i$ has a probability $p_i$ to occur where $0 < p_i \leq 1$ and $\sum_{i=1}^n p_i = 1$. Let $X$ be the set of all functions that assigns a real value to each state. In the analysis of choice under uncertainty, each member of $X$ is understood as an act or a prospect that describes a prize delivered to the decision maker contingent on states of the world. The decision maker exhibits his preference over acts, and this information is observed as a binary relation $\succ$ on $X$.

The major body of the economic analysis of risky outcomes relies on the expected utility theory developed by von Neumann and Morgenstern [17] according to which acts are compared by their expected utility values $\mathbb{E}(v \circ x)$ for some utility function $v : \mathbb{R} \to \mathbb{R}$. The theory has been acknowledged as the model of rational decision making under uncertainty, and the axiom system of the theory is often justified from the normative perspective.

Experimental and empirical studies, on the other hand, have consistently found the disparity between observed behavior in reality and the prediction of the expected utility theory. For instance, the celebrated work by Kahneman and Tversky [8] presented extensive evidence that shows violations of the expected utility hypothesis. To illustrate, consider the following two questionnaires each of which asks a subject to choose one from a pair of independent acts.
Questionnaire 1. Choose one from:

(a) $4000 with probability .80 and $0 with probability .20
(b) $3000 with certainty

Questionnaire 2. Choose one from:

(a) $4000 with probability .20 and $0 with probability .80
(b) $3000 with probability .25 and $0 with probability .75

They found that the majority of subject chooses the option (b) from Questionnaire 1 while many choose the option (a) from Questionnaire 2. This result is incompatible with the expected utility theory, and it in particular suggests that people has tendency of overweighting outcomes with certainty. This phenomenon is called a certainty effect.

To accommodate the observed violations of the expected utility theory, Loomes and Sugden [10] has proposed an alternative theory of decision making under uncertainty that reflects the experience of regret. The regret theory, upon a choice of an act over another, takes into account not only the utility derived from an outcome of the chosen act but also the relative status of well-being compared to a case where the other alternative were chosen instead. Once the state of the world is resolved, the decision maker may experience regret if an outcome of the chosen act happens to be worse than that of the alternative. Similarly, he may feel rejoicing if an outcome of the chosen act turns out more desirable than that of the alternative. The regret theory claims that the psychological factors as such by introspection affect the decision maker’s tastes over acts.

Formally, we define a representation by regret by two continuous functions $u : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$ such that, for all $x$ and $y$ in $X$,

$$x \succeq y \text{ iff } \sum_{i=1}^{n} p_i Q(u(x_i) - u(y_i)) \geq 0,$$  \hspace{1cm} (4)

while (i) $u$ is strictly increasing with $u(0) = 0$ and $u(\mathbb{R}) = \mathbb{R}$, (ii) $Q$ is strictly increasing and satisfies $Q(\xi) = -Q(-\xi)$ for all $\xi$, and (iii) $Q$ is convex on $\mathbb{R}_+$. We say that a complete binary relation $\succeq$ on $X$ admits a representation by regret if there exists such a pair of functions $(u, Q)$. Loomes and Sugden [10] show that the representation (4) explains various choice anomalies observed in reality when $Q$ is strictly convex.

Notice that the representation reduces to the expected utility representation when $Q$ is linear, in which case a preference relation of the decision maker is
<table>
<thead>
<tr>
<th>State</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2: Prize schedule

necessarily transitive. On the other hand, the representation may exhibit cyclic preferences if it is not compatible with the expected utility theory. For example, let $n = 3$ and $p_1 = p_2 = p_3 = 1/3$, and consider a preference relation represented by a pair $(u, Q)$ such that $u$ is the identity function and $Q$ is strictly convex. Let $x, y, z$ be three acts defined as in Table 2. Then, it follows that

$$\sum_{i=1}^3 p_i Q(u(x_i) - u(y_i)) = \frac{1}{3}Q(2) - \frac{1}{3}Q(1) - \frac{1}{3}Q(1) > 0$$

and thus $x \succ y$. As we can obtain similar results for pairs $(y, z)$ and $(z, x)$, it shows that a cycle $x \succ y \succ z \succ x$ occurs under the representation. What happens behind this example is that, even though two prospects $x$ and $y$ carry the identical distribution of returns, the decision maker knows that he will regret the fact that he has chosen $x$ over $y$ with two out of three times, and this fact reduces the appeal of $x$ relative to $y$ through the convexity of $Q$, making the decision maker prefer $y$ to $x$. The regret theory allows in this way the state-wise correlation of prospects to affect the decision maker’s preference relation.

The observation that the regret theory accounts for the preference cycles if not compatible with the expected utility representation holds in general. The next result in fact shows that a preference relation admitting a regret representation but not an expected utility representation must be cyclic.

**Proposition 5.** Suppose that a binary relation $\succeq$ on $X$ admits a representation by regret. Then, the following two statements are equivalent: (i) $\succeq$ is transitive; (ii) there exists a function $v : \mathbb{R} \to \mathbb{R}$ such that $x \succeq y$ iff $\mathbb{E}(v \circ x) \geq \mathbb{E}(v \circ y)$ for all $x, y \in X$.

Therefore, the entire descriptive power of the regret theory for observed violations of the expected utility theory is attributed to the intransitivity of the preference ordering. A regret representation not compatible with the expected utility theory must exhibit preference cycles, and consequently it is vulnerable to the issues of cyclic preference relations discussed in Section 3.

However, the attempt of rationalizing cyclic preference relations by taking their transitive closure brings a result far from intuition in the current setup. The following proposition shows that, for any preference relation admitting a regret representation, as far as it is not representable by an expected utility, and thus not transitive,
the transitive closure transforms the relation by making all the acts indifferent, irrelevant to what the decision maker originally manifests. Notice that, because the transitive closure is the smallest transitive extension, the same result follows for any extending rationalization rules. That means that, unless we accept the “all-indifferent” relation, a more natural method of rationalization must be sought in rules that refine the original preference relations.

**Proposition 6.** Any cyclic binary relation on $X$ admitting a representation by regret is extremely cyclic.

An important observation is that a decision maker following a regret representation is consistent in evaluating monotonically ordered acts: if $x \geq y$ then $x \succ y$. Moreover, it is obvious that such an order $\geq$ is transitive. Therefore, while a preference relation under a regret representation does not confirm transitivity in general, a certain normative order can be recognized within the preference relation. Therefore, one natural requirement on a rationalization rule may be to preserve such a normative order embedded in the decision maker’s preference relation.

Now, let $x \geq y$, and $z$ be any prospect in $X$ such that $y \succ z$. If $\succ$ admits a representation by regret $(u, Q)$, then we have $\sum_{i=1}^{n} Q(u(y_i) - u(z_i)) \geq 0$, which implies that $\sum_{i=1}^{n} Q(u(x_i) - u(z_i)) \geq 0$ as $u$ and $Q$ are monotonic, so $x \succ z$. Therefore, provided that $x$ is dominating $y$ with respect to the order $\geq$, any prospect that is less preferred to $y$ must be less preferred to $x$. Similarly, we can show that any prospect that is more preferred to $x$ must be more preferred to $y$, that is, that $z \succ x$ implies $z \succ y$. We have just observed that the transitive core preserves the order $\geq$.

**Proposition 7.** Let $\succ$ be a complete binary relation on $X$ that admits a representation by regret, and $x, y \in X$. Then, $x \geq y$ implies that $x \partial \succ y$, and $x \succ y$ implies that $x \partial \succ y$ and not $y \partial \succ x$.

Consequently, while the transitive core breaks some preferences from $\succ$ to obtain a transitive binary relation, it never changes the original relation where it exhibits consistency. Therefore, the result suggests that the preference relation left after the refinement is not only transitive but also has an appeal from the normative aspect.

### 4.4 Time preferences

Let $Z$ be a nondegenerate open interval in $\mathbb{R}$, and define $X := Z \times [0, \infty)$. In this section, a generic member $(x, t)$ of $X$ is interpreted as a dated outcome that the decision maker receives a prize $x$ at period $t$. A complete binary relation $\succ$ on $X$
is called a time preference if, for any \( t \), \((x, t) \succ (y, t)\) whenever \( x > y \). Note that a time preference is transitive as long as variations are made only in the prize dimension. Ok and Masatlioglu [13] examine a representation of time preferences by \textit{relative discounting}. In contrast to the model of absolute discounting, the relative discounting model allows a possibility that a comparison of dated outcomes cannot be made by means of their discounted present values evaluated at time 0. Instead, it compares two dated outcomes \((x, t)\) and \((y, s)\) through discounting them at time \( t \) or at time \( s \) but not any other periods in general.

To be precise, \textit{a relative discounting representation} involves a utility function \( U \) from \( \mathbb{Z} \) to \( \mathbb{R}^{++} \) and a relative discounting function \( \eta \) from \( \mathbb{R}^2_+ \) to \( \mathbb{R}^{++} \) that satisfy

\[
(x, t) \succeq (y, s) \quad \text{if and only if} \quad U(x) \geq \eta(s, t)U(y) \tag{5}
\]

for all \((x, t), (y, s) \in X\), where (i) \( U \) is an increasing homeomorphism, (ii) \( \eta \) is continuous and satisfies \( \eta(s, t)\eta(t, s) = 1 \) for any \( s, t \), and (iii) \( \eta(\cdot, t) \) is nonincreasing with \( \eta(\infty, t) = 0 \). We say that a time preference \( \succeq \) admits a relative discounting representation if such a pair of functions exists for \( \succeq \).

The map \( \eta \) in the representation computes the relative discount rate between any two dates. For example, \( \eta(s, t)U(y) \) obtains the value of a dated outcome \((y, s)\) evaluated at period \( t \). Therefore, the representation is defined by the criterion that a dated outcome \((x, t)\) is weakly preferred to another outcome \((y, s)\) if and only if the utility value of a prize \( x \) is at least as good as the relatively discounted utility value of \((y, s)\) evaluated at time \( t \).

Transitivity of a preference relation in the present model is equivalent to a property that \( \eta(r, t) = \eta(r, s)\eta(s, t) \) for all \( t, s, r \). As this property can fail in general, the discounting due to a delay may be magnified or reduced by dividing it into parts, and the representation consequently allows cyclic preferences in this case. Importantly, with this flexibility of the representation that encompasses non-transitive preferences, other known models of time preferences are obtained as subclasses of the relative discounting model. For example, the subadditive discounting model studied by Read [14] can be viewed as the relative discounting model where \( \eta(r, t) \geq \eta(r, s)\eta(s, t) \) holds for all \( t, s, r \) with \( t \leq s \leq r \). Moreover, time preferences induced from similarity relations studied by Rubinstein [16] are also shown to be represented by the relative discounting model ([13, Example 3]).

The relative discounting model is the descriptive attempts to account for a number of experimental and empirical findings that contradict with the traditional theory of time preferences such as the exponential and the hyperbolic discounting models. Ok and Masatlioglu note that “all in all, it seems clear that transitivity of time preferences, which is a precondition for the hyperbolic discounting model, is not really an innocuous assumption, especially from a descriptive angle.”
On the other hand, while the relative discounting theory provides an attractive foundation for the study of intertemporal decision problems where the decision maker’s preference relation is not necessarily transitive, another seemingly attractive question is left unanswered, namely, how the decision maker should evaluate the values of dated outcomes when his objective is in fact suffered from preference cycles. The first observation of this section reveals that applying the transitive closure in response to the previous question appears to be crumbled by the fact that it only provides the trivial relation.

**Proposition 8.** Any cyclic time preference admitting a relative discounting representation is extremely cyclic.

Consequently, as in the case of the regret theory, the transitive closure of any cyclic time preference within the class of the relative discounting model guides the decision maker to behave indifferently between any pair of alternatives. As we noted earlier, a decision maker under the relative discounting model has a coherent order on the outcome space so far as outcomes are delivered at the same date. In fact, under the present setup where each outcome is a prize on the real line, the order on the outcome space represents a simple normative principle that more is better. Arguably, any natural rationalization rule may well preserve this principle.

We can derive a further regularity from the relative discounting model. It is remarked by the authors of the original paper that the representation (5) allows a preference cycle only if the cycle involves outcomes with three or more different delivery dates. An immediate implication of this observation is that, for any \( x, y, z \in Z \) and \( t, r \in [0, \infty) \), \((x, t) \succeq (y, t) \succeq (z, r)\) implies \((x, t) \succeq (z, r)\) and that \((z, r) \succeq (y, t)\) implies \((z, r) \succeq (y, t)\). We have just shown that the transitive core of relative discounting time preferences preserves the natural order \( \geq \) on the outcome space.

More structures can be derived for the transitive core in the present model. It is known that if a time preference with a relative discounting representation happens to be transitive, it reduces to a multiplicative discounting representation studied by Fishburn and Rubinstein [6] according to which

\[
(x, t) \succeq (y, s) \quad \text{if and only if} \quad \delta(t)U(x) \geq \delta(s)U(y)
\]

for all \((x, t), (y, s) \in X\). The function \( \delta \) in the representation (6) is sometimes called a **general discounting function** in a sense that it generalizes both the exponential discounting and the hyperbolic discounting.

A similar result follows for the transitive core of relative discounting time preferences besides the fact that the transitive core is not necessarily complete. Note that the representation (6) necessitates a preference relation to be complete. Therefore, instead of represented by a single discounting function, the transitive core
is represented by a number of discounting functions. For a transitive binary relation $≽$ on $X$, a general multi-discounting representation of $≽$ consists of a function $U : Z \rightarrow \mathbb{R}_{++}$ and a collection $\mathcal{D}$ of functions $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ that satisfy

$$(x, t) \succsim (y, s) \text{ if and only if } \delta(t)U(x) \geq \delta(s)U(y) \text{ for all } \delta \in \mathcal{D}$$

(7)

for all $(x, t), (y, s) \in X$, where (i) $U$ is an increasing homeomorphism, (ii) every $\delta \in \mathcal{D}$ is continuous and nonincreasing with $\delta(\infty) = 0$. We say that $≽$ admits a general multi-discounting representation if such a pair $(U, \mathcal{D})$ exists for $≽$.

**Proposition 9.** The transitive core of any time preference that admits a relative discounting representation admits a general multi-discounting representation.

Notice that the previous observation, that the transitive core preserves the natural order $\succeq$ on the outcome space, is readily seen as a corollary of Proposition 9. Indeed, a general multi-discounting representation induces a constant linear order represented by $U$ on the outcome space if the delivery dates of the outcomes under considerations are identical.

As motivated by Ok [12], the representation of (possibly incomplete) transitive preference relations such as (7) facilitates the analytical study on the model while causing no information loss of preference relations. Additionally, in Proposition 9, the fact that the multiplicity of the representation appears on discount factors, but not on utility functions, may be viewed as a reflection from the relative discounting model where the decision maker’s preference relation exhibits cycles due to the passage of time.

## 5 Proofs

### 5.1 Proof of Theorem 1 and Proposition 2

First, we shall show that the transitive core $\partial$ satisfies Axioms 1 to 5. Fix any $S, T \in X$ with $T \subseteq S$ and any complete binary relations $\succsim, \succsim'$ on $S$.

(Axiom 1) Suppose that there exists a bijection $\pi$ on $S$ such that $x \succsim y$ iff $\pi(x) \succsim' \pi(y)$ for all $x, y \in S$. By symmetry, it is enough to show that $x \partial \succsim y$ implies $\pi(x) \partial \succsim' \pi(y)$ for all $x, y \in S$. To this end, fix any $x, y \in S$ with $x \partial \succsim y$. If $z \in S$ is such that $\pi(y) \succsim' z$, then $y \succsim \pi^{-1}(z)$ by the choice of $\pi$, which in turn implies that $x \succsim \pi^{-1}(z)$ as $x \partial \succsim y$. So, applying $\pi$ on both sides, we have $\pi(x) \succsim' z$. As we can similarly show that $z \succsim' \pi(x)$ implies $z \succsim' \pi(y)$, $\pi(x) \partial \succsim' \pi(y)$ follows.

(Axiom 2) Let $x, y \in S$ be such that $x \partial \succsim y$. Then, for all $z \in S$ with $x \preceq z$, since $z \succsim x$ and $x \partial \succsim y$, we have $z \succsim y$ and $y \preceq z$. As $z \preceq x$ similarly follows if $z \preceq y$, we obtain $y \partial \preceq x$ as desired. The converse implication holds by symmery.
rationalization rules. Also, the weak preference of $x \succ y \succ z$ as $\succ$ is reflexive.

Claim 1. Let $\sigma$ be a rationalization rule such that $\sigma(\succ)$ refines $\succ$ for all $\succ \in P$. If $\sigma$ satisfies Axioms 1 to 3, then $\sigma(\succ) = \partial \succ$ for all $\succ \in P$ whose domain is a set of three points.

Proof. Denote by $\Sigma'$ the collection of all rationalization rule $\sigma$ such that $\sigma(\succ)$ refines $\succ$ for all $\succ \in P$ and that it satisfies Axioms 1 to 3. Let $\succ \in P$ be a complete binary relation on a set of three points $\{x,y,z\}$. If $\succ$ is transitive, then the claim trivially holds as $\sigma(\succ) = \partial \succ$ for all $\sigma \in \Sigma'$. (Note that the definition of rationalization rules requires them to map any transitive binary relation to itself.) So, assume that $\succ$ is cyclic in what follows. Then, by Axioms 1 and 2, there are only three cases to be considered: (1) $x \succ y \succ z$, (2) $x \succ y \sim z$, and (3) $x \succ y \sim z$. Moreover, it is enough to show that, for each of these cases and any $\sigma \in \Sigma'$, Axioms 1 to 3 uniquely identifies $\sigma(\succ)$, for the transitive core is a member of $\Sigma'$. To this end, fix any $\sigma \in \Sigma'$.

First, consider $\succ$ of case (1). Then, for $\sigma(\succ)$ to be transitive, it must break either of the strict preference. However, by Axiom 1, we can show that $\sigma(\succ)$ breaks all the strict preference if it breaks any of them. So, we obtain $\sigma(\succ) = \emptyset$.

Next, let $\succ$ be of case (2). Then, the strict preference of $x$ to $y$ must be kept by $\sigma(\succ)$, for otherwise $\sigma$ would break the weak preference of $x$ to $y$ from a transitive binary relation $\succ \cup \{y,x\}$ by Axiom 3, which contradicts with the definition of rationalization rules. Also, the weak preference of $x$ to $z$ must be preserved by $\sigma(\succ)$, for otherwise the strict preference of $x$ to $z$ would be broken from a transitive
binary relation $\succ \setminus \{(z,x)\}$ by Axiom 3. A similar argument shows that the weak preference of $z$ to $y$ is preserved by $\sigma(\succ)$. Consequently, $\sigma(\succ)$ must break either the weak preference of $y$ to $z$ or the weak preference of $z$ to $x$ in order for $\sigma(\succ)$ to be transitive. However, by Axioms 1 and 2, we can show that $\sigma(\succ)$ breaks $y \succeq z$ if and only if it breaks $z \succ x$, and hence $\sigma(\succ)$ must break both. As a conclusion, we have obtained that $\sigma(\succ) = \{(x,y), (x,z), (z,y)\}$.

Finally, suppose that $\succ$ is of case (3). Let $\succ_{yx} := \succ \cup \{(y,x)\}$. Then, $x \preceq_{yx} y \succ_{yx} z \sim_{yx} x$. So, by Axiom 1 and the result from case (2), the weak preference of $x$ to $y$ is broken by $\sigma(\succ_{yx})$, which implies that the strict preference of $x$ to $y$ is broken by $\sigma(\succ)$ by Axiom 3. A similar argument shows that the strict preference of $y$ to $z$ is broken by $\sigma(\succ)$. Also, the weak preference of $z$ to $x$ must be broken by $\sigma(\succ)$, for otherwise $\sigma$ would preserve the strict preference of $z$ to $x$ for $\succ \setminus \{(x,z)\}$ by Axiom 3, which contradicts the result from case (1). Conversely, $\sigma(\succ)$ preserves the weak preference of $x$ to $z$, for otherwise $\sigma$ would break the strict preference of $x$ to $z$ from a transitive binary relation $\succ \setminus \{(z,x)\}$ by Axiom 3, which contradicts with the definition of rationalization rules. So, we have shown that $\sigma(\succ) = \{(x,z)\}$. The proof is now complete.

We are now ready to prove Proposition 2. Define $\Sigma'$ as in the proof of Claim 1. Suppose that $\sigma \in \Sigma'$ satisfies Axiom 4, and let $\succ$ be a complete binary relation on a set $S \subset X$. We wish to show that $\sigma(\succ) \subseteq \partial \succ$. To this end, fix any $x, y \in S$ with $x \sigma(\succ) y$. Then, by Axiom 4, we have $x \sigma(\succ_{xyz}) y$ for any $z \in S$, which in turn implies that $x \partial \succ_{xyz} y$ by Claim 1. Since $\partial$ satisfies Axiom 5, it follows that $x \partial \succ y$. We have shown that $x \partial \succ y$ whenever $x \sigma(\succ) y$.

Finally, we shall complete the necessity part of Theorem 1. Let $\sigma \in \Sigma'$ be a rationalization rule that satisfies Axioms 4 and 5, and fix any complete binary relation $\succ$ on a set $S \subset X$. As $\sigma(\succ) \subseteq \partial \succ$ follows by Proposition 2, we only need to show that $\partial \succ \subseteq \sigma(\succ)$. For this, take any $x, y \in S$ such that $x \partial \succ y$. Then, by Axiom 4, we have $x \partial \succ_{xyz} y$ for all $z \in S$. By Claim 1, this implies that $x \sigma(\succ_{xyz}) y$ for all $z \in S$. Then, it follows that $x \sigma(\succ) y$, for $\sigma$ satisfies Axiom 5. As $x \sigma(\succ) y$ whenever $x \partial \succ y$, we are done.

### 5.2 Proof of Proposition 3

Suppose that $x$ is unanimously preferred to $y$. Then, as $|\{i : x \succ_i y\}| = n$ and $|\{i : y \succ_i x\}| = 0$, $y \succ x$ does not hold by definition (1). As $\partial \succ$ refines $\succ$, it thus follows that $y \partial \succ x$ does not hold either. Next, we shall show that $x \partial \succ y$ holds on the other hand. Fix any $z \in X$, and suppose that $y \succ z$. Then, $|\{i : y \succ_i z\}| \geq |\{i : z \succ_i y\}|$. However, as $\succ_i$ is a linear order and $x \succ_i y$ for any $i$, we have $|\{i : y \succ_i z\}| \leq |\{i : x \succ_i z\}|$ and $|\{i : z \succ_i x\}| \leq |\{i : z \succ_i y\}|$. So,
\[|i : x \geq_i z| \geq |i : y \geq_i z| \geq |i : z \geq_i y| \geq |i : z \geq_i x|,\] and thus \(x \geq z\). As we can similarly show that \(z \geq y\) whenever \(z \geq x\), \(x \geq y\) follows as required.

### 5.3 Proof of Proposition 4

Let \(\geq\) be a complete binary relation on \(X\) that admits a semiorder representation \((u, e)\). Take any \(x, y \in X\) such that \(x >^L y\). Then, either (i) \(x > y\), (ii) there is a \(z \in X\) such that \(x \sim z\) and \(z > y\), or (iii) there is a \(z \in X\) such that \(x > z\) and \(z \sim y\).

In case (i), we have \(u(x) > u(y) + \epsilon \geq u(y)\). In case (ii), we have \(u(x) \geq u(z) - \epsilon\) and \(u(z) > u(y) + \epsilon\), so \(u(x) > u(y)\) follows. In case (iii), we have \(u(x) > u(z) + \epsilon\) and \(u(z) \geq u(y) - \epsilon\), so \(u(x) > u(y)\) follows. Consequently, \(u(x) > u(y)\) whenever \(x >^L y\). As a corollary of this observation, \(>^L\) is asymmetric.

To show the claim that the transitive core and the Luce’s order coincide, take any \(x, y \in X\). First, suppose that neither \(x >^L y\) nor \(x \sim^L y\) holds. Then, \(y >^L x\) follows, and thus either (i) \(y > x\), (ii) there is a \(z \in X\) such that \(y \sim z\) and \(z > x\), or (iii) there is a \(z \in X\) such that \(y > z\) and \(z \sim x\). As \(x \geq y\), (i) implies that not \(x \geq y\). Also, as \(x \geq z\) must follow whenever \(y \geq z\) for \(x \geq y\) to hold, (ii) implies that not \(x \geq y\). Lastly, \(z \geq y\) must hold whenever \(z \geq x\) for \(x \geq y\) to be true, so (iii) implies that not \(x \geq y\).

By contraposition, this shows that \(x \geq y\) implies that \(y \geq x\).

Next, let \(x, y \in X\) be such that not \(x \geq y\). Then, either (i) there exists a \(z \in X\) such that \(y \geq z\) and \(z > x\) or (ii) there exists a \(z \in X\) such that \(y \geq z\) and \(z \geq x\). Consider case (i). By the semiorder representation, it follows that \(u(y) > u(z) + \epsilon\) and \(u(z) \geq u(x) - \epsilon\). So, \(u(y) > u(x) \geq u(x) - \epsilon\) and thus \(y \geq x\). If \(y > x\), then \(y >^L x\) follows. If \(y \sim x\), since there exists a \(z \in X\) such that \(y \geq z\) and \(z > x\), \(y >^L x\) holds. So, \(y >^L x\) must hold in case (i). Then, not \(x \sim^L y\) by definition, and furthermore not \(x >^L y\), for \(>^L\) is asymmetric as shown above. Similarly, we can show that neither \(x \sim^L y\) nor \(x >^L y\) may hold in case (ii). By contraposition, it follows that \(x \geq y\) whenever either \(x >^L y\) or \(x \sim^L y\).

We shall now turn to showing that the transitive core is represented by \(u\). Choose any \(x, y \in X\). First, suppose that \(u(y) > u(x)\). As \(\geq\) is strongly nontrivial, there exists a \(z \in X\) such that (i) \(x > z\) or (ii) \(z > x\). In case (i), we have \(u(x) > u(z) + \epsilon\) by the representation. As \(u(X)\) is an interval (for \(X\) is connected and \(u\) is continuous), there is a \(z' \in X\) such that \(u(y) > u(x) = u(z') + \epsilon\). By the representation, this implies that \(y > z'\) and \(z' \sim x\). So, not \(x \geq y\). In case (ii), we have \(u(z) > u(x) + \epsilon\) by the representation. As \(u(X)\) is an interval, and \(u(y) + \epsilon > u(x) + \epsilon\), there is a \(z' \in X\) such that \(u(y) + \epsilon > u(x) + \epsilon\). By the representation, it follows that \(y > z'\) and \(z' > x\). So, not \(x \geq y\). By contraposition, we have shown that \(x \geq y\) implies \(u(x) \geq u(y)\).

Finally, let \(u(x) \geq u(y)\). If \(y \geq z\), \(u(y) \geq u(z) - \epsilon\), so \(u(x) \geq u(z) - \epsilon\), from which
we can infer \( x \succ z \). Similarly, if \( z \succ x \), \( u(z) \geq u(x) - \epsilon \), so \( u(z) \geq u(y) - \epsilon \), which implies that \( z \succ y \). Therefore, \( x \succeq y \) follows.

### 5.4 Proofs of Proposition 5 and Proposition 6

Let \( \succ \) be a binary relation on \( X \) that admits a representation by regret \( (u, Q) \), and denotes its transitive closure by \( \text{tr()} \). In this proof, the set \( X \) of functions from the state space \( \{1, \ldots, n\} \) to \( \mathbb{R} \) is identified as \( \mathbb{R}^n \), and a member of \( X \) is denoted as a vector in \( \mathbb{R}^n \). By the property that \( u \) is surjection, the next claim allows us to assume that \( u \) is the identity function without loss of generality.

**Claim 2.** Define a binary relation \( \succeq \) on \( X \) by \( x \succeq y \) iff \( \sum_{i=1}^{n} p_i Q(x_i - y_i) \geq 0 \). Then, \( \succeq \) is transitive if and only if \( \succ \) is, and \( \succeq \) admits an expected utility representation if and only if \( \succ \) does \( \succeq \).

**Proof.** Abusing notation, a vector \( (u(x_1), \ldots, u(x_n)) \) is written as \( u(x) \) for all \( x \in X \). Obviously, \( x \succ y \) iff \( u(x) \geq u(y) \). Suppose that \( \succeq \) is transitive, and let \( x \succeq y \succeq z \). Then, \( u(x) \geq u(y) \geq u(z) \), and therefore we have \( u(x) \geq u(z) \) and \( x \succeq z \). So, \( \succeq \) is transitive.

Suppose that there exists a function \( v : \mathbb{R} \to \mathbb{R} \) such that \( x \succeq y \) iff \( E(v \circ x) \geq E(v \circ y) \). Then, defining \( v' = v \circ u \), we have \( x \succeq y \) iff \( E(v' \circ x) \geq E(v' \circ y) \). So, \( \succeq \) also admits an expected utility representation. Repeating the proof with replacing the role of \( \succ \) and \( \succeq \) of \( u \) and \( u^{-1} \) yields the converse implication. The proof is complete. \( \square \)

In what follows, we shall assume that \( u \) is the identity function by Claim 2.

**Claim 3.** If \( \{x : x \sim 0\} \neq \{x : \sum_{i=1}^{n} p_i x_i = 0\} \), then \( \succ \) is extremely cyclic.

**Proof.** Let \( \{x : x \sim 0\} \neq \{x : \sum_{i=1}^{n} p_i x_i = 0\} \). Then, either (i) there exists an \( x^* \in X \) such that \( x^* \sim 0 \) and \( \sum_{i=1}^{n} p_i x_i^* = 0 \), or (ii) there exists an \( y^* \in X \) such that \( y^* \neq 0 \) and \( \sum_{i=1}^{n} p_i y_i^* = 0 \). Suppose that (ii) holds. Then, \( \sum_{i=1}^{n} p_i Q(y_i^*) \neq 0 \) by the representation. As \( Q(\mathbb{R}) = \mathbb{R} \), \( p_1 Q(y_1^* + \xi) + \sum_{i=2}^{n} p_i Q(y_i^*) = 0 \) for some \( \xi \neq 0 \). So, if we define \( x^* \in X \) by \( x_1^* = y_1^* + \xi \) and \( x_i^* = y_i^* \) for all \( i > 1 \), then it follows that \( x^* \sim 0 \) by the representation, and moreover \( \sum_{i=1}^{n} p_i x_i^* = \sum_{i=1}^{n} p_i y_i^* + p_1 \xi = p_1 \xi \neq 0 \).

This shows that the condition (i) always holds.

Let \( x^* \in X \) be such that \( x^* \sim 0 \) and \( \sum_{i=1}^{n} p_i x_i^* \neq 0 \). As the representation implies that \( -x \sim 0 \) whenever \( x \sim 0 \), we can assume that \( \epsilon := \sum_{i=1}^{n} p_i x_i^* > 0 \) without loss of generality. Define \( z^*_i \in X \) by \( z_i^* = \epsilon / 2 - x_i^* \) for all \( i \), and observe that \( \sum_{i=1}^{n} p_i z_i^* = -\epsilon / 2 < 0 \) and \( x_i^* + z_i^* = \epsilon / 2 > 0 \) for all \( i \). We claim that, for a sufficiently large \( m \in \mathbb{N} \), \( 0 \sim x^* > x^* + \frac{1}{2m} z^* > \cdots > x^* + \frac{m-1}{2m} z^* > x^* + z^* \). Note that the representation readily implies \( x \succ y \) iff \( x - y \succ 0 \), and hence all we have to show is that \( \frac{1}{m} z^* > 0 \) for a sufficiently large \( m \in \mathbb{N} \).
Define a function $F : \mathbb{R} \to \mathbb{R}$ by $F(\lambda) = \sum_{i=1}^{n} p_i Q(\lambda z_i^*)$ for all $\lambda \in \mathbb{R}$. Note that $Q$ is differentiable almost everywhere as it is convex on $\mathbb{R}_+$ and negatively symmetric around the $y$-axis. Hence, $F$ is differentiable almost everywhere, and $F'(\lambda) = \sum_{i=1}^{n} p_i z_i^* Q'(\lambda z_i^*)$ wherever it is differentiable. Moreover, the negative symmetry implies the differentiability of $Q(\xi)$ at $\xi = 0$, the strict monotonicity implies $Q'(0) > 0$, and the convexity implies that $Q'(\varepsilon_m) \to Q'(0)$ for any sequence $(\varepsilon_m)$ of differentiable points of $Q$ converging to 0. As a consequence, the value of $F'(\lambda)$ can be arbitrarily approximated by $\sum_{i=1}^{n} p_i z_i^* Q'(0) < 0$ around $\lambda = 0$, and in particular there exists an $\lambda^* > 0$ such that, for all $\lambda \in (-\lambda^*, \lambda^*)$, $F'(\lambda) < 0$. Now, it follows that 

$$F(\frac{1}{m}) = F(\frac{1}{m}) - F(0) = \int_{0}^{\frac{1}{m}} F'(\lambda) d\lambda < 0$$

by choosing $m \in \mathbb{N}$ so that $\frac{1}{m} < \lambda^*$. By the representation, $\frac{1}{m} z^* > 0$.

We have shown that, for some $\delta > 0$, there exists a finite sequence $(z_i^k)_{i=1}^{k}$ such that $0 \succ z_1 \succ \cdots \succ z_k \succ \delta e$, where $e = (1, \ldots, 1)$. However, as $x \succ y$ iff $x - y \succ 0$ by the representation, this result immediately extends to the following more general claim: for any $x \in X$ and any $\delta > 0$, there exists a finite sequence $(z_i^k)_{i=1}^{k}$ such that $x \succ z_1 \succ \cdots \succ z_k \succ x + \delta e$. With the immediate observation from the representation that $x \succ y$ whenever $x \geq y$, we obtain $\text{tr}(=)X \times X$ as desired. 

\textbf{Claim 4.} Let $\{x : x \sim 0\} = \{x : \sum_{i=1}^{n} p_i x_i = 0\}$. Then, for all $x, y \in X$, $x \succ y$ iff $\sum_{i=1}^{n} p_i x_i \geq \sum_{i=1}^{n} p_i y_i$. In particular, $\succ$ must be transitive.

\textbf{Proof.} Take any $x, y \in X$ with $x \succ y$. Then, $\sum_{i=1}^{n} p_i Q(x_i - y_i) \geq 0$ by the representation. As $Q$ is strictly increasing and $Q(\mathbb{R}) = \mathbb{R}$, we can find a $\xi \leq 0$ such that $p_1 Q(x_1 - y_1 + \xi) + \sum_{i=2}^{n} p_i Q(x_i - y_i) = 0$. So, if $x' \in X$ is defined by $x_i' = x_i + \xi$ and $x_1' = x_1$ for all $i > 1$, then $\sum_{i=1}^{n} p_i Q(x_i' - y_i) = 0$ and $x' \sim y \sim 0$. Now, by the premise of the claim, it follows that $\sum_{i=1}^{n} p_i x_i \geq \sum_{i=1}^{n} p_i y_i$, as desired.

Conversely, take any $x, y \in X$ with $\sum_{i=1}^{n} p_i x_i \geq \sum_{i=1}^{n} p_i y_i$. Choose $\xi \leq 0$ so that $p_1 (x_1 + \xi) + \sum_{i=2}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i$, and define $x'$ as in the last paragraph. Then, $\sum_{i=1}^{n} p_i Q(x_i' - y_i) = 0$, and thus $x' \sim y \sim 0$. Now, the representation implies that $\sum_{i=1}^{n} p_i Q(x_i - y_i) = 0$, from which we can obtain $\sum_{i=1}^{n} p_i Q(x_i - y_i) \geq 0$ and $x \succ y$ as $Q$ is strictly increasing. 

We are ready to complete the proof. For Proposition 5, if $\succ$ is transitive, then it may not be extremely cyclic, so $\{x : x \sim 0\} = \{x : \sum_{i=1}^{n} p_i x_i = 0\}$ by Claim 3, which in turn implies that $\succ$ admits an expected utility representation by Claim 4. That the compatibility with a utility representation implies transitivity is obvious. For Proposition 6, if $\succ$ is cyclic, then $\{x : x \sim 0\} \neq \{x : \sum_{i=1}^{n} p_i x_i = 0\}$ by Claim 4, and thus $\succ$ is extremely cyclic by Claim 3.
5.5 Proof of Proposition 8

Let $\geq$ be a time preference admitting a relative discounting representation $(U, \eta)$, and denote its transitive closure by $\geq^*$. The first claim characterizes the transitive closure $\geq^*$ in terms of the pair $(U, \eta)$.

Claim 5. For any dated outcomes $(x, t), (y, s) \in X$, $(x, t) \geq^* (y, s)$ if and only if $U(x) \geq \eta(s, r_1)\eta(r_1, r_2)\cdots\eta(r_n, t)U(y)$ for some finite sequence $(r_j)$ in $[0, \infty)$.

Proof. Let $(x, t) \geq^* (y, s)$. Then, there exists a finite sequence $(z_j, r_j)$ in $X$ such that $(x, t) \geq (z_n, r_n) \geq \cdots \geq (z_1, r_1) \geq (y, s)$. By the representation, we obtain

$$U(x) \geq \eta(r_n, t)U(z_n),$$
$$U(z_j) \geq \eta(r_{j-1}, r_j)U(z_{j-1}) \text{ for } j = 2, \ldots, n$$
$$U(z_1) \geq \eta(s, r_1)U(y)$$

from which an inequality $U(x) \geq \eta(s, r_1)\eta(r_1, r_2)\cdots\eta(r_n, t)U(y)$ follows at once.

Conversely, suppose that there exists a finite sequence $(r_j)$ in $[0, \infty)$ that satisfies an inequality $U(x) \geq \eta(s, r_1)\eta(r_1, r_2)\cdots\eta(r_n, t)U(y)$. As $U$ is a homeomorphism and thus a surjection, we may inductively construct a sequence $(z_j)$ such that $U(z_j) = \eta(s, r_1)U(y)$ and $U(z_{j+1}) = \eta(r_j, r_{j+1})U(z_j)$ for all $j < n$. Then, the representation implies that $(z_1, r_1) \sim (y, s)$ and $(z_{j+1}, r_{j+1}) \sim (z_j, r_j)$ for all $j < n$. In addition, substituting these equalities into the hypothesized inequality obtains $U(x) \geq \eta(r_n, t)U(z_n)$. So, $(x, t) \geq (z_n, r_n)$ and $(x, t) \geq^* (y, s)$ as desired. □

Claim 6. If $\geq$ is cyclic, then there exist $t^*, s^* \geq 0$ such that, for all $x, y \in Z$, $(x, t^*) \geq^* (y, s^*)$.

Proof. Suppose that $\geq$ is not transitive. Then, $\not\geq \not\geq^*$. As $\geq \subseteq \geq^*$, it must be the case that $(x^*, t^*) \geq^* (y^*, s^*)$ while not $(x^*, t^*) \geq^* (y^*, s^*)$ for some $(x^*, t^*), (y^*, s^*) \in X$. By the representation and Claim 5, this guarantees the existence of a finite sequence $(r_j)$ that satisfies $\eta(s^*, t^*) > \eta(s^*, r_1)\eta(r_1, r_2)\cdots\eta(r_n, t^*)$. As $\eta(t^*, s^*) = 1/\eta(s^*, t^*)$, we obtain

$$1 > \eta(s^*, r_1)\eta(r_1, r_2)\cdots\eta(r_n, t^*)\eta(t^*, s^*).$$

This shows that a product $\eta(s^*, q_1)\eta(q_1, q_2)\cdots\eta(q_m, t^*)$ can be taken arbitrarily close to 0 by choosing a sequence $(q_j)$ that repeats the sequence (8) many times within. Consequently, for any $x, y \in X$, there exists a finite sequence $(q_j)$ such that $U(x) \geq \eta(s^*, q_1)\eta(q_1, q_2)\cdots\eta(q_m, t^*)U(y)$. By Claim 5, we are done. □

Claim 7. If $\geq$ is cyclic, then there exist $t^*, s^* \geq 0$ such that, for all $x, y \in Z$, $(x, t^*) \sim^* (y, s^*)$. 

24
Proof. Suppose that $\succ$ is cyclic, and find two dates $t^*, s^* \geq 0$ as in Claim 6. Then, clearly, $(x, t^*) \succeq^* (y, s^*)$ for any $x, y \in X$. To obtain the inverse relation, fix any $x, y \in X$. As $U$ is a homeomorphism, we can find $x', y' \in X$ such that $U(y) \geq \eta(t^*, s^*)U(x')$ and $U(y') \geq \eta(t^*, s^*)U(x)$, which in turn implies $(y, s^*) \succeq (x', t^*)$ and $(y', s^*) \succeq (x, t^*)$ by the representation. But $(x', t^*) \succeq^* (y', s^*)$ by the choice of two periods, and hence it follows that $(y, s^*) \succeq^* (x, t^*)$. \hfill $\square$

Now, we are ready to complete the proof of Theorem 8. Assume that $\succ$ is cyclic, and find two period $t^*, s^*$ as in Claim 7. Then, from the property given by Claim 7, we can readily derive $(x, t^*) \sim^* (x', t^*)$ for all $x, x' \in Z$. Moreover, for all $(z, r) \in X$, we can find an $x \in Z$ such that $(z, r) \sim^* (x, t^*)$ as $U$ is a homeomorphism. Applying transitivity of $\succeq^*$, we obtain $\succeq^* = X \times X$.

### 5.6 Proof of Proposition 9

Let $\succ$ be a time preference with a relative discounting representation $(U, \eta)$. Define $\mathcal{D} := \{\eta(\cdot, r) : r \in [0, \infty)\}$. Then, any $\delta \in \mathcal{D}$ is continuous and nonincreasing with $\delta(\infty) = 0$. We shall show that $(U, \mathcal{D})$ represents $\partial \succ$ as in (7). First, take any $(x, t), (y, s) \in X$ such that $(x, t) \partial_\succ (y, s)$. Fix any $\delta \in \mathcal{D}$, and let $r \in [0, \infty)$ be such that $\delta(\cdot) = \eta(\cdot, r)$. As $U$ is a homeomorphism from $Z$ to $\mathbb{R}^{++}$, there exists a $z \in Z$ such that $U(z) = \eta(s, r)U(y)$. Then, by the representation of $\succ$, $(y, s) \succ (z, r)$, which in turn implies that $(x, t) \succ (z, r)$ as $(x, t) \partial_\succ (y, s)$. Applying the representation of $\succ$ again, we obtain $\eta(t, r)U(x) \geq U(z) = \eta(s, r)U(y)$. So, $\delta(t)U(x) \geq \delta(s)U(y)$. As $\delta \in \mathcal{D}$ is arbitrary, we have shown that $(x, t) \partial_\succ (y, s)$ implies $\delta(t)U(x) \geq \delta(s)U(y)$ for all $\delta \in \mathcal{D}$. For the converse, take any $(x, t), (y, s) \in X$ such that $\delta(t)U(x) \geq \delta(s)U(y)$ for all $\delta \in \mathcal{D}$. Choose any $(z, r) \in X$, and suppose that $(y, s) \succ (z, r)$. Then, by the representation of $\succ$, $\eta(s, r)U(y) \geq U(z)$. As $\eta(\cdot, r) \in \mathcal{D}$, we have $\eta(t, r)U(x) \geq \eta(s, r)U(y)$, which implies that $\eta(t, r)U(x) \geq U(z)$. So, by the representation of $\succ$, $(x, t) \succ (z, r)$. As a similar proof shows that $(z, r) \succ (x, t)$ implies $(z, r) \succeq (y, s)$, we obtain $(x, t) \partial_\succ (y, s)$.

### References


