Time and No Lotteries: A Simple Axiomatization of Maxmin Expected Utility*

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Abstract

The paper provides a simple axiomatization of the maxmin expected utility model in a multi-period Savage-style framework of purely subjective uncertainty.

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1 Introduction

The prevalent approach to modeling choice under uncertainty was introduced by Savage [19]. In this approach uncertainty is represented by a space of states of the world and actions take the form of mappings from states into outcomes. This paper provides a simple axiomatization of maxmin expected utility in a multi-period specification of the Savage framework. There are a number of reasons why such an axiomatization is useful. Most importantly, it gives a sharp understanding of the empirical content of the maxmin model. The seminal paper on the subject, Gilboa and Schmeidler [12], assumes that outcomes take the form of lotteries with exogenously specified probabilities. This specification of the domain was introduced by Anscombe and Aumann [3] and will be henceforth referred to as the AA framework. The AA framework is tractable and simplifies the analysis but is rarely used in applications. Moreover, it is now clear that stark differences in predictions may arise if one substitutes the AA for the Savage framework. The fact is evidenced by the behavior of the multiplier model recently introduced in economics by Anderson et al. [2].

Most applications of the maxmin, or multiple-prior, model in finance and macroeconomics involve intertemporal decisions. The key feature of the model, which has been termed ‘preference for hedging’, interacts nontrivially with other important aspects of intertemporal behavior such as stationarity. This interaction is exploited in many applications. We refer the reader to the recent survey by Epstein and Schneider [8]. Characterizing the maxmin model in an intertemporal setting brings this interaction to the fore and delineates it precisely.

Several other papers have recognized the first problem outlined above and characterized the multiple-prior model in a general Savage framework without time or lotteries, e.g., [1, 4, 11]. These characterizations typically involve many axioms or axioms of technical nature that are not directly verifiable. The axiomatization provided here is arguably as simple as the original one by Gilboa and Schmeidler [12]. In this sense, adopting an intertemporal framework turns out to be not only relevant to applied work but also amenable to axiomatic analysis.

Even though the focus of this paper is on the multiple-prior model, the arguments can be easily adapted to characterize other models of ambiguity such as the Choquet model of Schmeidler [20], and the multiplier model mentioned
above and recently axiomatized in an Anscombe-Aumann setting by Strza-lecki [21]. The derivation of subjective expected utility is also technically easier in the present multi-period setting and requires no restrictions on the cardinality of the state space. This point has been made by Gorman [14] in a little known paper. It is the subject of ongoing work to see if the setting can be used to study the more general models of ambiguity aversion developed by Maccheroni et al. [18] and Cerreia-Vioglio et al. [5]. It is nonetheless reasonable to conclude that the multi-period setting provides a viable alternative to the Anscombe-Aumann setup. It is tractable and more relevant to applied work.

2 Domain

Time is discrete and varies over an infinite horizon $T := \{0, 1, 2, \ldots\}$. The information structure is described by a filtered space $(\Omega, \{\mathcal{F}_t\}_t)$ where $\Omega$ is an arbitrary set of states of the world and $\{\mathcal{F}_t\}_t := \mathcal{F}$ is an increasing sequence of algebras such that $\mathcal{F}_0 = \{\Omega, \emptyset\}$. If $\Omega$ is finite, one may think of the information structure as an event tree whose nodes correspond to time-event pairs $(t, \omega) \in T \times \Omega$.

In every period, outcomes lie in a connected, separable, compact space $X$. An act is an $X$-valued, $\mathcal{F}$-adapted processes. Each such process is the multi-period counterpart of a Savage act; it has the form $h = (h_t)$ where $h_t : \Omega \to X$ is $\mathcal{F}_t$-measurable for every $t$.

Let $\mathcal{A}'$ be a subalgebra of $\cup_t \mathcal{F}_t$. An act $h := (h_t)_t$ is $\mathcal{A}'$-adapted if $h_t$ is $\mathcal{A}'$-measurable for every $t$. An act $h$ is simple if $h$ is $\mathcal{A}'$-adapted for some finitely-generated algebra $\mathcal{A}' \subset \cup_t \mathcal{F}_t$. The domain of choice is the set $\mathcal{H}$ of all simple acts. The domain excludes acts whose outcomes may depend on the realization of tail events. Such events do not take place in finite time and are therefore unobservable.

An act is deterministic if its outcomes do not depend on the state of the world. The space of deterministic acts may be identified with the space $X^\infty$. Generic elements of $X^\infty$ are denoted by bold-faced letters, e.g, $x = \ldots$.

\footnote{A tail event is an event in $\sigma(\cup_t \mathcal{F}_t) \setminus \cup_t \mathcal{F}_t$. A preference on the set of simple acts can be extended to the set of all acts using the techniques developed in Epstein and Wang [10].}
(x_0, x_1, ...). Other important subsets of \( \mathcal{H} \) include the subspaces \( X^{t+1} \times \mathcal{H} \) where \( t \in T \). Generic elements of \( X^{t+1} \times \mathcal{H} \) will be denoted as tuples \((x_0, x_1, ..., x_t, h)\).

3 Axioms

The primitive of the model is a preference relation over the space of simple acts \( \mathcal{H} \).

Order The preference relation \( \succeq \) is complete and transitive.

If \( \cup_t \mathcal{F}_t \) is finitely generated, it is standard to require that preference be continuous in the product topology. This is no longer the case if there are infinitely many events. Consider a sequence of events \( A_n \) that increases monotonically to \( \Omega \). Continuity in the product topology would require that the certainty equivalent of a bet on \( A_n \) changes continuously as \( A_n \uparrow \Omega \). In the context of ambiguity, a discontinuity may arise since \( \Omega \) is unambiguous whereas all of the events \( A_n \) may be ambiguous. The next axiom imposes continuity on every subspace of acts that is adapted with respect to a finitely-generated algebra.

Continuity For every finitely generated subalgebra \( A' \) of \( \cup_t \mathcal{F}_t \), the restriction of \( \succeq \) to the subspace of \( A' \)-adapted acts is continuous in the respective product topology.

For every act \( h \) and state \( \omega \), let \( h(\omega) := (h_0(\omega), h_1(\omega), ...) \in X^\infty \) denote the deterministic act corresponding to the sample path of \( h \) in state \( \omega \). The next axiom is a monotonicity requirement. It says that an act \( h \) is preferred to \( h' \) whenever \( h(\omega) \) is preferred to \( h'(\omega) \) for every \( \omega \). The axiom is less innocuous in the the present multi-period setting than it is in the atemporal model of Gilboa and Schmeidler [12]. In particular, it presumes indifference to the timing of uncertainty as made clear by the seminal work of Kreps and Porteus [17].

\[\text{See the related discussion in Epstein and Seo [9, p.347].}\]
Monotonicity For all \( h, h' \in \mathcal{H} \), \( h(\omega) \succeq h'(\omega) \) for every \( \omega \in \Omega \) implies that \( h \succeq h' \).

The next two assumptions are the core of the present axiomatization. The first axiom captures a form of intertemporal hedging. It says that combining some uncertain bet, \( h_t \), paying in period \( t \), with another uncertain bet, \( h_{t+1} \), paying in period \( t+1 \), is at least as good as combining their respective certainty equivalents \( z_t, z'_{t+1} \). The intuition is that the two uncertain bets, \( h_t \) and \( h_{t+1} \) may hedge one another eliminating any ambiguity about the overall outcome. In contrast, combining their certainty equivalents produces no such complementarity.

Uncertainty Aversion For every \( t, h \in \mathcal{H} \), \( z, z' \in X \) and \( x \in X^\infty \),

\[
(x_{-t}, h_t) \sim (x_{-t}, z) \text{ and } (x_{-(t+1)}, h_{t+1}) \sim (x_{-(t+1)}, z')
\]

\[
\Rightarrow (x_{-t,t+1}, h_t, h_{t+1}) \succeq (x_{-t,t+1}, z, z')
\]

Intertemporal hedging generates a complementarity between bets in different time periods and may thus lead to nonstationary behavior. This interplay between hedging and stationarity is delineated by the next axiom. First, the axiom requires that \( h_t \) and \( h_{t+1} \) do not hedge one another if one of them is constant. In all these instances, the axiom requires that behavior be stationary.

Certainty Independence For every \( t, h, h' \in \mathcal{H} \) and \( x_0, ..., x_t \in X \):

\[
h \succeq h' \iff (x_0, ..., x_t, h) \succeq (x_0, ..., x_t, h')
\]

The next axiom is arguably restrictive but common in modeling intertemporal behavior. Since it deals with deterministic act only and the primary focus is on ambiguity, the axiom is a natural starting point. The axiom rules out complementarities among the utility levels of outcomes in different time periods. Remarkably, Section 4.2 shows that the axiom is redundant except in certain nongeneric cases.

Time Separability For every \( x, x', y, y' \in X \) and \( z, \tilde{z} \in X^\infty \),

\[
(x, y, z) \succeq (x', y', z) \iff (x, y, \tilde{z}) \succeq (x', y', \tilde{z}).
\]
The final axiom is a weak nontriviality condition.

**Nontriviality** There exist \( x, y \in X \) and \( z \in X^\infty \) such that \((x, z) > (y, z)\).

## 4 Result and Discussion

Endow the space of finitely additive probability measures on \( \cup_t \mathcal{F}_t \) with the weak\(^*\) topology, i.e., the topology generated by all real-valued bounded measurable functions on \( \Omega \).

**Theorem 1** The preference \( \succeq \) satisfies Order, Continuity, Monotonicity, Certainty Independence, Uncertainty Aversion, Time Separability and Nontriviality if and only if there exist a discount factor \( \beta \in (0, 1) \), a nonconstant continuous function \( u : X \to \mathbb{R} \) and a nonempty weak\(^*\)-closed convex set \( P \) of finitely additive probability measures on \( \cup_t \mathcal{F}_t \) such that:

\[
h \succeq h' \iff \min_{p \in P} \int_{\Omega} \left[ \sum_t \beta^t u(h_t) \right] dp \geq \min_{p \in P} \int_{\Omega} \left[ \sum_t \beta^t u(h'_t) \right] dp. \tag{4.1}
\]

Moreover, \( P \) and \( \beta \) are unique and \( u \) is unique up to positive affine transformations.

The simplicity of the axiomatization given in Theorem 1 depends critically on the time-additivity of the ranking of deterministic acts. The framework, however, leaves the door open for more general yet well-structured preferences on \( X^\infty \). For example, the stationary rankings studied in Koopmans [15] have a number of separability properties that are jointly weaker but akin to time additivity. It is an interesting question if those properties can be exploited to generalize Theorem 1. Of course, a similar trade-off between generality and simplicity arises in the context of the Gilboa and Schmeidler [12] axiomatization in which the linearity of the ranking of objective lotteries plays an analogous role.

Despite the obvious similarities, the proof of Theorem 1 is not a straightforward application of the arguments in [12]. A key property of the multiple-prior functional is that it is positively homogenous.\(^3\) In an AA framework,
outcomes are objective lotteries and utility is linear in their probabilities. By varying the latter, one obtains homogeneity with respect to any given positive constant. In the present setting, the utility of deterministic acts is additive across time periods and one may think of the discount factor $\beta^t$ as the ‘probability’ of the period-$t$ outcome. The problem is that $\beta$ is fixed and therefore homogeneity obtains in a limited sense only, that is, with respect to powers of the fixed constant $\beta$. In the appendix, this property is called $\beta$-homogeneity. The gist of the proof is to show that a $\beta$-homogenous, monotone, translation invariant and superadditive functional is in fact positively homogenous.

4.1 Dynamic Extension

Even though the paper develops the static model only, the analysis can be extended to a dynamic model of choice. In particular, it is possible to obtain the prior-by-prior Bayes rule and the recursive utility formulation of Epstein and Schneider [7] without assuming an AA framework. Because the necessary changes to their proof are minimal and can be easily deduced from the proof of Theorem 1, it is enough to mention the primitives and additional axioms. As in [7], restrict the information structure so that every $\mathcal{F}_t$ is finitely generated. To model dynamic choice, take as primitive a process of conditional preferences $\{\succeq_{t,\omega}\}$ and assume that each $\succeq_{t,\omega}$ satisfies appropriate versions of the above axioms. The main additional axiom is Dynamic Consistency which links together preferences at different nodes. The axiom is standard and may be formulated in a manner identical to [7]. The same applies to the other two axioms needed to derive the recursive formulation in Epstein and Schneider [7], namely, Consequentialism and Conditional State Independence.\footnote{In [7], Consequentialism is called Conditional Preference whereas Conditional State Independence is subsumed in an axiom called Risk Preference. The full force of the latter is not needed here since it overlaps with Certainty Independence.}

4.2 Essential Events

Say that an event $A \in \bigcup_t \mathcal{F}_t$ is strongly essential if, for all $\{A, A^c\}$-adapted acts $h, h'$, $h$ is strictly preferred to $h'$ whenever $h(\omega) \succ h'(\omega)$ for all $\omega \in \Omega$.\footnote{In [7], Consequentialism is called Conditional Preference whereas Conditional State Independence is subsumed in an axiom called Risk Preference. The full force of the latter is not needed here since it overlaps with Certainty Independence.}
and the latter ranking is strict for some $\omega'$. That strongly essential events exist generically becomes evident if we consider the representation in (4.1). In that case, an event $A$ is strongly essential if and only if $p(A) \in (0, 1)$ for all $p \in P$. In dynamic settings and many applications, it is common to assume that all events $A \notin \{\Omega, \emptyset\}$ are strongly essential. The assumption is made to avoid the problem of updating on null events. Remarkably, the next lemma shows that Time Additivity is redundant whenever a strongly essential event exists.

**Lemma 2** If there exists a strongly essential event, Monotonicity, Continuity, Certainty Independence and Nontriviality imply Time Additivity.

It is important to note that strongly essential events play an indispensable role in characterizations of the multiple-prior model which do not impose a specific structure on the outcome space. See, for instance, [1, 4, 11]. In these studies, the existence of a strongly essential event is used to ‘calibrate’ a cardinal utility index over outcomes. In turn, cardinal utility is necessary to derive a unique set of priors. Theorem 1 does not require the existence of a strongly essential event. As in Gilboa and Schmeidler [12], a cardinal utility index is derived from the richness of the outcome space, in conjunction with an appropriate form of additivity.

## 5 Proofs

### 5.1 Theorem 1

Necessity of the axioms is straight-forward. Focus on sufficiency.

The first lemma is due to Koopmans [16]. It delivers an additively separable utility function for the set of deterministic acts $X^\infty$.

**Lemma 3** There exists a continuous function $u : X \to \mathbb{R}$ and a discount factor $\beta \in (0, 1)$, such that the restriction of $\succeq$ to $X^\infty$ is represented by the utility function:

$$U(x) := (1 - \beta) \sum \beta^t u(x_t)$$

(5.1)

Moreover, $\beta$ is unique and $u$ is unique up to positive affine transformations.
Proof. There are some minor differences in the domain and axioms used by Koopmans \cite{16} to derive (5.1). First, he assumes that $X$ is metrizable and that $\succeq$ is continuous in the uniform topology on $X^\infty$. Second, he assumes a monotonicity axiom (P5) which is not made here. It is obvious that continuity in the product topology, assumed here, implies uniform continuity. Moreover, because $X^\infty$ is compact in the product topology, there exist best and worst elements in $X^\infty$. As Koopmans \cite[p.84]{16} notes, the existence of best and worst elements implies his monotonicity postulate P5. Finally, Koopmans' arguments are based on the results of \cite{6, 13} for which metrizability is not needed.

Because $X$ is compact and connected, the range of $u$ is a compact interval. Rescaling appropriately, it is without loss of generality to set $u(X) = [-1, 1]$. Let $x^* \in X$ be an outcome such that $u(x^*) = 0$.

The product space $X^\infty$ inherits the properties of $X$. In particular, $X^\infty$ is compact, separable and connected.

Let $\mathbf{\overline{x}}$ and $\mathbf{x}$ be the best and worst elements in $X^\infty$, respectively. Fix an arbitrary $h \in \mathcal{H}$. It follows from Monotonicity that $\mathbf{\overline{x}} \succeq h \succeq \mathbf{x}$. Because $X^\infty$ is connected, a standard argument shows that there exists some $x_h \in X^\infty$ such that $x_h \sim h$. This allows us to extend the utility function $U$ from $X^\infty$ to the space of all acts $\mathcal{H}$. For every $h \in \mathcal{H}$, define $\bar{U}(h) := U(x_h)$. The transitivity of $\succeq$ implies that $\bar{U} : \mathcal{H} \to \mathbb{R}$ is well-defined and represents the preference relation $\succeq$.

For every act $h \in \mathcal{H}$, define the utility act $U \circ h : \Omega \to \mathbb{R}$:

$$U \circ h : \omega \mapsto (1 - \beta) \sum_t \beta^t u(h_t(\omega))$$

(5.2)

Let $\mathcal{U} = \{U \circ h : h \in \mathcal{H}\}$ be the set of all utility acts. Define the functional $I : \mathcal{U} \to \mathbb{R}$:

$$I(U \circ h) = \bar{U}(h).$$

Notice that, by Monotonicity, $I$ is well-defined.

For any $t$, let $B_t^\circ$ be the set of simple $\mathcal{F}_t$-measurable functions from $\Omega$ into the closed interval $[-\beta^t, \beta^t]$.

**Lemma 4** For every $t$, $B_t^\circ \subset \mathcal{U}$. 

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Proof. Let $a = \sum_{i=1}^{k} \alpha_{i}1_{A_{i}}$ be the canonical representation of $a \in B_{t}^{o}$ and note that, for every $i = 1, \ldots, k$, $|\alpha_{i}| \leq 1$. Let $x_{i}$ be such that $u(x_{i}) = \frac{\alpha_{i}}{\beta_{t}}$. Define $f(\omega) = x_{i}$ for every $\omega \in A_{i}$, $i \in \{1, \ldots, k\}$. Define an act $h$ by setting $h_{\tau} := x^{*}$ for all $\tau < t$ and $h_{\tau} = f$ for all $\tau \geq t$. It is easy to check that $U(h) = a$. ■

Before stating the next lemma, notice that $a \in B_{t}^{o}$ implies that $\beta_{k}a \in B_{t}^{o}$ for every $k,t$.

**Lemma 5** For every $t,k \geq 0$, $a \in B_{t}^{o}$, $I(\beta_{k}a) = \beta_{k}I(a)$.

**Proof.** Fix some $t$ and $a \in B_{t}^{o}$. It suffices to show that the result holds for $k = 1$. The rest of the proof follows by induction. By the previous lemma, there exists an act $h \in \mathcal{H}$ such that $U(h) = a$. Let $x_{h}$ be a deterministic act such that $x_{h} \sim h$. Consider the act $(x^{*}, h)$ and note that $U(x^{*}, h) = \beta a$. By Certainty Independence, $(x^{*}, h) \sim (x^{*}, x_{h})$. Conclude that

$$I(\beta a) = I(U \circ (x^{*}, h)) = \hat{U}(x^{*}, h) = U(x^{*}, x_{h}) = \beta U(x_{h}) = \beta \hat{U}(h) = \beta I(U \circ h) = \beta I(a)$$

■

Given any real number $\alpha$, let $\alpha^{*} : \Omega \to \mathbb{R}$ be the constant function such that $\alpha^{*}(\omega) = \alpha$ for every $\omega$.

**Lemma 6** For every $t$, $a,a^{*},a + \alpha^{*} \in B_{t}^{o}$, $I(a + \alpha^{*}) = I(a) + \alpha$.

**Proof.** By Lemma 5, it is without loss of generality to assume that $\alpha \in [\beta^{t} - 1, 1 - \beta^{t}]$. Let $(x_{0}, \ldots, x_{t-1})$ be such that

$$(1 - \beta) \sum_{\tau=0}^{t-1} u(x_{\tau}) = \alpha$$

(5.3)

and let $h = (x^{*}, \ldots, x^{*}, h_{t}, h_{t+1}, \ldots)$ be such that $a = U(h)$. If we define $h' = (x_{0}, \ldots, x_{t-1}, h_{t}, h_{t+1}, \ldots)$, it follows that $U(h') = a + \alpha$. Let $x_{h}$ be such that

$$h = (x^{*}, \ldots, x^{*}, h_{t}, h_{t+1}, \ldots) \sim (x^{*}, \ldots, x^{*}, x_{h})$$

(5.4)
By Certainty Independence,
\[ h' = (x_0, ..., x_{t-1}, h_t, h_{t+1}) \sim (x_0, ..., x_{t-1}, x_h) \]  \hspace{1cm} (5.5)

Conclude that
\[ I(a + \alpha^*) = U(x_0, ..., x_{t-1}, x_h) = \alpha + U(x^*, ..., x^*, x_h) = \alpha + \bar{U}(h) = \alpha + I(a). \]

**Lemma 7** For every \( t \) and \( a, b, a + b \in B_t^\circ \), \( I(a + b) \geq I(a) + I(b) \).

**Proof.** By Lemma 5, it is without loss of generality to assume that \( a, b \) take values in the interval \([-\beta^t + \frac{1}{1-\beta}, \beta^t + \frac{1}{1-\beta}] \). If that is the case, there exist \( h_t, h_{t+1} \) such that \( U(x^*_{-t}, h_t) = a \) and \( U(x^*_{-(t+1)}, h_{t+1}) = b \). Let \( z_a, z_b \in X \) be such that
\begin{align*}
(x^*_{-t}, h_t) &\sim (x^*_{-t}, z_a) \quad \text{and} \quad (x^*_{-(t+1)}, h_{t+1}) \sim (x^*_{-(t+1)}, z_b)
\end{align*}

Then,
\[ I(a + b) = U(x^*_{-t,(t+1)}, h_t, h_{t+1}) \geq U(x^*_{-t,(t+1)}, z_a, z_b) \]
\[ = U(x^*_{-t}, z_a) + U(x^*_{-(t+1)}, z_b) = I(a) + I(b), \]
where the inequality follows from Uncertainty Aversion.

Let \( B^\circ \) be the set of all simple, real-valued, \( \cup_t \mathcal{F}_t \)-measurable functions on \( \Omega \).

**Lemma 8** \( \mathcal{U} \) is an absorbing subset of \( B^\circ \).

**Proof.** Fix some \( a \in B^\circ \). Because \( a \) is simple, it must be \( \mathcal{F}_t \)-measurable for some \( t \). Moreover, for \( k \) large enough, \( \beta^k a(\omega) \in [-\beta^t, \beta^t] \) for every \( \omega \in \Omega \). Conclude that \( \beta^k a \in \mathcal{U} \). \( \blacksquare \)

Fix any \( a \in B^\circ \) and let \( k, t \) be such that \( \beta^k a \in B^\circ_t \). Define,
\[ \tilde{I}(a) := \frac{1}{\beta^k} I(\beta^k a) \]
\hspace{1cm} (5.6)

To see that \( \tilde{I} \) is well-defined, let \((k', t')\) be another pair such that \( \beta^{k'} a \in B^\circ_{t'} \). If \( k' \geq k \), then \( \beta^{k'} a \in B^\circ_t \). If \( k > k' \), then \( \beta^k a \in B^\circ_{t'} \). It is therefore without
loss of generality to assume that $t = t', k' > k$ and $\beta^{k'}a, \beta^ka \in B^\circ_t$. Lemma 5 then implies that:

$$I(\beta^{k'}a) = I(\beta^{k'-k}\beta^ka) = \beta^{k'-k}I(\beta^ka),$$

(5.7)

and hence that $\bar{I}$ is well-defined.

The next lemma shows that the extension $\bar{I}$ defined in (5.6) inherits the properties of $I$ established in Lemmas 5-7. Its proof is straightforward and therefore omitted.

**Lemma 9** The functional $\bar{I} : B^\circ \to \mathbb{R}$ is $\beta$-homogenous, translation-invariant, monotone and superadditive.

The next lemma is the main step of the argument.

**Lemma 10** The functional $\bar{I}$ is positively homogenous.

**Proof.** By translation-invariance, it suffices to show that $\bar{I}(aa) = 0$ for all $\alpha > 0$ and all $a$ such that $\bar{I}(a) = 0$. Suppose by way of contradiction that $\lambda := \bar{I}(aa) \neq 0$. By translation-invariance, again, it is without loss of generality to assume that $\lambda < 0$. Else, set $b := \alpha a - \lambda^*$ and note that $\bar{I}(b) = 0$ whereas $\bar{I}(\frac{1}{\alpha}b) = -\frac{\lambda}{\alpha} < 0$. Moreover, by $\beta$-homogeneity, it is also without loss of generality to assume that $\alpha < 1$.

So let $\alpha \in (0, 1)$ and let $a$ be such that $\bar{I}(a) = 0$ and $\lambda := \bar{I}(aa) < 0$. Define $b := \alpha a - \lambda^*$ and note that $\bar{I}(b) = 0$. Let $A$ be the positive ray through $a$:

$$A = \{\gamma a : \gamma \geq 0\}$$

(5.8)

Let $V$ be the span of $a, b$ and let $B = \{b' \in V : b \gg b'\}$. Because $a$, and hence $b$, is simple, the set $B$ is open in $V$. Moreover, for every $b' \in B$, there exists a $\gamma > 0$, sufficiently small, such that $b \gg b' + \gamma^* \gg b'$. By translation-invariance and monotonicity, conclude that:

$$0 = \bar{I}(b) > \bar{I}(b')$$

(5.9)

for all $b' \in B$. By construction, $aa$ belongs to $A \cap B$. Since $A$ is a ray and $B$ is convex and open, $A \cap B$ is an open line segment. Let $(\gamma_1, \gamma_2)$ be the open interval such that $\gamma a \in A \cap B$ for every $\gamma \in (\gamma_1, \gamma_2)$. If $\gamma_1 = 0$, then
\[
\beta^k a \in A \cap B, \text{ for } k \text{ sufficiently large. By } \beta\text{-homogeneity, } \bar{I}(\beta^k a) = \beta^k \bar{I}(a) = 0 = \bar{I}(b), \text{ for every } k, \text{ which contradicts (5.9)}. \text{ So suppose that } \gamma_1 > 0 \text{ and fix any } k \text{ such that } \gamma_1 > \beta^k > 0. \text{ By superadditivity and } \beta\text{-homogeneity in turn,}
\]
\[
\bar{I}(\beta^k a + \beta^t b) \geq \bar{I}(\beta^k a) + \bar{I}(\beta^t b) = 0, \quad (5.10)
\]
for every \( t \). Applying superadditivity inductively, conclude that:
\[
\bar{I}(\beta^k a + n\beta^t b) \geq 0 \quad (5.11)
\]
for every \( n, t \in \mathbb{N} \). Let
\[
A' := \{ \beta^k a + \gamma b : \gamma \geq 0 \} \quad (5.12)
\]
be the ray through \( \beta^k a \) parallel to \( b \). By construction \( \alpha > \gamma_1 > \beta^k \) which implies that \( \frac{\beta^k}{\alpha} \in (0, 1) \). Recall that \( b = \alpha a - \lambda^* \gg \alpha a \), since \( \lambda < 0 \). Conclude that
\[
b \gg \frac{\beta^k}{\alpha} (\alpha a) + (1 - \frac{\beta^k}{\alpha}) b \quad (5.13)
\]
and so \( A' \cap B \) is a nonempty open subset of \( A' \). It is not difficult to see that
\[
\{ \beta^k a + n\beta^t b : n \in \mathbb{N}, t \in \mathbb{N} \} \quad (5.14)
\]
is dense in the \( A' \). Conclude that \( b \gg \beta^k a + n\beta^t b \) for some \( n, t \). But then
\[
0 = \bar{I}(b) > \bar{I}(\beta^k a + n\beta^t b) \quad (5.15)
\]
contradicting (5.11). \( \blacksquare \)

### 5.2 Proof of Lemma 2

The result is an application of Gorman’s overlapping theorem \[13\]. Partition the set of time periods into \( T' := \{0, 1\} \) and \( T'' := \{2, 3, \ldots\} \). Let \( A \) be a strongly essential event and \( \mathcal{H}(A) \) be the subset of \( \{A, A^c\}\)-adapted acts. To recast our framework into Gorman’s setup, it is convenient to think of an act \( h \) as a mapping from \( \Omega \times T \) into \( X \).
Consider the following partition of \( \Omega \times T \):
\[
\{ A \times T', A \times T'', A^c \times T', A^c \times T'' \} \tag{5.16}
\]
which we enumerate as \( \{ B_1, B_2, B_3, B_4 \} \). Given this partition, we can identify \( \mathcal{H}(A) \) with the product space:
\[
X^2 \times X^2 \times X^\infty \times X^\infty, \tag{5.17}
\]
where \( X^2 \) is the restriction of \( \mathcal{H}(A) \) to \( A \times T' \), etc. Accordingly, every act \( h \in \mathcal{H}(A) \) can be written as a vector \( h = (h_{B_1}, h_{B_2}, h_{B_3}, h_{B_4}) \). Notice that every element in the factorization (5.17) is a connected, separable space, as needed to apply Gorman’s overlapping theorem. Because the event \( A \) is strongly essential, it follows that
\[
(h_{B_1}, h_{B_2}, h_{B_3}, h_{B_4}) \succeq (\hat{h}_{B_1}, \hat{h}_{B_2}, h_{B_3}, h_{B_4}) \Leftrightarrow (5.18)
\]
\[
(h_{B_1}, h_{B_2}, h'_{B_3}, h'_{B_4}) \succeq (\hat{h}_{B_1}, \hat{h}_{B_2}, h'_{B_3}, h'_{B_4}) \tag{5.19}
\]
for all \( h, h', \hat{h} \in \mathcal{H}(A) \). In the language of Gorman, the set \( Y := B_1 \cup B_2 \subset \Omega \times T \) is separable. It is not difficult to see that, by Stationarity, the set \( Y' := B_2 \cup B_4 \) is separable. If one more condition is satisfied, Gorman \cite{13} proved that the set difference \( Y \setminus Y' = B_1 \) must be separable as well, which is equivalent to Time Additivity.

The final condition is that for every \( h \in \mathcal{H}(A) \), there exist \( h', h'' \in \mathcal{H}(A) \) such that
\[
(h_{B_1}, h_{B_2}, h_{B_3}, h'_{B_4}) \succ (h_{B_1}, h_{B_2}, h_{B_3}, h''_{B_4}) \tag{5.20}
\]
But this is easily seen to follow from Nontriviality, Stationarity and the fact that \( A \) is strongly essential.

References


