Fear of loss, inframodularity, and transfers

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Received 3 September 2010; final version received 12 January 2011; accepted 1 February 2011

Abstract

There exist several characterizations of concavity for univariate functions. One of them states that a function is concave if and only if it has nonincreasing differences. This definition provides a natural generalization of concavity for multivariate functions called inframodularity. Inframodular transfers are defined and it is shown that a finite lottery is preferred to another by all expected utility maximizers with an inframodular utility if and only if the first lottery can be obtained from the second via a sequence of inframodular transfers. This result is a natural multivariate generalization of Rothschild and Stiglitz’s construction based on mean preserving spreads.

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JEL classification: D81

Keywords: Mean preserving spread; Integral stochastic orders; Risk aversion; Ultramodularity; Dual cones

1. Introduction

Aversion to risk represents preference for a sure amount of money \( w \) versus a random amount having expectation \( w \). In other words, a risk averse decision maker does not like to add to her sure wealth \( w \) a random variable \( \varepsilon \) having zero mean. What is the reason for disliking randomness? Obviously, it has to do with the possibility of ending up with less than the initial wealth \( w \) once

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1 The work of this author was partially supported by MIUR-COFIN.

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\( \varepsilon \) is realized. In fact, any nondegenerate random variable with zero mean can assume negative values with positive probability, and therefore can give rise to a loss. A risk averter fears losses, and the possibility of getting a positive gain does not compensate for these possible losses. In a univariate von Neumann–Morgenstern expected utility context, fear of loss coincides with risk aversion, which coincides with concavity of the agent’s utility function.

A random variable \( Y \) is riskier than a random variable \( X \) if all risk averters prefer \( X \) to \( Y \). Rothschild and Stiglitz [12] study comparison of risks using some balayage results previously unknown in the economics literature. They focus on the idea of a mean preserving spread, i.e., a transfer of probability mass from inside a finite interval to outside the interval that does not alter the mean of a distribution. They show that a mean preserving spread increases risk and, more importantly, that mean preserving spreads are the building blocks of distribution comparisons, because \( X \) is preferred to \( Y \) by all risk averters if and only if the distribution of \( Y \) can be obtained from the distribution of \( X \) via a sequence of mean preserving spreads. Rothschild and Stiglitz’s paper has had a tremendous impact on the literature, reducing many comparative static results to analyzing the effect of a single mean preserving spread.

Often decisions involve several commodities that are not necessarily valued using prices. For instance, when comparing two job offers, a person takes into account the salary, the type of job, the working environment, the commuting time from home, etc. Most of these quantities involve randomness of some sort, so that a truly multivariate evaluation is necessary.

It is also often assumed in the multivariate case that a decision maker who prefers any sure multivariate wealth \( w \) to a random vector \( w + \varepsilon \), where \( \varepsilon \) has mean vector \( 0 \), is risk averse, so her utility function is concave. But does this embody the same rationale that exists in the univariate case? Since the natural order on \( \mathbb{R}^d \) is only partial, a random vector \( \varepsilon \) can have zero expectation even if it never assumes values that are smaller than zero in all coordinates. For instance, in \( \mathbb{R}^2 \) a random vector that assumes with equal probability the values \((1, -1)\) and \((-1, 1)\) has zero mean. Is there any good reason to fear this random variable? This paper argues that the spirit of the univariate case may be better kept by limiting aversion to zero-mean random vectors that involve possible losses and gains in all variables, for instance, a vector that assumes with equal probability the values \((-1, -1)\) and \((1, 1)\).

To characterize this risk posture, we consider two multivariate lotteries having the same mean. Four multidimensional baskets are involved: \( x, w, z, y \) with \( x \leq w \leq y \) and \( x \leq z \leq y \). The first lottery gives with equal probability either \( x \) or \( y \), the second gives with equal probability either \( w \) or \( z \), and the two lotteries have the same mean, i.e., \( x + y = z + w \). Preference of the second lottery over the first represents a fear of loss because the decision maker shuns the worst possible outcome \( x \) and the fear of \( x \) is not compensated by the possibility of obtaining the best outcome \( y \). A von Neumann–Morgenstern decision maker expresses this preference over all such pairs of lotteries if and only if her utility function is inframodular, i.e., it has nonincreasing finite differences. Since the property of having nonincreasing finite differences characterizes univariate concave functions, we see that inframodularity is a bona fide extension of concavity to a multivariate setting. Moreover, unlike standard multivariate concavity, it represents fear of loss and not just aversion to randomness.

As mentioned in [10], a function \( u(x_1, \ldots, x_n) = \psi(\sum_{i=1}^{n} \alpha_i x_i) \), with \( \psi \) concave and \( \alpha_1, \ldots, \alpha_n \geq 0 \), is inframodular. Therefore, inframodular utility functions are the natural tool to compare portfolios of commodities for any given nonnegative vector of prices. Multivariate concave utility functions would not do the job because they compare portfolios with any vector of prices, including negative ones.
Infra-modular functions can be used also to model the preferences of a risk averse decision maker facing lotteries that involve substitutable, but incompatible, commodities. Think of the owner of a tanker that can transport only one type of liquid. The more liquid is transported the better (up to capacity), but it has to be of only one type. Such an extreme case of preferences corresponds to an inframodular utility function \( u(x_1, \ldots, x_n) = \psi(\max\{\alpha_1 x_1, \ldots, \alpha_n x_n\}) \), with \( \psi \) concave and \( \alpha_1, \ldots, \alpha_n \geq 0 \).

On the other hand, an inframodular function cannot be used to model preferences for lotteries that involve complementary commodities, such as left gloves and right gloves. A utility function of the type \( u(x_1, \ldots, x_n) = \psi(\min\{\alpha_1 x_1, \ldots, \alpha_n x_n\}) \), \( \alpha_1, \ldots, \alpha_n \geq 0 \), cannot be inframodular, no matter what \( \psi \) is. Notice that in this case, if \( \psi \) is concave, then \( u \) is concave as well.

In order to compare multivariate risks in term of this inframodular order, we consider a class of transfers that mimic the mean preserving spread, i.e., transfers that move mass from inside a set to outside a set, but is not allowed by all decision makers with an inframodular utility function. The central result of this paper is the converse of this statement. If a random vector \( X \) is preferred to another random vector \( Y \) and all decision makers with an inframodular utility function like these transfers, then the distribution of \( X \) is better (up to capacity), but it has to be of only one type. Such an extreme case of preferences can be obtained from the distribution of \( X \) via a sequence of such transfers.

This paper is organized as follows. Section 2 describes different types of transfers. Section 3 states the main results. Section 4 contains the proofs.

2. Transfers

This section introduces a general definition of transfer. To do this, the definitions of some useful classes of functions are needed. The following notation is used:

\[
x \vee y := (\max\{x_1, y_1\}, \ldots, \max\{x_d, y_d\}), \quad x \wedge y := (\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\}).
\]

**Definition 2.1.**

(a) Let \( A \subset \mathbb{R}^d \) be convex. A function \( f : A \to \mathbb{R} \) is *concave* if for all \( x, y \in A \) and all \( \alpha \in [0, 1] \),

\[
f(\alpha x + (1 - \alpha) y) \leq \alpha f(x) + (1 - \alpha) f(y).
\]

A function is *concave* if the reverse inequality holds.

(b) Let \( A \subset \mathbb{R}^d \) be convex. A function \( f : A \to \mathbb{R} \) is *component-wise convex* if (2.1) holds for all \( x, y \in A \) such that \( x_i = y_i \) for \( j \neq i \), for some \( i \in \{1, \ldots, d\} \). A function is *component-wise concave* if the reverse inequality holds.

(c) Let \( B \subset \mathbb{R}^d \) be a lattice. A function \( f : B \to \mathbb{R} \) is *supermodular* if for all \( x, y \in B \), the inequality \( f(x) + f(y) \leq f(x \vee y) + f(x \wedge y) \) holds. A function is *submodular* if the reverse inequality holds.

(d) Let \( C \subset \mathbb{R}^d \) be a convex lattice. A function \( f : C \to \mathbb{R} \) is *ultramodular* if for all \( x, y, w, z \in C \) such that \( x + y = z + w, x \leq w \leq y \) and \( x \leq z \leq y \) the inequality \( f(z) + f(w) \leq f(x) + f(y) \) holds. A function is *inframodular* if the reverse inequality holds.

A twice differentiable function is submodular if all its mixed second derivatives are nonpositive; it is inframodular if all its second derivatives (mixed and pure) are nonpositive. Topkis [16] is the classical reference for properties and applications of supermodular functions. The term...
“ultramodular” has been coined by Marinacci and Montrucchio [8], who provide a thorough analysis of these functions, previously known under different, sometimes misleading, names, such as “directionally convex.”

Let $S \subset \mathbb{R}^d$ be compact and let $\mathcal{S}$ be the Borel-$\sigma$-algebra on $S$. For a signed measure $\mu$ on $(S, \mathcal{S})$, its positive and negative parts are denoted $\mu^+$ and $\mu^-$, respectively, and $\|\mu\| := \mu^+(S) + \mu^-(S)$ is the total variation norm. Denote by $\mathcal{M}$ the set of all signed measures on $S$ with finite total variation norm $\|\mu\| < \infty$ and with the property that $\mu^+(S) = \mu^-(S)$. Notice that for any two probability measures $P$, $Q$, the difference $Q - P \in \mathcal{M}$, and that in fact $\mathcal{M}$ is the linear space spanned by the differences of probability measures.

A degenerate probability measure on $x$ is denoted $\delta_x$. Given two probability measures $P$, $Q$ supported on a finite subset of $\mathbb{R}^d$, call the signed measure $Q - P$ a transfer from $P$ to $Q$. If

$$(Q - P)^- = \sum_{i=1}^{n} \alpha_i \delta_{x_i} \quad \text{and} \quad (Q - P)^+ = \sum_{i=1}^{m} \beta_i \delta_{y_i},$$

then the transfer $Q - P$ removes probability mass $\alpha_i$ from point $x_i$, $i = 1, \ldots, n$ and adds probability mass $\beta_i$ to $y_i$, $i = 1, \ldots, m$. To indicate this transfer we write

$$\sum_{i=1}^{n} \alpha_i \delta_{x_i} \rightarrow \sum_{i=1}^{m} \beta_i \delta_{y_i}.$$

**Definition 2.2.** Consider a set $M \subset \mathcal{M}$ of transfers and the class $\mathcal{F} \subset \mathcal{C}$ of continuous functions $f$ such that

$$\sum_{i=1}^{n} \beta_i f(y_i) \geq \sum_{i=1}^{m} \alpha_i f(x_i)$$

whenever $\mu \in M$, where $\mu := \sum_{i=1}^{n} \beta_i \delta_{y_i} - \sum_{i=1}^{m} \alpha_i \delta_{x_i}$. The class $\mathcal{F}$ is said to be induced by $M$.

This definition has the following economic interpretation. Any decision maker using expected utility theory with a utility function $u \in \mathcal{F}$ will prefer $Q$ to $P$ if $Q - P \in M$, i.e., if $Q$ is obtained from $P$ by a transfer in $M$.

Next comes the definition of simple transfers that induce the classes of functions of Definition 2.1. In the following definition, the terminology of Elton and Hill [5] is adopted.

**Definition 2.3.** For a measure $P = \sum_{i=1}^{m} \alpha_i \delta_{x_i}$ with finite support, the barycenter of $P$ is given by

$$\text{bar}(P) := \frac{1}{\sum_{i=1}^{m} \alpha_i} \sum_{i=1}^{m} \alpha_i x_i.$$ 

2.1. Simple transfers

A simple transfer $\mu$ has the form $\mu = \beta_1 \delta_{y_1} + \beta_2 \delta_{y_2} - \alpha_1 \delta_{x_1} - \alpha_2 \delta_{x_2}$, where it is possible that $y_1 = y_2$ or $x_1 = x_2$. Therefore a simple transfer involves the move of some probability mass from at most two points to at most two other points. In the sequel, only simple transfers that preserve the barycenter are considered.
Simple convex transfer: Given \( x, y, w, z \in \mathbb{R}^d \) and \( \alpha, \beta, \gamma, \varepsilon, \eta \in [0, 1] \) such that \( z = \alpha x + (1-\alpha) y, w = \beta y + (1-\beta) x \) and \( \gamma x + (1-\gamma) y = \varepsilon z + (1-\varepsilon) w \), a simple transfer \( \eta(\varepsilon \delta z + (1-\varepsilon) \delta y) \rightarrow \eta(\gamma \delta x + (1-\gamma) \delta y) \) is called convex. The reverse transfer is called concave. When \( \alpha = \beta \), hence \( \gamma = \varepsilon = 1/2 \), the transfer is called symmetric. Notice that, if \( \alpha = 1 - \beta \), then \( w = z \).

Simple component-wise convex transfer: A simple convex transfer is called component-wise convex if \( x_j = y_j \) for all \( j \neq i \) for some fixed \( i \).

Simple supermodular transfer: Given \( x, y, w, z \in \mathbb{R}^d \) with the property that \( x = z \wedge w \) and \( y = z \vee w \), a simple transfer \( \eta(\varepsilon \delta z + (1-\varepsilon) \delta y) \rightarrow \eta(\frac{1}{2} \delta x + \frac{1}{2} \delta y) \) is called supermodular. The reverse transfer is called submodular.

Simple ultramodular transfer: Given \( x, y, w, z \in \mathbb{R}^d \) and \( \gamma, \varepsilon, \eta \in [0, 1] \) such that \( x \leq w, z \leq y \), and \( \gamma x + (1-\gamma) y = \varepsilon z + (1-\varepsilon) w \), a simple transfer \( \eta(\varepsilon \delta z + (1-\varepsilon) \delta y) \rightarrow \eta(\gamma \delta x + (1-\gamma) \delta y) \) is called ultramodular. The reverse transfer is called inframodular. When \( \gamma = \varepsilon = 1/2 \), the transfer is called symmetric.

Component-wise convex and supermodular transfers are particular cases of ultramodular transfers. The following proposition shows a stronger property.

**Proposition 2.4.** Any simple ultramodular transfer can be obtained by combining simple supermodular and component-wise convex transfers.

It is immediate to see that the classes of convex, concave, component-wise convex, component-wise concave, supermodular, submodular, ultramodular, and inframodular functions are induced by the set of simple symmetric transfers with the same name.

General (non-simple) transfers are obtained by iterating simple transfers. In dimension \( d = 1 \), a convex transfer is nothing else than a mean-preserving spread, as studied by Rothschild and Stiglitz [12]. In dimension \( d \geq 2 \), concave transfers are related to fusions [4] and to mean preserving bifurcations [6]. Simple supermodular transfers and their iterations have been studied by Tchen [15].

3. Main results

3.1. General ultramodular transfers

For univariate distributions Müller [9] and Machina and Pratt [7] show that mean-preserving spreads correspond to taking mass from some bounded interval and moving it above and below this interval, without affecting the mean. The following theorem shows that something similar holds for ultramodular transfers in the multivariate case. The following notation is used: given \( x \in \mathbb{R}^d \), define the upper set \( U(x) := \{ z \in \mathbb{R}^d : z \geq x \} \) and lower set \( L(x) := \{ z \in \mathbb{R}^d : z \leq x \} \), and for two ordered points \( x \leq y \), call \( B(x, y) := \{ z \in \mathbb{R}^d : x \leq z \leq y \} \) the interval between \( x \) and \( y \).

**Theorem 3.1.** Let \( P, Q \) be two probability measures having finite support on \( \mathbb{R}^d \) with \( \text{bar}(P) = \text{bar}(Q) \) such that for some \( x \leq y \) we have \( \text{supp}(P) \subset B(x, y) \) and \( \text{supp}(Q) \subset L(x) \cup U(y) \). Then \( Q \) can be obtained from \( P \) via a sequence of simple ultramodular transfers.

Theorem 3.1 justifies the following definition.
Definition 3.2. A transfer $\mu := \sum_{i=1}^{n} \beta_i \delta_{z_i} - \sum_{i=1}^{m} \alpha_i \delta_{w_i}$ is called ultramodular if there exist $x \preceq y$ such that $w_1, \ldots, w_m \in B(x, y)$, $z_1, \ldots, z_n \in L(x) \cup U(y)$, and $\sum_{i=1}^{n} \beta_i z_i = \sum_{i=1}^{m} \alpha_i w_i$.

The reverse transfer is called inframodular. (See Fig. 1.)

It is interesting to note that the concept of an inframodular (or ultramodular) transfer involves both the vector space and the order structure of $\mathbb{R}^d$, whereas the concave (or convex) transfer is based only on the vector space structure of $\mathbb{R}^d$. An ultramodular transfer moves probability mass from some points in an interval to other points that are either smaller or larger than all points in the interval. A convex transfer just moves mass away from a point. In a univariate setting the difference between the two concepts disappears, but in the multivariate case they represent two different attitudes towards randomness.

3.2. Integral orders and transfers

Definition 3.3. A probability measure $P$ is dominated by a probability measure $Q$ with respect to the integral order $\preceq_{\mathcal{F}}$ (denoted $P \preceq_{\mathcal{F}} Q$) if

$$\int u \, dP \preceq \int u \, dQ \quad \text{for all } u \in \mathcal{F}.$$ 

The economic meaning of this definition is that any expected utility maximizer with a utility function $u \in \mathcal{F}$ prefers the lottery $Q$ to the lottery $P$. 

Fig. 1. General inframodular transfer.
For the general theory of stochastic orders, the reader is referred to Müller and Stoyan [11] and Shaked and Shanthikumar [14]. Arlotto and Scarsini [1] study a family of integral orders $\leq_F$ where $F$ can be, among others, any of the classes in Definition 2.1.

Rothschild and Stiglitz [12] prove (under some regularity conditions) that if a measure $P$ on the real line dominates $Q$ in terms of the concave order, then $Q$ can be obtained from $P$ via a sequence of mean preserving spreads. Machina and Pratt [7] refine this result using a more general definition of mean preserving spread. Elton and Hill [5] and Grant et al. [6] prove analogous results for measures on $\mathbb{R}^d$. The following theorem proves a similar result for the inframodular order.

**Theorem 3.4.** Let $F$ be the class of inframodular functions and let $P$ and $Q$ be two measures on $\mathbb{R}^d$ with finite support. Then the following statements are equivalent:

(a) $P \leq_F Q$,
(b) $Q$ can be obtained from $P$ by a finite number of simple inframodular transfers,
(c) $Q$ can be obtained from $P$ by a finite number of inframodular transfers as in Definition 3.2.

4. Proofs

4.1. General transfers

A set $S \subset \mathbb{R}^d$ is called comonotonic if it is totally ordered in the natural component-wise order of $\mathbb{R}^d$. Given a convex set $A \in \mathbb{R}^d$, the set of its extreme points is denoted by $\text{Ex}(A)$.

**Lemma 4.1.** Let $P$ be any measure supported on $B(x, y)$, and call $P^*$ a probability measure supported on $\text{Ex}(B(x, y))$, such that $\text{supp}(P^*)$ is comonotonic and $\text{bar}(P^*) = \text{bar}(P)$. Then $P^*$ can be obtained from $P$ via a sequence of ultramodular transfers.

**Proof.** First, existence of $P^*$ is shown. Call $P_1, \ldots, P_d$ the univariate marginals of $P$. For each $P_i$, there exists a measure $P_i^*$ supported on the extreme points $x_i, y_i$ and such that $\text{bar}(P_i^*) = \text{bar}(P_i)$. Consider the upper Fréchet bound of $d$-variate measures with marginals $P_1^*, \ldots, P_d^*$. This is $P^*$.

Take each point $z \in \text{supp}(P)$ and split its mass into the two points $(x_1, z_2, \ldots, z_d)$ and $(y_1, z_2, \ldots, z_d)$ in such a way that the barycenter is preserved (there is only one way to do this). Now all the points in the support of the new measure have their first coordinate equal to either $x_1$ or $y_1$. Repeat the operation for all the remaining coordinates. Now the new measure $\tilde{P}$ is supported only on extreme points of $B(x, y)$. For any pair of points $s, t \in \text{supp}(\tilde{P})$, move as much mass as possible to $s \land t$ and $s \lor t$ keeping the barycenter fixed. In the end, the obtained measure is exactly $P^*$. □

Lemma 4.1 says that using a sequence of ultramodular transfers, any measure on a compact interval can be transformed into the unique measure whose univariate marginals are maximal with respect to the convex order (and are therefore supported on the extreme points of the interval), and whose joint distribution is the upper Fréchet bound in the class of $d$-variate distributions with these marginals.
Proof of Theorem 3.1. Consider each point in $\text{supp}(Q) \cap L(x)$ one by one and move its mass along the first coordinate upwards towards $x_1$, while moving the mass of points in $\text{supp}(Q) \cap U(y)$ downwards towards $y_1$, all this without changing the barycenter. Stop when no mass can be moved further, namely when either all mass in $L(x)$ rests on points having first coordinate equal to $x_1$ or all mass in $U(y)$ rests on points having first coordinate $y_1$. Call the obtained probability measure $Q_1$. Repeating the same procedure with the other coordinates yields a probability measure $Q_d$ with the property that there is an index set $I \subseteq \{1, \ldots, d\}$ such that for all $z \in \text{supp}(Q_d)$, it is either $z_i = x_i$ for all $i \in I$ (if $z \in L(x)$) or $z_i = y_i$ for all $i \not\in I$ (if $z \in U(y)$).

In light of Lemma 4.1, the proof can be completed by showing that $Q_d$ can be obtained from $P^*$ via a sequence of ultramodular transfers. To do so it is sufficient to show that for a fixed $z \in \text{supp}(Q_d)$, a measure $P_z$ can be obtained from $P^*$ via a sequence of ultramodular transfers, where $P_z$ is comonotonic and has the same mass in $z$ as $Q_d$ and $\text{supp}(P_z) \subseteq \{z\} \cup \prod_{i=1}^d \{x_i, y_i\}$. The proof then follows by induction.

Without loss of generality assume $z \in L(x)$ and distinguish two cases.

If $\delta := Q_d((z)) \leq P^*(\{x\})$ then $P_z$ can be obtained from $P^*$ by moving the mass $\delta$ from the point $x$ to the point $z$ using a sequence of ultramodular transfers indexed over $j \not\in I$ that move mass along the $j$-th coordinate from $x_j$ to $y_j < x_j$ and at the same time move mass from some point $s$ in the support of $P^*$ with $s_j = x_j$ along the same coordinate from $x_j$ to $y_j$. As a consequence, the $j$-th marginal is transformed from the one of $P^*$ (supported on $x_j$ and $y_j$) to the one of $P_z$ (supported on $z_j, x_j$, and $y_j$). Once this is done some supermodular transfers within $\prod_{i=1}^d \{x_i, y_i\}$ may be necessary to get the comonotonic probability measure $P_z$.

In the other case $\eta := P^*(\{x\}) < Q_d((z))$. Then move all the mass $\eta$ from the point $x$ to the point $z$ as above. Then continue moving mass from the smallest point $x' \geq x$ with $x' \in \text{supp}(P^*)$ to the point $z$ in the same fashion. Iterate this procedure as many times as necessary to move mass $\delta$ to the point $z$. Again, at the end of this procedure, some supermodular transfers within $\prod_{i=1}^d \{x_i, y_i\}$ may be necessary to obtain the comonotonic probability measure $P_z$. □

The proof of Theorem 3.4 requires some known results from functional analysis and some theory of discrete inframodularity. These results are described in the next subsections.

4.2. Duality theory

For $S \subset \mathbb{R}^d$ compact, denote by $C$ the set of continuous functions on $S$. By the compactness assumption on $S$, these functions are all bounded and therefore integrable with respect to any $\mu \in \mathcal{M}$.

Integrals are often written as a bilinear form $\langle f, \mu \rangle = \int f \, d\mu = \int f \, d\mu^+ - \int f \, d\mu^-$. Some results from functional analysis are presented. The details can be found, e.g., in Choquet [3, §22].

A pair $(E, F)$ of vector spaces is said to be in duality if there is a bilinear mapping $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{R}$. The duality is said to be strict if for each $0 \neq x \in E$ there is a $y \in F$ with $\langle x, y \rangle \neq 0$ and if for each $0 \neq y \in F$, there is an $x \in E$ with $\langle x, y \rangle \neq 0$.

Unfortunately, the duality $(\mathcal{M}, C)$ is not strict, as $\langle f, \mu \rangle = 0$ for all $\mu \in \mathcal{M}$ only implies that $f$ is constant. But strict duality can be obtained by identifying functions which differ only by a constant. Formally, define an equivalence relation $f \sim g$ if $f - g$ is constant. Equivalently, fix some $s_0 \in S$ and require $f(s_0) = 0$. With utility functions in mind, it is quite natural to identify...
functions that differ only by a constant, as they lead to the same preference relation. Denote the corresponding quotient space by $\mathcal{C}_\sim$.

Lemma 4.2. $\mathcal{M}$ and $\mathcal{C}_\sim$ are in strict duality under the bilinear mapping

$\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{C}_\sim \to \mathbb{R},$

$\langle \mu, f \rangle = \int f \, d\mu.$

A crucial role in our further investigations is played by the bipolar theorem for convex cones. Given a vector space $V$, a subset $K \subset V$ is called a cone if $x \in K$ implies $\alpha x \in K$ for all $\alpha \geq 0$. Given any subset $M \subset V$, the convex cone $\text{co}(M)$ generated by $M$ is the smallest convex cone that contains $M$.

Define the dual cone of an arbitrary set $M \subset E$ by $M^\ast = \{ y \in F : \langle x, y \rangle \geq 0 \text{ for all } x \in M \}$. It is easy to see that $M^\ast$ is a convex cone.

For any duality $(E, F)$ define the weak topology $\sigma(E, F)$ on $E$ as the weakest topology on $E$ such that the mappings $x \mapsto \langle x, y \rangle$ are continuous for all $y \in F$. Now the bipolar theorem for convex cones can be stated as follows, see [3, Corollary 22.10].

Theorem 4.3. Suppose $E$ and $F$ are in strict duality and $X \subset E$ is an arbitrary set. Then $X^{**}$ is the weak closure of the convex cone generated by $X$.

For more results about duality theory and its applications in the economic literature, we refer to [2].

4.3. Discrete inframodularity

We now define discrete inframodular functions on a finite lattice $S \subset \mathbb{R}^d$. Assume that $S = \bigtimes_{i=1}^d S_i = \bigtimes_{i=1}^d \{ x_{i,1}, \ldots, x_{i,n_i} \}$ is a finite lattice, where the elements of $S_i$ are ordered, i.e., $x_{i,1} < \cdots < x_{i,n_i}$. Define the difference operator in direction $i$ computed at point $x = (x_{1,k_1}, \ldots, x_{d,k_d})$ as

$$\Delta_i f(x) := \frac{f(x_{1,k_1}, \ldots, x_{i-1,k_{i-1}}, x_{i,k_i+1}, x_{i+1,k_{i+1}}, \ldots, x_{d,k_d}) - f(x)}{x_{i,k_i+1} - x_{i,k_i}}, \quad x \in S, \ k_i < n_i.$$

The function $f : S \to \mathbb{R}$ is discrete component-wise concave if the mapping $x_i \mapsto \Delta_i f(x_1, \ldots, x_i, \ldots, x_d)$ is decreasing for all $i = 1, \ldots, d$, for any fixed $x_j \in S_j, j \neq i$. As in the univariate case, this is equivalent to requiring for any three consecutive points $x_{i,k_i}, x_{i,k_i+1}, x_{i,k_i+2}$ and for any fixed $x_j \in S_j, j \neq i$, that

$$f(x_1, \ldots, x_{i,k_i+1}, \ldots, x_d) \geq \alpha f(x_1, \ldots, x_{i,k_i+2}, \ldots, x_d) + (1 - \alpha) f(x_1, \ldots, x_{i,k_i}, \ldots, x_d),$$

where $\alpha = (x_{i,k_i+1} - x_{i,k_i})/(x_{i,k_i+2} - x_{i,k_i}),$ i.e., $\alpha$ is such that $x_{i,k_i+1} = \alpha x_{i,k_i+2} + (1 - \alpha) x_{i,k_i}$. The classical definition of submodularity is valid on any lattice and is equivalent to the requirement that $\Delta_i \Delta_j f(x) \leq 0$ for all $i \neq j$ and all $(x_{1,k_1}, \ldots, x_{d,k_d}) \in S$ with $k_i < n_i, k_j < n_j$.

A function $f$ on a finite lattice $S \subset \mathbb{R}^d$ is called discrete inframodular if it is discrete component-wise concave and submodular.
Lemma 4.4. The following conditions are equivalent:

(a) The function $f$ is inframodular.
(b) $\Delta_i f(x)$ is a decreasing function of $x$.
(c) The function $f : S \to \mathbb{R}$ is submodular and component-wise concave.

For the proof of the above lemma, see, e.g., Marinacci and Montrucchio [8]. The following consistency result holds.

Lemma 4.5.

(a) If $f : \mathbb{R}^d \to \mathbb{R}$ is inframodular, then its restriction to $S$ is discrete inframodular.
(b) Any discrete inframodular function $f : S \to \mathbb{R}$ can be extended to an inframodular function $f : \mathbb{R}^d \to \mathbb{R}$.

Proof. Part (a) is obvious. For (b) the extension has to be defined. Between grid points, this is done in a component-wise linear fashion. For $z \in \times_{i=1}^d [x_{i,k_i}, x_{i,k_i+1}]$ write the coordinates as $z_i = \alpha_ix_{i,k_i} + (1 - \alpha_i)x_{i,k_i}$, $i = 1, \ldots, d$. The coordinate-wise linear extension can then be defined iteratively, starting with

$$f(z_1, x_{2,k_2}, \ldots, x_{d,k_d}) := \alpha_1 f(x_{1,k_1+1}, x_{2,k_2}, \ldots, x_{d,k_d})$$

$$+ (1 - \alpha_1) f(x_{1,k_1}, x_{2,k_2}, \ldots, x_{d,k_d}).$$

In step $i$, define

$$f(z_1, \ldots, z_{i-1}, z_i, x_{i+1,k_i+1}, \ldots, x_{d,k_d}) := \alpha_i f(z_1, \ldots, z_{i-1}, x_{i,k_i+1}, \ldots, x_{d,k_d})$$

$$+ (1 - \alpha_i) f(z_1, \ldots, z_{i-1}, x_{i,k_i}, \ldots, x_{d,k_d}).$$

Thus a piecewise linear extension of $f : S \to \mathbb{R}$ to $\text{conv}(S) = \times_{i=1}^d [x_{i,k_i}, x_{i,k_i}]$ has been obtained. It is straightforward to see that this piecewise linear extension is component-wise concave and submodular (this construction is similar to the extension of a subcopula to a copula: see [13]). Outside of $\text{conv}(S)$, extend the function by component-wise linear extrapolation as in Müller and Scarsini [10, proof of Theorem 2.7], which leads to a function that is inframodular on all of $\mathbb{R}^d$. □

The two properties of Lemma 4.5 together imply that the set of discrete inframodular functions $f : S \to \mathbb{R}$ is equivalent to the set of restrictions of inframodular functions $f : \mathbb{R}^d \to \mathbb{R}$ to $S$ if $S$ is a finite lattice. It follows, therefore, that for probability measures $P$ and $Q$ with finite support in $\mathbb{R}^d$, the following statements are equivalent:

(i) $\int f \, dP \leq \int f \, dQ$ for all inframodular functions $f : \mathbb{R} \to \mathbb{R}$;

(ii) $\int f \, dP \leq \int f \, dQ$ for all discrete inframodular functions $f : S \to \mathbb{R}$, where $S$ is the smallest finite lattice containing the supports of $P$ and $Q$.

4.4. Stochastic orders and transfers

Using the properties of the two previous subsections, Theorem 3.4 can now be proved.
Proof of Theorem 3.4. The equivalence of (b) and (c) follows from Theorem 3.1 and it is clear that (b) implies (a). Thus, it remains to show that (a) implies (b). Assume that $P$ and $Q$ are probability measures on $\mathbb{R}^d$ with finite support fulfilling $\int f \, dP \leq \int f \, dQ$ for all inframodular functions $f : \mathbb{R}^d \to \mathbb{R}$. It follows from Lemma 4.5 that this is equivalent to the statement that $\int f \, dP \leq \int f \, dQ$ for all inframodular functions $f : S \to \mathbb{R}$, where $S$ is the smallest lattice containing the supports of $P$ and $Q$. Using the terminology of duality theory as described in Section 4.2 the condition can be rewritten as $Q - P \in \mathcal{F}^*$, where $\mathcal{F}$ is the set of all inframodular functions $f : S \to \mathbb{R}$. The fact that $\mathcal{F}$ is induced by the set $M$ of inframodular transfers can be rewritten as $\mathcal{F} = M^*$. Thus $Q - P \in M^{**}$.

Therefore it follows from Theorem 4.3 that $Q - P$ is in the weak closure of the convex cone generated by $M$. As $S$ is finite, the set $M$ of inframodular transfers on this set has finite dimension and is compact, and therefore the convex cone generated by $M$ is weakly closed. Thus, $Q - P = \sum_{i=1}^n \gamma_i \mu_i$ with $\gamma_i > 0$ and $\mu_i \in M$. As $P$ and $Q$ are probability measures, it is possible to choose $\gamma_i \leq 1$. But this means that $Q$ can be obtained from $P$ by a finite number of inframodular transfers $\gamma_i \mu_i$.

Acknowledgments

The authors thank Alessandro Arlotto for interesting discussions, Davide Crapis for a careful reading, and the referees and the Associate Editor for useful suggestions on how to make the paper more readable.

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